

## H-INFINITY CONTROLLER DESIGN FOR A DC MOTOR MODEL WITH UNCERTAIN PARAMETERS

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*The proposed article deals with uncertain description of permanent-magnet DC motor Maxon RE 35 via parametric uncertainty and H-infinity controller design. There was analyzed influence of uncertainties of the particular motor parameters on the model behavior. Consequently there was designed an H-infinity controller via Matlab functions. The behavior of the obtained controller was analyzed on the step responses and course tracking of the closed loop with the nominal system and the system with perturbed parameters.*

Keywords: DC motor, parametric uncertainty, H-infinity

### 1. Introduction

Models describing dynamics of systems typically contain some inaccuracies when compared with the real device. This is mostly caused by simplifications of the model, neglecting of some factors influencing the dynamics or general modeling inaccuracy. However this might be a problem when designing a control of the system – the precise model is needed for the proper design of a controller.

The approach dealing with this problem is based on modeling of the real system as a set of linear time-invariant models built around a nominal one, i.e. the model is built as uncertain within known boundaries. The benefit of such a representation of the model is the possibility of designing a robust controller stabilizing a closed loop system even with uncertainties. The ideal goal is to design a controller capable of stabilizing even the ‘the worst case scenario’ representing the most degraded model. Such a controller is then able to stabilize also the real system.

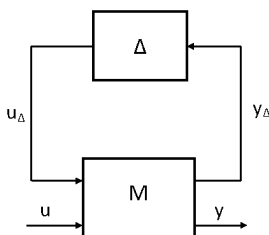


Fig.1: M– $\Delta$  configuration of a model with uncertainty

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The typical representation of the model with uncertainty is called  $M-\Delta$ , Fig. 1. The  $M$  matrix represents the augmented system obtained from the nominal system and  $\Delta$  matrix represents variations of the system parameters. The  $\Delta$  matrix is a diagonal matrix  $\Delta = \text{diag}\{\delta_1, \dots, \delta_m\}$  in case of the parametric uncertainty. The augmented model  $M$  is typically obtained by upper linear transformation [1].

The approach of the uncertainty modeling is widely used in many technical areas (e.g. [2], [3]) where a robust control is needed or a model precise description is impossible. The proposed article presents its application on specific Maxon RE 35 PMDC.

The proposed controller for the single PMDC motor is based on a simple state-space model transformed to the uncertain form and on the H-infinity controller design techniques in Matlab.

## 2. Standard representation of the unloaded PMDC motor

The unloaded PMDC motor model is based on well known description

$$\begin{aligned} \frac{di}{dt} &= -\frac{R}{L}i - \frac{K_b}{L}\omega + \frac{1}{L}u, \\ \frac{d\omega}{dt} &= -\frac{1}{J}K_f\omega + \frac{1}{J}K_m i, \end{aligned} \quad (1)$$

which may be in a state-space form presented as

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} i \\ \omega \end{bmatrix} &= \begin{bmatrix} -\frac{R}{L} & -\frac{K_b}{L} \\ \frac{K_m}{J} & -\frac{K_f}{J} \end{bmatrix} \begin{bmatrix} i \\ \omega \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t), \\ y(t) &= [0 \quad 1] \begin{bmatrix} i \\ \omega \end{bmatrix} + [0] u(t). \end{aligned} \quad (2)$$

The meaning of particular terms is following:  $K_m$  is the torque constant,  $J$  is the rotor inertia,  $K_f$  is coefficient of viscous friction,  $i$  is the instantaneous value of the electrical current,  $\omega$  is the instantaneous angular velocity of the shaft,  $K_b$  is the voltage constant (inverse speed constant),  $R$  is the terminal resistance,  $L$  is the terminal inductance and finally  $u$  is the instantaneous value of a supply voltage.

The values of the terms are for the Maxon RE 35 (catalogue number 273754) following:  $J = 7.2 \times 10^{-6} \text{ kg m}^2$ ,  $R = 2.07 \text{ } \Omega$ ,  $L = 0.00062 \text{ H}$ ,  $K_m = 0.052 \text{ N m A}^{-1}$ ,  $K_b = 0.052 \text{ V s rad}^{-1}$  and  $K_f = 0.000048 \text{ N m s rad}^{-1}$ .

## 3. Model of the motor Maxon RE 35 with uncertain parameters

The goal of the uncertain model is to cover possible difference between the model and the reality by defining of the uncertainty for selected parameters. The following PMDC motor model with uncertain parameters is based on description (1) and principles of uncertain modeling [4].

For  $x_1 = i$ ,  $x_2 = \omega$  and by introducing the parametric uncertainty, the equations (1) may be transformed into a form

$$\begin{aligned} \dot{x}_1 &= \frac{1}{\bar{L} + \delta_L} [- (\bar{R} + \delta_R) x_1 - (\bar{K}_b + \delta_{Kb}) x_2 + u] , \\ \dot{x}_2 &= \frac{1}{\bar{J} + \delta_J} [(\bar{K}_m + \delta_{Km}) x_1 - (\bar{K}_f + \delta_{Kf}) x_2] \end{aligned} \tag{3}$$

where  $\bar{L}$ ,  $\bar{R}$ ,  $\bar{K}_b$ ,  $\bar{J}$ ,  $\bar{K}_m$ ,  $\bar{K}_f$  are nominal parameters and  $\delta_L$ ,  $\delta_R$ ,  $\delta_{Kb}$ ,  $\delta_J$ ,  $\delta_{Km}$ ,  $\delta_{Kf}$  are uncertainties of the nominal parameters. The model with uncertainties is then described by the following scheme, Fig. 2. Let's note that the general nominal state-space model may be described by matrices  $\bar{\mathbf{A}}$ ,  $\bar{\mathbf{B}}$ ,  $\bar{\mathbf{C}}$  and  $\bar{\mathbf{D}}$ .

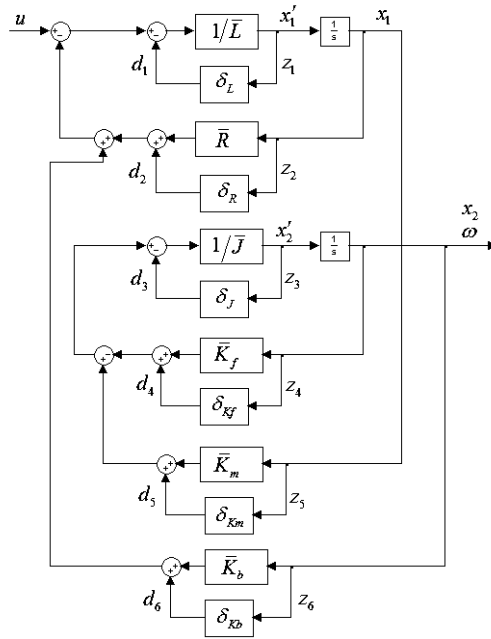


Fig.2: Scheme of the PMDC motor with uncertain parameters

The uncertain model in a matrix form is then obtained according to the scheme as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{\bar{R}}{\bar{L}} & -\frac{\bar{K}_b}{\bar{L}} \\ \frac{\bar{K}_m}{\bar{J}} & -\frac{\bar{K}_f}{\bar{J}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{\bar{L}} & -\frac{1}{\bar{L}} & 0 & 0 & 0 & -\frac{1}{\bar{L}} \\ 0 & 0 & -\frac{1}{\bar{J}} & -\frac{1}{\bar{J}} & \frac{1}{\bar{J}} & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} + \begin{bmatrix} \frac{1}{\bar{L}} \\ 0 \end{bmatrix} [u] , \\ \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} &= \begin{bmatrix} -\frac{\bar{R}}{\bar{L}} & -\frac{\bar{K}_b}{\bar{L}} \\ \frac{1}{\bar{L}} & 0 \\ \frac{\bar{K}_m}{\bar{J}_0} & -\frac{\bar{K}_f}{\bar{J}} \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{\bar{L}} & -\frac{1}{\bar{L}} & 0 & 0 & 0 & -\frac{1}{\bar{L}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\bar{J}} & -\frac{1}{\bar{J}} & \frac{1}{\bar{J}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} + \begin{bmatrix} \frac{1}{\bar{L}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} [u] , \\ \begin{bmatrix} i \\ \omega \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} . \end{aligned} \tag{4}$$

The compact form corresponding with the M–Δ configuration is then for matrix **M**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ \omega \end{bmatrix} = \begin{bmatrix} -\frac{\bar{R}}{L} & -\frac{\bar{K}_b}{L} & -\frac{1}{L} & -\frac{1}{L} & 0 & 0 & 0 & -\frac{1}{L} & \frac{1}{L} \\ \frac{\bar{K}_m}{J} & -\frac{\bar{K}_f}{J} & 0 & 0 & -\frac{1}{J} & -\frac{1}{J} & \frac{1}{J} & 0 & 0 \\ -\frac{\bar{R}}{L} & -\frac{\bar{K}_b}{L} & -\frac{1}{L} & -\frac{1}{L} & 0 & 0 & 0 & -\frac{1}{L} & \frac{1}{L} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\bar{K}_m}{J_0} & -\frac{\bar{K}_f}{J} & 0 & 0 & -\frac{1}{J} & -\frac{1}{J} & \frac{1}{J} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ u \end{bmatrix} \tag{5}$$

and **Δ** matrix as

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} = \begin{bmatrix} \delta_L & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_R & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_J & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_{Kf} & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_{Km} & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_{Kb} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \tag{6}$$

The matrix (5) may be transformed into a general form

$$\mathbf{M} = \left[ \begin{array}{c|cc} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \hline \mathbf{M}_{21} & \mathbf{M}_{22} \end{array} \right] = \left[ \begin{array}{c|cc} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{D}_{22} \end{array} \right] \tag{7}$$

Figure 3 shows the Simulink scheme of the obtained uncertain model. The matrix **M** may be presented in Matlab as  $\mathbf{M} = \text{ss}(\mathbf{A}, [\mathbf{B}_1 \ \mathbf{B}_2], [\mathbf{C}_1; \mathbf{C}_2], [\mathbf{D}_{11} \ \mathbf{D}_{12}; \mathbf{D}_{21} \ \mathbf{D}_{22}])$ .

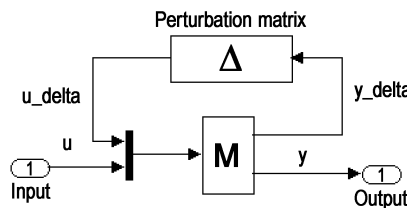


Fig.3: Simulink scheme of the uncertain model of the Maxon RE 35 PMDC motor

#### 4. Frequency characteristics of the uncertain model

The following figures (Fig. 4–9) present the impact of the uncertainty of the particular parameters on the behavior of the model. The uncertainty was randomly changed between –100% and 100% of the nominal value for the selected parameter. The model behavior is presented on Bode diagrams for twenty random samples from the uncertainty range.

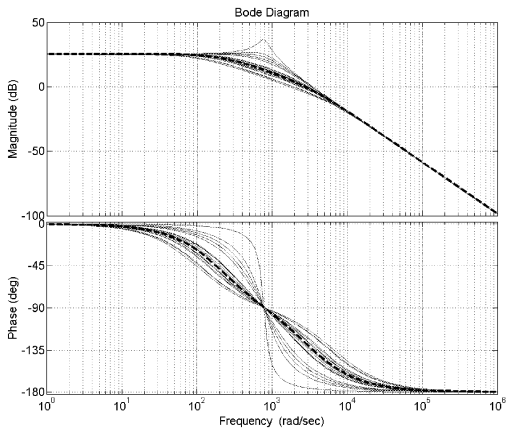


Fig.4: Uncertainty in parameter  $R$

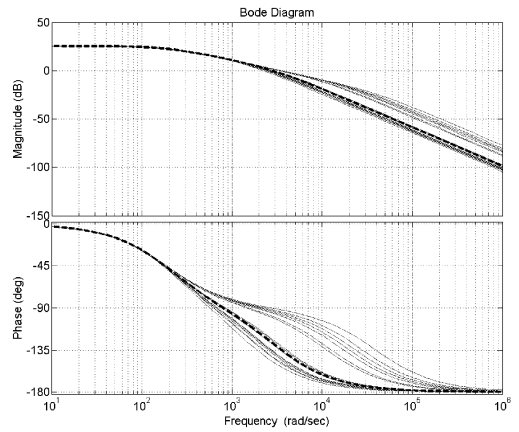


Fig.4: Uncertainty in parameter  $L$

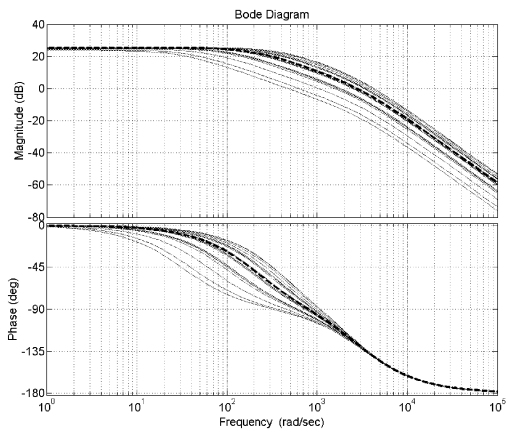


Fig.6: Uncertainty in parameter  $K_m$

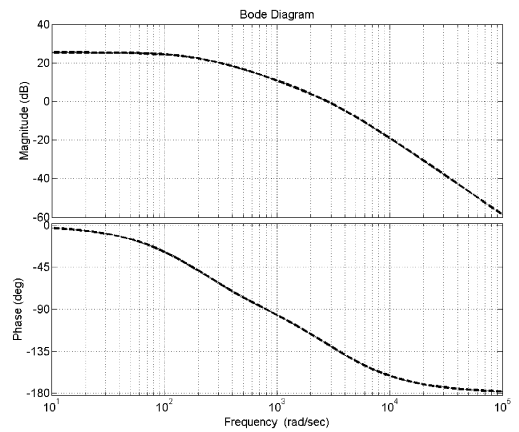


Fig.7: Uncertainty in parameter  $K_f$

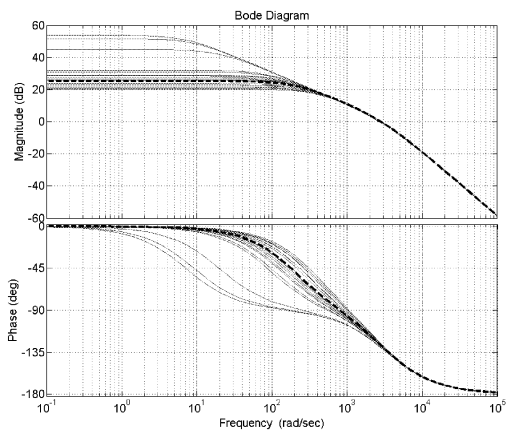


Fig.8: Uncertainty in parameter  $K_b$

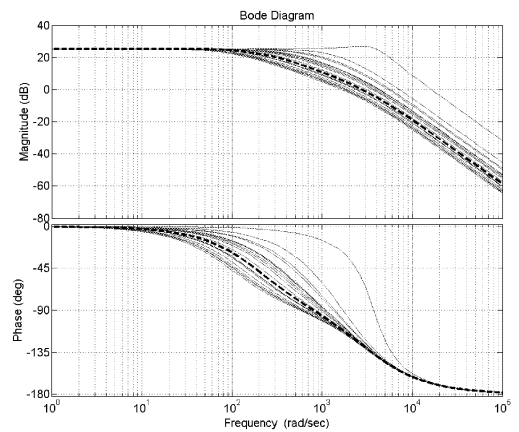


Fig.9: Uncertainty in parameter  $J$

It is obvious that the least impact on the model behavior has the uncertainty of the parameter  $K_f$  (Fig. 7). On the other hand uncertainty of the parameter  $K_b$  (Fig. 8) is changing the model behavior dramatically even at very low frequencies. The uncertainties in the rest of parameters brings higher changes to the model behavior of the model from the frequencies about 20 rad/s, except the uncertainty of parameter  $L$  (Fig. 5) which starts to influence the behavior approximately at 300 rad/s.

## 5. H-infinity controller design

The H-infinity controller design is in general based on minimization of H-infinity norm [5] of the selected closed loop system described as

$$\|F_1(\mathbf{M}, \mathbf{K})_\infty = \sup_{\omega} \bar{\sigma}(F_1(\mathbf{M}, \mathbf{K})(j\omega))$$

where  $\bar{\sigma}$  is the singular value of the function  $F_1(\mathbf{M}, \mathbf{K})(j\omega)$  and  $\mathbf{K}$  is the controller. The expression  $F_1(\mathbf{M}, \mathbf{K})$  is called lower fractional transformation [1]. It is defined as  $F_1(\mathbf{M}, \mathbf{K})(j\omega) = M_{11} + M_{12} \mathbf{K}(I - M_{22} \mathbf{K})^{-1} M_{21}$ . The obtaining of the controller is then based on solution of Riccati equations [6], [7]. This is quite complicated thus some of available robust control design tools for the numerical solution is often used, e.g. Matlab 'hinfsv' function.

It is suitable to transform the obtained uncertain model (5), (6) according to [8], [4] for the controller design purposes as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{z}_A \\ \mathbf{z}_B \\ \mathbf{z}_C \\ \mathbf{z}_D \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}} & \mathbf{A}_\Delta & \mathbf{B}_\Delta & \mathbf{0} & \mathbf{0} & \bar{\mathbf{B}} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \bar{\mathbf{C}} & \mathbf{0} & \mathbf{0} & \mathbf{C}_\Delta & \mathbf{D}_\Delta & \bar{\mathbf{D}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{d}_A \\ \mathbf{d}_B \\ \mathbf{d}_C \\ \mathbf{d}_D \\ \mathbf{u} \end{bmatrix}, \quad (8)$$

$$\begin{bmatrix} \mathbf{d}_A \\ \mathbf{d}_B \\ \mathbf{d}_C \\ \mathbf{d}_D \end{bmatrix} = \begin{bmatrix} \Delta_{\Delta A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Delta_{\Delta B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Delta_{\Delta C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \Delta_{\Delta D} \end{bmatrix} \begin{bmatrix} \mathbf{z}_A \\ \mathbf{z}_B \\ \mathbf{z}_C \\ \mathbf{z}_D \end{bmatrix}$$

where  $\mathbf{x} = [x_1 \ x_2]^T$ ,  $\dot{\mathbf{x}} = [\dot{x}_1 \ \dot{x}_2]^T$ ,  $\mathbf{z}_A = \mathbf{z}_C = \mathbf{x}$ ,  $\mathbf{z}_B = \mathbf{z}_D = \mathbf{u}$ ,  $\mathbf{y} = \omega$ ,  $\mathbf{u} = u$ ,  $\mathbf{d}_A = \Delta_{\Delta A} \mathbf{z}_A$ ,  $\mathbf{d}_B = \Delta_{\Delta B} \mathbf{z}_B$ ,  $\mathbf{d}_C = \Delta_{\Delta C} \mathbf{z}_C$ ,  $\mathbf{d}_D = \Delta_{\Delta D} \mathbf{z}_D$  and finally  $-\mathbf{I} \leq \Delta_{\Delta A, B, C, D} \leq \mathbf{I}$ .

$\mathbf{A}_\Delta = \mathbf{A}_{\max} - \bar{\mathbf{A}}$  where  $\mathbf{A}_{\max}$  is the state-space matrix of the model with maximal uncertainty in its parameters.  $\mathbf{B}_\Delta$ ,  $\mathbf{C}_\Delta$  and  $\mathbf{D}_\Delta$  are obtained similarly.

The transformed matrix  $\mathbf{M}$  then contains information about the uncertainty. It was contained only in the  $\Delta$  matrix before the transformation. The transformed matrix  $\mathbf{M}$  is the input parameter for the Matlab functions (e.g. hinfsv) solving the Riccati equations in order to design a robust controller  $\mathbf{K}$ .

### 6. Examples of the obtained robust controller behavior

The example shows the behavior of the theoretically designed controller  $\mathbf{K}$  based on the uncertain model with following experimental values of parameters  $\delta_L = 0.02$ ,  $\delta_{Kb} = 0.3$ ,  $\delta_{Kf} = 1$ . Let's note that the controller is of the same order as the controlled system. The control scheme is standard feedback control scheme.

The first example (Fig. 10) shows the step response of the closed loop with the nominal system, i.e. without any uncertainty in parameters. The action time of the controller is in this case approximately 0.042s. Almost the same action time is then for the step response of the uncertain system with uncertainty  $\delta_L = 0.02$ , Fig. 11. It is then approximately 0.045s for the uncertainty  $\delta_{Kf} = 1$ ; 0.062s for the uncertainty in  $\delta_{Kb} = 0.3$  and 0.064s for the combination of mentioned uncertainties, Fig. 11. Thus the action time of the controller gets longer with the increasing amount of the uncertainty in the controlled system. The system response is without any overshoot for all of these examples.

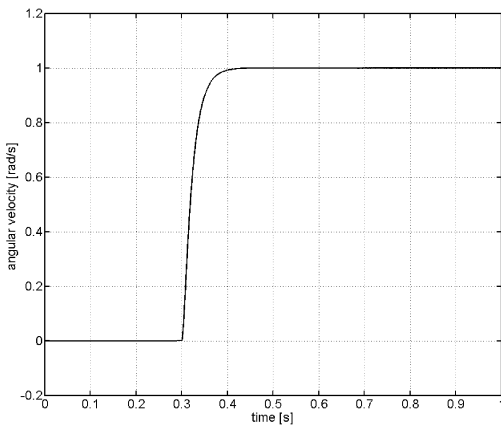


Fig.10: Step response of the nominal model

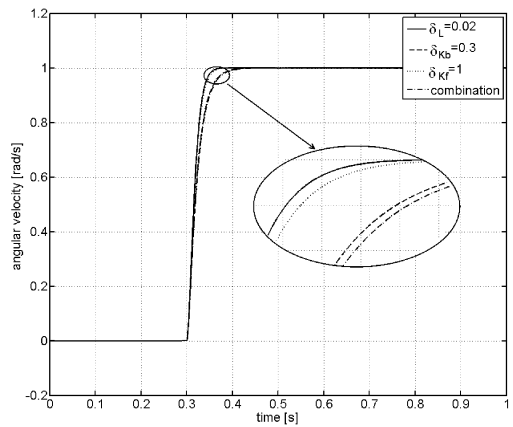


Fig.11: Step responses of perturbed models

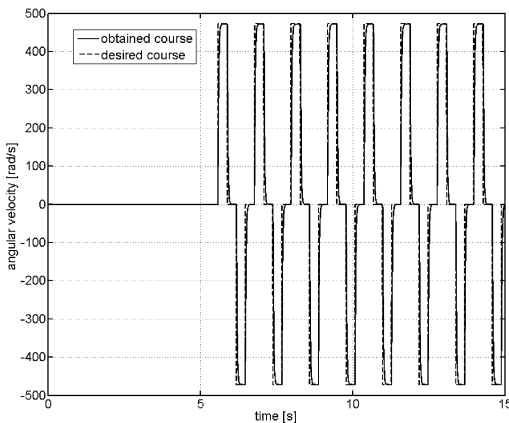


Fig.12: Comparison between desired and obtained angular velocity course

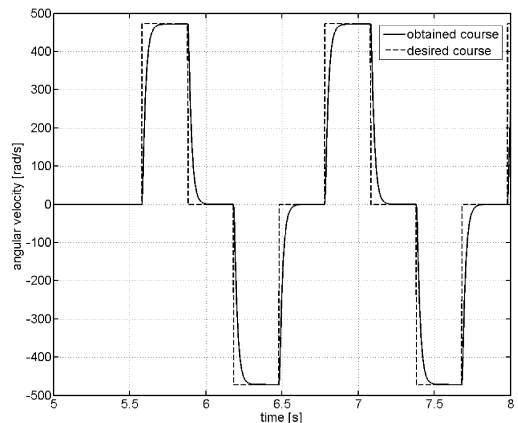


Fig.13: Comparison between desired and obtained angular velocity course – detail

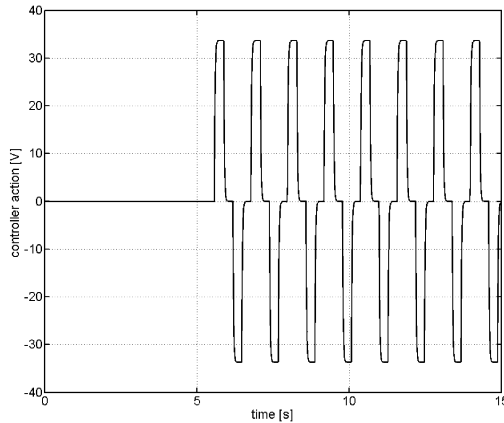


Fig.14: The controller action

The following example (Fig.12–13) simulates sharp increasing and decreasing of the angular velocity in approximate range of  $\pm 470$  rad/s (maximum for the given motor is ca. 780 rad/s) with observed energy requirements (controller action), Fig.14. The example is performed for the uncertain model with combination of all of mentioned uncertainties.

The controller action moves between  $\pm 33$  V which still satisfies the maximal motor supply voltage 42 V. The tracking ability is given by the mentioned action time of the controller.

## 7. Conclusions

The article presents quite simple approach for the robust controller design for the PMDC motor Maxon RE 35. The approach is based on uncertain model of the motor. Consequently the H-infinity controller was designed via Matlab functions. The controller is of the second order, thus quite simple. It was tested in a simulation and it was proved that the controller is able to stabilize even the most degraded model within the given uncertainty range. Such a controller should be theoretically able to stabilize also the real system with behavior covered by the uncertainty.

The approach is quite simple and possibly applicable to other systems where it is impossible to create precise model for the control design.

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