ELLIPTICAL CONTACT ON ELASTIC INCOMPRESSIBLE COATINGS

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The study addresses a contact problem of a thin elastic (isotropic, incompressible and homogeneous) layer bonded to a rigid plane foundation and indented by a rigid frictionless punch in the form of an elliptical paraboloid. Given the total load, approximate analytical results for the contact ellipse, penetration depth and contact pressure distribution are presented.

Key words : *elastic isotropic incompressible coating, ellipsoidal frictionless indenter, elliptical contact*

1. Introduction

Contact problems involving bodies which are covered with a coating whose elastic properties differ from those of the substrate are frequently encountered in industry. Protective adhesive coatings are used in tribosystems, microelectronic industry, polygraphic, textile and paper machinery, naval vessels and airplanes, etc. The coatings increase the wear resistance, decrease the energy losses, suppress noise, quell vibrations and protect the components from impact damage. Incompressible elastomers are successful partly because of their large strain to fracture, resulting in the ability to absorb large amounts of impact energy in elastic deformation. They are reinforced by adhesion to the substrate. Good adhesion reduces the strains in the coating and is essential for their optimal performance.

The mathematical problem of the contact of a rigid punch pressed against a compliant layer bonded to a rigid half-space is of great practical interest. Cylindrical, spherical and flat-ended indenters were discussed by the Russian writers Lebedev, Ufliand, Aleksandrov, Vorovich et al. in the early 1960s. Axi-symmetrical contact problems for an elastic layer of arbitrary thickness are studied in Aleksandrov and Pozharskii [1]. Meijers [2] obtained an asymptotic solution to the problem of a rigid cylinder pressed on an isotropic elastic plane layer of any thickness and Poisson's ratio. Using Legendre polynomials, Jaffar [3] examined the case of circular and flat-ended indenters. The method of matched asymptotic expansions was used for the indentation by a rigid elliptical paraboloid of an elastic compressible layer bonded to a rigid foundation by Argatov [4] who tackled also the case of a thin layer [5]. The solutions were obtained in an explicit form for the axi-symmetric problem.

Matthewson [6] published an interesting solution for the indentation by a rigid, axially symmetric, frictionless punch of an elastic coating, bonded to a rigid half-space. The essence of the method lies in the fact that a simple polynomial approximation across the layer thickness for the displacement vector is assumed and an averaging technique through the layer thickness is applied. Due to this approximation, most equations and conditions can be

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satisfied only averaged through the coating thickness. Matthewson obtained simple analytic formulas for the contact pressure distribution and contact radius. Especially, in the case of an incompressible compliant coating, these analytic results agreed surprisingly well, when compared to numerical results of other authors.

In the current paper, a thin elastic isotropic incompressible layer bonded to a rigid foundation is indented by a rigid frictionless punch in the form of an elliptical paraboloid. Simple approximations for the contact ellipse, penetration depth and contact pressure distribution are presented for a given total load. The above averaging method used by Matthewson [6] for the axially symmetric case is applied. Any contact of two rigid highly congruent bodies, both coated with thin deformable layers of the same thickness, can be replaced with a contact of a coated rigid half space with a rigid uncoated indenter. This indenter can be replaced by an elliptic paraboloid with the same main curvatures. The results obtained in the current paper for an incompressible coating are more general compared to the axially symmetric case studied by Matthewson [6]. In Par. 2 the mathematical model is presented, Par. 3 shows results of the model in Figures and the conclusion is in Par. 4.

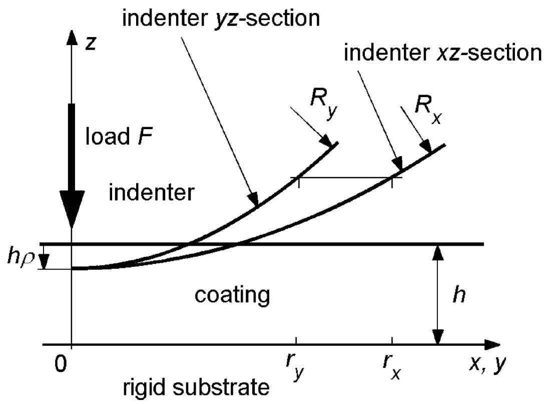

Fig.1: Indentation of an elastic isotropic incompressible coating by a rigid frictionless ellipsoidal surface (elliptical paraboloid)

2. Mathematical model

2.1. Formulation of the problem

Cartesian coordinates x, y, z are introduced in an elastic plane layer of thickness h, bonded to a rigid substrate at the plane $z = 0$ (Fig. 1). Material of the coating is isotropic, homogeneous and incompressible. The layer is indented by a rigid frictionless ellipsoidal surface, with its axis situated in z axis. The radii of the principal curvatures at the indenter vertex are R_x, R_y, and xz, yz planes are the planes of symmetry of the contact. A constant compressive force F is applied in z axis.

For a thin layer, the displacements u_x, u_y, u_z in the contact region of the layer (referred to the rigid substrate) are approximated as

$$\begin{aligned} u_x(x,y,z) &= \alpha_x(x,y)\,z + \beta_x(x,y)\,z^2 \;, \\ u_y(x,y,z) &= \alpha_y(x,y)\,z + \beta_y(x,y)\,z^2 \;, \\ u_z(x,y,z) &= \gamma(x,y)\,z \;. \end{aligned} \tag{1}$$

The form of (1) corresponds to that for the displacements in Matthewson [6]. Only the Cartesian x and y coordinates are used instead of the polar coordinates, while the dependence on z coordinate remains the same. Functions $\alpha_x(x,y)$, $\alpha_y(x,y)$, $\beta_x(x,y)$, $\beta_y(x,y)$ are to be found and function $\gamma(x,y)$ is determined except for a constant by the indenter profile. In fact, setting $z = h$ in $(1)_3$ and approximating quadratically the surface of the ellipsoidal indenter yields

$$\gamma(x,y) = \varrho + \frac{x^2}{2\,h\,R_x} + \frac{y^2}{2\,h\,R_y} \ . \tag{2}$$

ϱ is an unknown dimensionless constant, $h\,\varrho$ means the penetration depth. Note that (1) meet the conditions $u_x = u_y = 0$ at the rigid interface $z = 0$. The Hooke law of an elastic isotropic incompressible coating is of the form

$$\begin{gathered} \sigma_{xx} = -p + 2\,\mu\,\varepsilon_{xx} \ , \qquad \sigma_{yy} = -p + 2\,\mu\,\varepsilon_{yy} \ , \qquad \sigma_{zz} = -p + 2\,\mu\,\varepsilon_{zz} \ , \\ \sigma_{xy} = 2\,\mu\,\varepsilon_{xy} \ , \qquad \sigma_{xz} = 2\,\mu\,\varepsilon_{xz} \ , \qquad \sigma_{yz} = 2\,\mu\,\varepsilon_{yz} \ . \end{gathered} \tag{3}$$

Here, σ_{ij} and $\varepsilon_{ij} = u_{(i,j)}$ $(i,j = x,y,z)$ are the stress tensor and small deformation tensor, while μ and $p(x,y,z)$ denote the shear modulus and the spherical part of the stress tensor, respectively. It follows from $(1)_3$ that $\varepsilon_{zz} = \gamma(x,y)$.

The equilibrium equations and the incompressibility condition are $\sigma_{ij,i} = 0$ and $\varepsilon_{ii} = 0$, respectively. A comma followed by a subscript denotes a partial differentiation and summing is taken over repeated pairs of subscripts i. Due to the simple approximation (1), most equations and boundary conditions can be satisfied only averaged through the coating thickness h.

Assume that the vertical projection of the contact edge is formed by an ellipse with the semi-axes r_x, r_y in x and y direction, respectively, and, moreover, let $r_x, r_y \gg h$. Introduce elliptical coordinates r, ϕ in xy plane: $x = r_x\,r\,\cos\phi$, $y = r_y\,r\,\sin\phi$. For the border ellipse $r = 1$ and $\overline{x} = r_x\,\cos\phi$, $\overline{y} = r_y\,\sin\phi$. Denote

$$\frac{r_y}{r_x} = \delta \ . \tag{4}$$

In what follows, the value of a function $f(x,y)$ at point $(\overline{x},\overline{y})$ is denoted by $\overline{f}$ and the average of a function $f(z)$ through h is denoted by $\underline{f}$.

The contact is assumed frictionless, i.e. $\sigma_{xz}(x,y,h) = \sigma_{yz}(x,y,h) = 0$, and (1), (3) yield

$$\alpha_x + 2\,\beta_x\,h + \gamma_{,x}\,h = 0 \ , \qquad \alpha_y + 2\,\beta_y\,h + \gamma_{,y}\,h = 0 \ . \tag{5}$$

The average incompressibility condition, $\underline{\varepsilon}_{ii} = 0$, gives, after eliminating β_x and β_y, by means of (5),

$$\frac{h}{3}\,(\alpha_{x,x} + \alpha_{y,y}) - \frac{h}{6}\left(\frac{1}{R_x} + \frac{1}{R_y}\right) + \gamma = 0 \ . \tag{6}$$

The equilibrium equations in x and y directions, averaged through h, give by means of (1), (3) and (5)

$$\begin{aligned} \frac{1}{2\,\mu}\,\underline{p}_{,x} &= \frac{h}{3}\left(\alpha_{x,xx} + \frac{1}{2}\,\alpha_{x,yy} + \frac{1}{2}\,\alpha_{y,xy}\right) - \frac{1}{2\,h}\,\alpha_x \ , \\ \frac{1}{2\,\mu}\,\underline{p}_{,y} &= \frac{h}{3}\left(\alpha_{y,yy} + \frac{1}{2}\,\alpha_{y,xx} + \frac{1}{2}\,\alpha_{x,yx}\right) - \frac{1}{2\,h}\,\alpha_y \ . \end{aligned} \tag{7}$$

Note that the averaging procedure through h makes the equilibrium equation in z direction irrelevant.

2.2. Solution to the problem

Symmetry of the displacement (1) with respect to the planes $x = 0$ and $y = 0$ yields the conditions $\alpha_x(x,y) = -\alpha_x(-x,y) = \alpha_x(x,-y)$ and $\alpha_y(x,y) = -\alpha_y(x,-y) = \alpha_y(-x,y)$. Take α_x, α_y in the form of infinite polynoms in x, y. Then, α_x can contain only odd powers in x and even powers in y, while α_y contains only odd powers in y and even powers in x. Thus,

$$\begin{aligned} \alpha_x(x,y) &= b_1\,x + b_2\,x^3 + b_3\,x\,y^2 + \dots\ , \\ \alpha_y(x,y) &= c_1\,y + c_2\,y^3 + c_3\,x^2\,y + \dots\ , \end{aligned} \tag{8}$$

where $b_1, b_2, b_3, \dots, c_1, c_2, c_3, \dots$ are unknown constants. Insert (8) into (6), valid for any x, y, and use (2). It follows that

$$\begin{gathered} b_1 + c_1 - \frac{1}{2}\left(\frac{1}{R_x} + \frac{1}{R_y}\right) + \frac{3}{h}\,\varrho = 0\ , \\ 3\,b_2 + c_3 + \frac{3}{2\,h^2\,R_x} = 0\ , \qquad 3\,c_2 + b_3 + \frac{3}{2\,h^2\,R_y} = 0 \end{gathered} \tag{9}$$

and higher order coefficients in (8) are zero. Insert (8) into (7) and integrate to obtain

$$b_3 = c_3 \tag{10}$$

and $\underline{p}(x,y)$, by means of of $(9)_{2,3}$, becomes

$$\begin{aligned} \frac{1}{2\,\mu}\,\underline{p} &= p_0 + p_1\,x^2 + p_2\,y^2 + p_3\,x^2\,y^2 + p_4\,x^4 + p_5\,y^4\ , \\ p_1 &= \frac{h}{3}\,(3\,b_2 + b_3) - \frac{b_1}{4\,h} = -\frac{b_1}{4\,h} - \frac{1}{2\,h\,R_x}\ , \\ p_2 &= \frac{h}{3}\,(3\,c_2 + b_3) - \frac{c_1}{4\,h} = -\frac{c_1}{4\,h} - \frac{1}{2\,h\,R_y}\ , \\ p_3 &= -\frac{b_3}{4\,h}\ , \qquad p_4 = -\frac{b_2}{8\,h}\ , \qquad p_5 = -\frac{c_2}{8\,h}\ . \end{aligned} \tag{11}$$

The condition of zero averaged normal stress,

$$\overline{\underline{\sigma}}_{zz} = \underline{\sigma}_{zz}(\overline{x},\overline{y}) = -\overline{\underline{p}} + 2\,\mu\,\overline{\gamma} = 0\ ,$$

at the contact edge at $\overline{x} = r_x\cos\phi$ and $\overline{y} = r_y\sin\phi$, valid for any ϕ, yields by means of (2), (11)

$$\begin{gathered} p_0 + p_1\,r_x^2 + p_4\,r_x^4 = \varrho + \frac{r_x^2}{2\,h\,R_x}\ , \\ -p_1\,r_x^2 + p_2\,r_y^2 + p_3\,r_x^2\,r_y^2 - 2\,p_4\,r_x^4 = -\frac{r_x^2}{2\,h\,R_x} + \frac{r_y^2}{2\,h\,R_y}\ , \\ -p_3\,r_x^2\,r_y^2 + p_4\,r_x^4 + p_5\,r_y^4 = 0\ . \end{gathered} \tag{12}$$

$(12)_3$ yields by means of $(11)_{4–6}$ and (4)

$$b_2 - 2\,\delta^2\,b_3 + \delta^4\,c_2 = 0\ ,$$

which, together with $(9)_{2\text{–}3}$, gives b_2, b_3, c_2 in the form

$$b_2 = -\frac{\delta^2}{2\,h^2\,D}\left(\frac{\delta^2+6}{R_x} - \frac{\delta^2}{R_y}\right) , \qquad b_3 = -\frac{3}{2\,h^2\,D}\left(\frac{1}{R_x} + \frac{\delta^4}{R_y}\right) ,$$
$$c_2 = \frac{1}{2\,h^2\,D}\left(\frac{1}{R_x} - \frac{1+6\,\delta^2}{R_y}\right) , \tag{13}$$

where

$$D = 1 + 6\,\delta^2 + \delta^4 \ . \tag{14}$$

$(12)_2$ yields with the aid of $(11)_{2-5}$ and (4)

$$b_1 - \delta^2\,c_1 = r_x^2\,(\delta^2\,b_3 - b_2) - \frac{4}{R_x} + \frac{4\,\delta^2}{R_y} ,$$

which, together with $(9)_1$, gives b_1, c_1 in the form

$$b_1 = \frac{1}{1+\delta^2}\left[-\frac{3\,\delta^2}{h}\,\varrho + \frac{r_x^2\,\delta^2\,d_1}{2\,h^2\,D} + \frac{\delta^2-8}{2\,R_x} + \frac{9\,\delta^2}{2\,R_y}\right] ,$$
$$c_1 = -\frac{1}{1+\delta^2}\left[\frac{3}{h}\,\varrho + \frac{r_x^2\,\delta^2\,d_1}{2\,h^2\,D} - \frac{9}{2\,R_x} + \frac{8\,\delta^2-1}{2\,R_y}\right] , \qquad d_1 = \frac{\delta^2+3}{R_x} - \frac{\delta^2\,(1+3\,\delta^2)}{R_y} \ . \tag{15}$$

Now, using $(3)_3$, (11), $(12)_1$, (13) and (15), the averaged total normal stress $\underline{\sigma}_{zz}$ in the contact ellipse can be obtained for a thin layer as

$$\frac{1}{\mu}\,\underline{\sigma}_{zz} = \frac{3\,r_x^2\,\delta^2\,\varrho}{2\,h^2\,(1+\delta^2)}\,(1-r^2) + \frac{r_x^4\,\delta^2\,d_1}{4\,h^3\,D}\,(r^2-r^4)\,\cos^2\phi +$$
$$+ \frac{r_x^4\,\delta^4}{8\,h^3\,D\,(1+\delta^2)}\left[2\,d_1\,(1-r^2) - d_2\,(1-r^4)\right] , \tag{16}$$
$$d_2 = (1+\delta^2)\left(\frac{1}{R_x} - \frac{1+6\,\delta^2}{R_y}\right) \ .$$

2.3. Parameters $\boldsymbol{\delta}$, $\boldsymbol{\varrho}$, $\mathbf{r_x}$

It remains to find unknown parameters δ, ϱ and r_x. Quotient $\delta = r_y/r_x$ defines the shape of the contact. In his model of the indentation of a thin elastic incompressible layer by an ellipsoidal punch, Barber [7] assumed that both the contact pressure and its gradient across the contact boundary are zero there. This gives for the contact form the following variation of δ with $k = R_y/R_x$

$$\delta^2 = \frac{1}{6}\left\{k - 1 + \left[(k-1)^2 + 36\,k\right]^{\frac{1}{2}}\right\} \ .$$

In the classical Hertzian solution for a layer of an infinite thickness, δ also depends only on k, however, through elliptical integrals [8] and for $0.2 < k \leq 1$ this variation can be approximated as $\delta = k^{2/3}$. Argatov [5] investigated asymptotic expansions for the indentation of a thin plane layer by a rigid elliptical paraboloid. The first approximation yields

$$\delta = k^{\frac{1}{2}} , \tag{17}$$

which corresponds to a plane section of the indenter parallel to the layer surface. The Hertzian solution gives a higher (or lower) swelling of the coating for more (or less) distant edge points, respectively, compared to the plane section. Barber's model gives quite the opposite. Thus, the Hertzian solution gives a thinner and the Barber solution a thicker contact ellipse compared to that of a plane section of the indenter. In the present paper relation (17) is accepted.

In order to determine the unknown parameters ϱ and r_x, special cases are considered first, i.e. axial symmetry and plane strain.

2.4. Axial symmetry

Matthewson [6] considered an axi-symmetric case of the spherical indenter. In this case, the problem formulated in the polar coordinates r, ϕ ($x = r_a\, r \cos\phi$, $y = r_\mathrm{a}\, r \sin\phi$), with a simple approximation for u_r, u_z analogical to (1), leads to ordinary differential equations with one variable, r, and general solutions can be found both inside and outside the contact region. The conditions at the contact edge r_a ($r = 1$), i.e. the continuity conditions for $\underline{\varepsilon}_{rr}$, $\underline{\varepsilon}_{zz}$ (incompressibility also yields that for $\underline{\varepsilon}_{\phi\phi}$) and the total load condition

$$-F = r_\mathrm{a}^2 \int_0^{2\pi} \int_0^1 \underline{\sigma}_{zz}\, r \,\mathrm{d}\phi \,\mathrm{d}r \tag{18}$$

yields an equation for r_a as

$$\frac{2^5\, h^3\, R_\mathrm{a}\, F}{\pi\, \mu\, r_\mathrm{a}^6} = 1 + \frac{24}{\eta_\mathrm{a}^2} + \frac{6\, K_1(\eta_\mathrm{a})}{2\, K_1(\eta_\mathrm{a}) + \eta_\mathrm{a}\, K_0(\eta_\mathrm{a})}\,, \qquad \eta_\mathrm{a} = \left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{r_\mathrm{a}}{h}\,. \tag{19}$$

Here R_a is the indenter radius, and K_0, K_1 are the modified Bessel functions of zero and first order, respectively. By replacing the varied radial coordinate r in the coefficients of the differential equation for the radial displacement u_r outside the contact by its fixed value at the contact edge r_a (see [9], equation (A2) for isotropy), the following equation for an approximate value of r_a was obtained in this modification of the Matthewson approach

$$\frac{2^5\, h^3\, R_\mathrm{a}\, F}{\pi\, \mu\, r_\mathrm{a}^6} = 1 + \frac{6}{\eta_\mathrm{a}} + \frac{2^6}{3\, \eta_\mathrm{a}^2}\,. \tag{20}$$

No special functions are used in (20). r_a in (20) is a good approximations of r_a in (19) for $h \ll r_\mathrm{a}$ (see Fig. 3). The reason for it is the fact that stress and strain decay quickly near r_a with the increasing $r > r_\mathrm{a}$.

2.5. Plane strain case

In the case of plane strain (cylindrical indenter), the corresponding value of the contact half-width, denoted as r_p, takes the form [10]

$$\frac{3 \cdot 5\, h^3\, R_\mathrm{p}\, F_\mathrm{p}}{2\, \mu\, r_\mathrm{p}^5} = 1 + \frac{2^3\, 5}{3^2\, \eta_\mathrm{p}} + \frac{3 \cdot 5}{\eta_\mathrm{p}^2}\,, \qquad \eta_\mathrm{p} = \left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{r_\mathrm{p}}{h}\,. \tag{21}$$

Here, R_p and F_p are the radius of the cylindrical indenter and the load per one meter of the cylinder, respectively. (21) has been obtained using the Matthewson approach [6] as a special case in an analysis of the contact of two parallel circular coated cylinders [10], if the radii tend to infinity, with the equivalent contact radius R_p kept constant.

2.6. General case

For a general ellipsoidal case, it is not possible to proceed exactly as Matthewson [6] did in the axi-symmetric case in order to find the contact region. In fact, the system of partial differential equations (6–7) for $\alpha_x(x,y)$, $\alpha_y(x,y)$, $\gamma(x,y)$ should be also solved outside the contact region, with boundary conditions at the contact boundary varying with ϕ. This solution for an ellipsoidal indenter is not available. However, the total load condition,

$$-F = r_x\, r_y \int_0^{2\pi}\!\!\int_0^1 \underline{\sigma}_{zz}\, r \,\mathrm{d}\phi\,\mathrm{d}r \ , \tag{22}$$

can be used to find ϱ as a function of r_x, r_y. By using $(3)_3$ averaged through h, (2), (4), $(15)_3$ and (16), then, condition (22) yields ϱ for a thin layer in the form

$$\varrho = -\frac{4\,h^2\,F\,(r_x^2+r_y^2)}{3\,\pi\,\mu\,r_x^3\,r_y^3} - \frac{1}{12\,h}\left(\frac{r_x^2}{R_x}+\frac{r_y^2}{R_y}\right) \ . \tag{23}$$

Now, it remains to find δ and r_x or r_y. For the spherical indenter, some authors used the condition of zero volume change of the material in the contact region [8], [11], [12]. For the ellipsoidal case, this condition is

$$r_x\, r_y \int_0^{2\pi}\!\!\int_0^1 h\,\gamma\, r\,\mathrm{d}\phi\,\mathrm{d}r = 0 \ . \tag{24}$$

In (24), the volume change of the layer outside the contact is not taken into account. (The upper bound 1 is used for r in (24) instead of infinity). For the spherical and cylindrical indenters, condition (24) is equivalent to the condition of zero gradient (normal to the edge) of the contact pressure at the contact edge. Barber [7] also used this condition for an ellipsoidal indenter. However, it has been shown [13] that for spherical and cylindrical indenters the above conditions yield the contact width considerably different from the values obtained numerically by Meijers [2], McCormick [14] and Jaffar [3]. See also Figs. 2–3. The difference increases quickly with the decreasing contact width-to-layer thickness ratio. Condition (24), with the use of (2), (4) and (23), gives a simple relation

$$\frac{2^4\,h^3\,(R_x+R_y)\,F}{\pi\,\mu\,r_x^3\,r_y^3} = 1 \ . \tag{25}$$

For the circular contact ($r_x = r_y = r_\mathrm{a}$, $R_x = R_y = R_\mathrm{a}$), (25) takes the form of $2^5\,h^3\,R_\mathrm{a}\,F/(\pi\,\mu\,r_\mathrm{a}^6) = 1$, which differs from (19) and (20). Thus, for axial symmetry, condition (24) yields an equation for r_a different from (19) or (20).

The following way is suggested in the general case. Instead of (25) write the equation (symmetric in x and y)

$$\frac{2^4\,h^3\,F\,(R_x+R_y)}{\pi\,\mu\,r_x^3\,r_y^3} = 1 + 3\left(\frac{1}{\eta_x}+\frac{1}{\eta_y}\right) + \frac{2^5}{3}\left(\frac{1}{\eta_x^2}+\frac{1}{\eta_y^2}\right) \ , \tag{26}$$

where

$$\eta_x = \left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{r_x}{h} \qquad \text{and} \qquad \eta_y = \left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{r_y}{h} \; .$$

For $k = 1$ (spherical indenter) equation (26) takes the form of (20). Write (26) for the limiting case realizing the line contact (cylindrical indenter). Let $R_y = R_\mathrm{p}$ be fixed. Consider a series of R_x and that of F with $R_x \to \infty$, $F \to \infty$ such that (26) yields $F/r_x \to F_\mathrm{p}$ with $0 < F_\mathrm{p} < \infty$. The limit corresponds to plane strain in yz plane (cylindrical punch) with a loading F_p in Newtons per meter. Now, $k \to 0$, $\eta_x \to \infty$, and due to (17), the series of r_y obtained from (26) tends to a fixed value r_p with $0 < r_\mathrm{p} < \infty$. (26) becomes in this limit

$$\frac{2^4\, h^3\, R_\mathrm{p}\, F_\mathrm{p}}{\pi\, \mu\, r_\mathrm{p}^5} = 1 + \frac{2}{\eta_\mathrm{p}} + \frac{2^5}{3\, \eta_\mathrm{p}^2} \; . \tag{27}$$

To sum up, with (4) and (17) taken into account, equations (26), (23) and (16) for the contact semi-axis r_y, the penetration depth $h\,\varrho$ and the contact pressure distribution $S(r,\phi) = -\underline{\sigma}_{zz}$, respectively, can be written in the form

$$\begin{aligned}
\frac{2^4\, h^3\, R_y\, F\, k^{\frac{1}{2}}\,(1+k)}{\pi\, \mu\, r_y^6} &= 1 + \frac{1}{\eta_y}\left(1 + k^{\frac{1}{2}}\right) + \frac{2^5}{3\, \eta_y^2}\,(1+k) \; , \\
-\frac{3\, R_y\, h}{r_y^2}\, \varrho &= \frac{2^2\, h^3\, R_y\, F\, k^{\frac{1}{2}}\,(1+k)}{\pi\, \mu\, r_y^6} + \frac{1}{2} - \frac{3\,(1+k)}{\eta_y^2} \; , \\
\frac{2\, h^3\, R_y\,(1+k)}{\mu\, r_y^4}\, S(r,\phi) &= -\frac{3\, h\, R_y}{r_y^2}\, \varrho\,(1-r^2) - \\
&\quad - \frac{1}{1 + 6\,k + k^2}\Big[(1-k^2)\,(r^2 - r^4)\,\cos^2\phi + \\
&\quad + \frac{1}{4}\,(1+k)\,(1+5\,k)\,(1-r^4) + k\,(1-k)\,(1-r^2)\Big] \; ,
\end{aligned} \tag{28}$$

where $k = R_y/R_x$. Remind that r, ϕ are the elliptic coordinates, i.e. $x = r_x\, r\, \cos\phi$, $y = r_y\, r\, \sin\phi$. Due to (17), radius r_x is given as $r_x = r_y\, k^{-1/2}$.

With the known contact pressure distribution, $S(r,\phi) = -\underline{\sigma}_{zz}(r,\phi)$, an analytic asymptotic solution for the displacement vector, the stress and strain tensors in the coating inside the contact region can be obtained, using a perturbation method [15] with a small parameter $\varepsilon \ll h/r_x \ll 1$. Introduce non-dimensional (primed) variables and functions

$$\begin{aligned}
x &= x'\, r_x \; , \qquad y = y'\, r_y \; , \qquad z = z'\, h \; , \qquad S = S'\, \mu \; , \\
u_x &= u_x'\, r_x \; , \qquad u_y = u_y'\, r_y \; , \qquad u_z = u_z'\, h \; , \qquad p = p'\, \mu
\end{aligned}$$

into the equations of equilibrium (with the use of Hooke's law), the incompressible condition and the boundary conditions $u_x = u_y = u_z = 0$ at $z = 0$ and $-S = -p + 2\,\mu\,\varepsilon_{zz}$, $\varepsilon_{xz} = \varepsilon_{yz} = 0$ at $z = h$. Assume that u_x', u_y', u_z', p' can be represented by asymptotic series in terms of powers of ε and let r_x, r_y be of the same order of magnitude. Collect the terms of the same power of ε. The system of the differential equations is replaced by a series of simple differential systems thus obtained for the expansion coefficients that can be easily solved. The solution for the dimensional quantities u_x, u_y, u_z, p to order ε^2 becomes in the end

$$u_x = \frac{h^2}{2\,\mu}\, S_{,x}\left(z'^2 - 2\,z'\right)\ , \qquad u_y = \frac{h^2}{2\,\mu}\, S_{,y}\left(z'^2 - 2\,z'\right)\ ,$$
$$u_z = \frac{h^2}{2\,\mu}\,(S_{,xx} + S_{,yy})\left(z'^2 - \frac{z'^3}{3}\right)\ , \qquad p = S + \frac{h^2}{2}\,(S_{,xx} + S_{,yy})\left(1 + 2\,z' - z'^2\right)\ . \tag{29}$$

Note that u_z in (29) is different from the expression in $(1)_3$. The forms of (1) were just used to obtain the contact pressure distribution $S = -\underline{\sigma}_{zz}$. Then, this traction is applied to solve asymptotically the boundary-value problem in this Paragraph for u_x, u_y, u_z and p in one thin layer. (29) can be used to obtain the stress and strain in the layer.

3. Results

Let $r_{\rm pH} = [R_{\rm p}\, F_{\rm p}/(\pi\,\mu)]^{1/2}$ denote the Herzian contact width of a rigid cylinder on an incompressible layer of infinite thickness [8]. Fig. 2 shows a variation of $r_{\rm p}/r_{\rm pH}$ with $r_{\rm pH}/h$ for the cylindrical indenter. The thick solid line represent the numerical asymptotic values obtained by Meijers [2], valid for $r_{\rm pH}/h > 2$. The dashed line is calculated from (21), valid for a thin layer and obtained in [10] by the method of Matthewson [6]. Both curves differ less than 5 % for $r_{\rm pH}/h > 4$, which justifies the Matthewson approach. The thin solid curve calculated from (27) differs very little from the dashed line obtained using (21) for the whole range of $r_{\rm pH}/h$. Good coincidence of the dashed, thin solid and thick solid lines obtained for the cylindrical case corroborates the choice of (26) for general elliptical contacts.

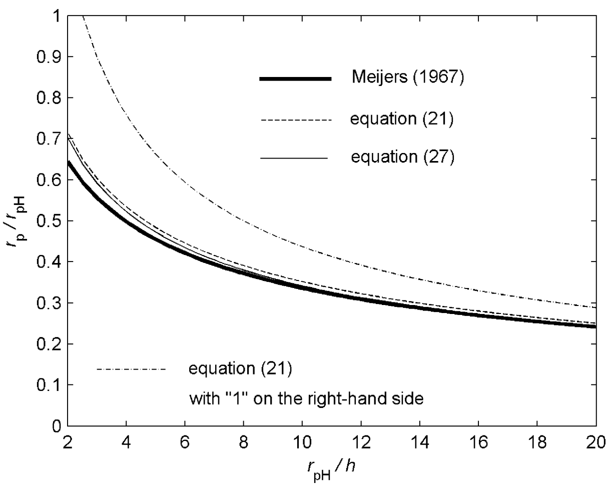

Fig.2: Variation of $r_{\rm p}/r_{\rm pH}$ with $r_{\rm pH}/h$ for a cylindrical indenter; $r_{\rm pH}$ is the Hertzian contact width

Fig. 3 is an analogy to Fig. 2 for the spherical indenter. This time, $r_{\rm aH} = [3\,R_{\rm a}\,F/(2^4\mu)]^{1/3}$ denotes the Hertzian contact radius for the spherical rigid indenter on an elastic incompressible layer of infinite thickness [8]. The thick solid line gives the numerical results by McCormick [14] and Jaffar [3] – see also Fig. 5 in Ref. [6]. The thin solid and dashed lines are calculated from equations (20) and (19), respectively, and approximate well the numerical results, which again justifies the Matthewson approach as well as its modification [9].

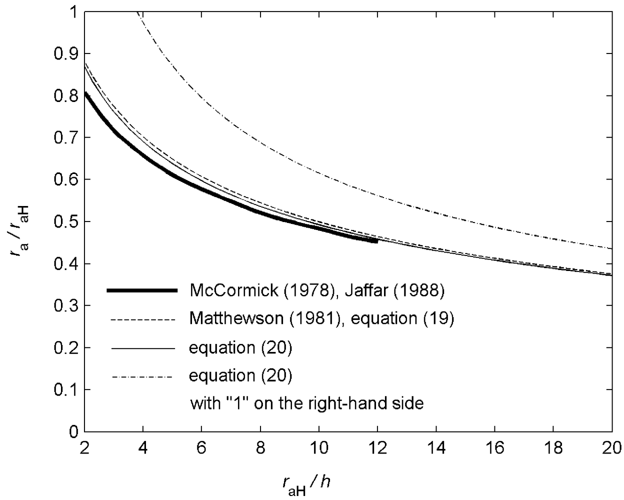

Fig.3: Variation of $r_{\mathrm{a}}/r_{\mathrm{aH}}$ *with* r_{aH}/h *for a spherical indenter;* r_{aH} *is the Hertzian contact radius*

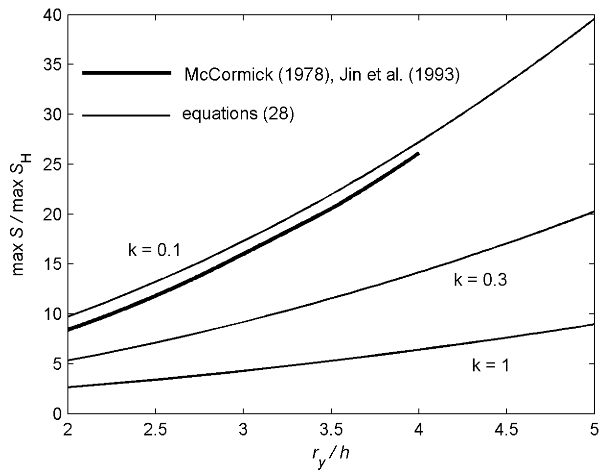

Fig.4: Variation of $\max S/\max S_{\mathrm{H}}$ *with* r_y/h *for an ellipsoidal indenter with* $k = R_y/R_x = 0.1$, 0.3 *and* 1; $\max S_{\mathrm{H}}$ *is the maximum Hertzian contact pressure for the spherical indenter with radius* R_x

The dash-dot lines calculated in Fig. 2 from (21) and in Fig. 3 from (20), but in both cases with the second and third terms on the right-hand side left out, lie much apart from the thick solid lines representing numerical asymptotic values by Meijers [2] and McCormick [14], respectively. The dash-dot lines correspond to the condition of zero gradient of the contact pressure across the contact edge (or, equivalently, to (24)), used by some other authors [12], [11], [8], [7]. It is apparent in Figs. 2–3 that for axial symmetry and plane strain the condition of zero gradient of the contact pressure across the contact edge does not give good results. On the contrary, (26) yields for these two special cases much better results.

Fig. 4 shows the variation of the quotient of the maximum contact pressures $\max S$ to $\max S_{\mathrm{H}}$ for $k = 1$, 0.5 and 0.1 as a function of r_y/h. $\max S$ is calculated from (28) for a rigid ellipsoid with the curvatures R_x, R_y. $\max S_{\rm H}$ is the maximum Hertzian contact pressure (for the layer of infinite thickness) for a rigid spherical indenter of radius R_x [8], i.e.

$$\max S_{\rm H} = \frac{2^2}{\pi} \left(\frac{3\,F\,\mu^2}{2\,R_x^2} \right)^{\frac{1}{3}} \ .$$

The curves are independent of μ. The curve for $k = 0.1$ is compared to the numerical results of McCormick [14] – for this curve see also [16] – and good agreement is obtained, which again speaks in favor of (26) applied to an ellipsoidal punch.

Fig. 5 shows a variation of the contact pressure $S(x, y)$ as calculated from (28). Here, $\mu = 0.3\,$MPa, $h = 2\,$mm, $R_y = 0.5\,$m, $k = R_y/R_x = 0.1$ and $F = 10\,$N. Axial plane cross-sections of $S(x, y)$ are shown and indicate an elliptic-bell-shaped pressure distribution. The gradient of S at the boarder ellipse in the normal direction takes a finite negative value. Numerical results for the spherical indenter yield this gradient value infinite [3]. For a thin layer, however, the above analytic and numeric forms of S for the spherical indenter are close to each other, with the exception of a narrow boundary strip of a high numerical gradient of S [13].

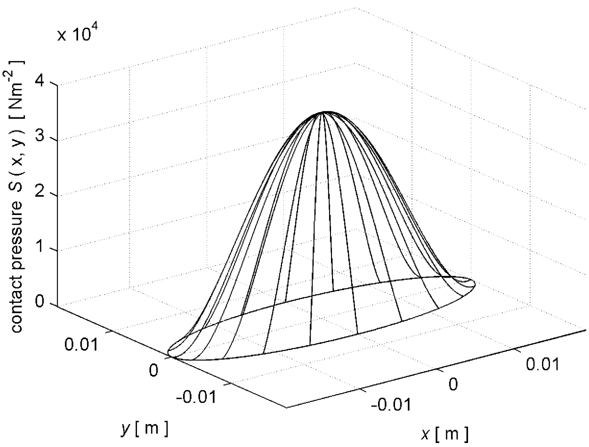

Fig.5: The contact pressure distribution $S(x, y)$ for an ellipsoidal indenter with $\mu = 0.3\,MPa$, $h = 2\,mm$, $R_y = 0.5\,m$, $k = R_y/R_x = 0.1$ and $F = 10\,N$

4. Conclusion

Equations (28) present an analytic approximation of the contact ellipse, penetration depth and contact pressure distribution for an elastic isotropic incompressible coating bonded to a rigid backing and indented by a rigid frictionless indenter in the form of an ellipsoid (or an elliptical paraboloid). In accord with Argatov [5], the form of the contact ellipse is given by the relation $(r_y/r_x)^2 = R_y/R_x$. The contact radius $r_{\rm a}$ and the contact width $r_{\rm p}$ for the spherical and cylindrical indenters, obtained from $(28)_1$ respectively for $k = 1$ and $k \to 0$, are good approximations of the numerical values by McCormick [14], Jaffar [3] and Meijers [2]. The maximum contact pressure from (28) for the ellipsoidal indenter of $k = 1/10$, normalized by that for infinite layer thickness (the Hertz solution), also agrees well to the numerical values by McCormick [14] in this severe check. All this substantiates the choice of equation (26) for the contact ellipse.

It has been shown that the condition of zero change in volume of the material under the indenter (or equivalently, the condition of zero gradient of the contact pressure across the contact edge), as used by Johnson [8], Barber [7], Jaffar [11] and Ateshian et al. [12], does not yield acceptable estimates for the contact surface (see the dash-dot lines in Figs. 2–3 for the cylindrical and spherical indenter). The error increases with the decreasing quotient of the Hertzian value of the contact surface to the layer thickness.

The results obtained can be used as simple approximate solutions to contacts of coated rigid congruent bodies of any form, with measured main curvatures at the contact center. If a more exact solution is looked for, the present solution can serve as a starting estimation for a numerical solution (e.g., using FEM) to this difficult problem with the contact region unknown in advance.

Nomenclature

F	total load
F_{p}	load per unit length of cylinder
h	coating thickness
k	R_y/R_x
$\max S$	maximum contact pressure for rigid ellipsoid
$\max S_{\mathrm{H}}$	$2^2\,[3\,F\,\mu^2/(2\,R_x^2)]^{1/3}/\pi$, maximum Hertzian contact pressure for rigid sphere of radius R_x
p	spherical part of stress tensor
r, ϕ	elliptical coordinates
r_{aH}	$[3\,R_{\mathrm{a}}\,F/(2^4\,\mu)]^{1/3}$, Hertzian contact radius for rigid sphere of radius R_{a}
r_{pH}	$[R_{\mathrm{p}}\,F_{\mathrm{p}}/(\pi\,\mu)]^{1/2}$, Hertzian contact width for rigid cylinder of radius R_{p}
$r_x, r_y, r_{\mathrm{a}}, r_{\mathrm{p}}$	semi-axes, radius and width of contact ellipse, circle and strip, respectively
$R_x, R_y, R_{\mathrm{a}}, R_{\mathrm{p}}$	main curvature radii of rigid ellipsoidal, spherical and cylindrical indenters
S	$-\underline{\sigma}_{zz}(r,\phi)$, contact pressure for rigid ellipsoidal indenter
x, y, z	Cartesian coordinates
$\alpha_x, \alpha_y, \beta_x, \beta_y, \gamma$	functions of x, y defined in (1)
δ	r_y/r_x
$\varepsilon_{ij}\ (i,j = x,y,z)$	strain tensor
$\sigma_{ij}\ (i,j = x,y,z)$	stress tensor
η_y	$(2/3)^{1/2} r_y/h$; similarly for η_x, η_{a}, η_{p} using r_x, r_{a}, r_{p}, respectively
μ	shear modulus
ϱ	penetration depth divided by h
$\underline{f}$	average of f through layer thickness h
$\overline{\overline{f}}$	value of f at contact edge
f'	dimensionless variable of f

subscripts:

a	value for axial symmetry (for spherical indenter)
H	Hertzian value (for layer of infinite thickness)
p	value for plane strain (for cylindrical indenter)

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