THE LIMITS OF THE BEAM SAG UNDER INFLUENCE OF STATIC MAGNETIC AND ELECTRIC FORCE

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Utilization of a magnetic force can be found in many mechatronic applications, where e.g. a slender beam or plate is subjected to static magnetic force generated by an electromagnetic actuator consisting of a solenoid wound on a ferromagnetic core and a ferromagnetic armature, fixed to the beam. The static magnetic force, acting perpendicularly onto the beam, causes sag (downwards bending) of the beam. If the magnitude of the magnetic force surpasses some threshold value the armature and hence the beam is completely attracted to the core of the solenoid. For small deflections the mathematical expression of the magnetic force can be linearised and approximated by a polynomial dependence on the distance to the electromagnet. In practical applications, it is important to analyse the nature of the sag and to determine the limits of the linear approximation, as well as the limits leading to the full attraction to the electromagnet. The mathematical generalisation of the sag is valid for electrostatic force between planar electrodes, too.

Keywords: *clamped beam sag, electromagnetic actuator, sag approximation, threshold current, sag limit, electrostatic case*

1. Introduction

Let us analyse a mechatronic system, which consists of a slender beam or plate of length L, subjected to static magnetic force F_{M}, generated by an electromagnetic actuator. The actuator consists of a solenoid wound on a pot-form ferromagnetic core and an armature (of length $L_{\mathrm{m}} \ll L$), fixed to the beam at its midpoint (Fig. 1). The magnetic force F_{M} is acting in the middle of the beam at distance $L/2$ from rigid fixtures on both ends and induces a sag (downwards deflection) $z_{\max}$. If the intensity of the magnetic force FM exceeds certain threshold, the beam is permanently attracted to the end-stops [1].

2. Mathematical model of the equilibrium state

The deflection, $z_{\max}$, at the beam midpoint (i.e. approximately at the distance $L/2$ from the rigid fixture), due to a general force, P, localized at midpoint of the clamped-clamped slender beam of length L acting in the perpendicular direction to the beam longest axis is [1], [2], [4] :

$$z_{\max}(P) = \frac{P}{192}\,\frac{L^3}{E_{\mathrm{b}}\,I_{\mathrm{b}}}\ , \tag{1}$$

where E_{b} is the modulus of elasticity (Young's module) of the beam material, I_{b} is the

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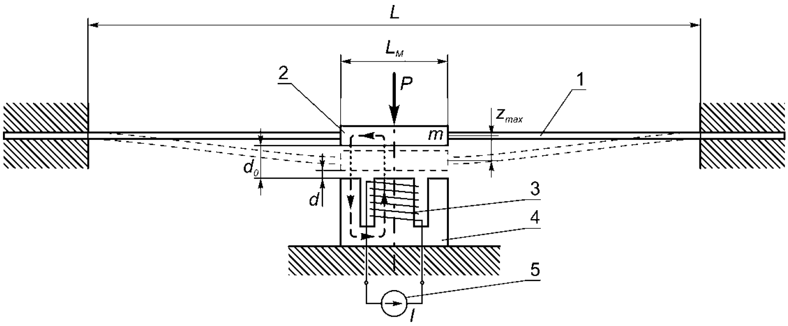

Fig.1: Schematics of the clamped-clamped beam with electromagnet : 1 the beam, 2 ferromagnetic armature, 3 electromagnet coil, 4 electromagnet core, 5 current source; a thick dashed line denotes the middle flux line

second moment of inertia of the beam cross-section. The displacement $z_{\max}$ is collinear with the acting force P and has the same direction. For a beam with a rectangular cross-section, the second moment of inertia is given by the formula $I_{\mathrm{b}} = 1/12\, b\, h^3$, where b is the width of the beam and h is its height. Formula (1) can be re-formulated as $P = z_{\max}\, k$, where the stiffness of the clamped-clamped slender beam loaded in the midpoint is $k = 192\,(E_{\mathrm{b}}\, I_{\mathrm{b}})/L^3$ [1], [2], [4].

Energizing the electromagnet with a coil of N turns, wound on a ferromagnetic core of cross-section S with a steady state (DC) current I a magnetic force F_{M} is generated in the air gap. The magnitude of magnetic force F_{M} is described by the equation [1], [2], [3] :

$$F_{\mathrm{M}}(d, I) = \frac{\mu_0\, S\, N^2\, I^2}{\left(2\, d + \dfrac{l_\Phi}{\mu_{\mathrm{r}}}\right)^2} \ . \tag{2}$$

The magnetic flux lines are crossing twice the air gap of width d, as shown in Fig. 1; μ_0 is the permeability of air and l_Φ is the middle flux line length in the ferromagnetic material of relative permeability μ_{r}. A more thorough magnetic field analysis by FEM approach would be beyond the scope of this contribution. The flux line length l_Φ can be transformed into an equivalent half flux line length in d_{Fe}, assuming *linear properties* of the core magnetic material : $d_{\mathrm{Fe}} = 1/2\, l_\Phi/\mu_{\mathrm{r}}$. This is also a simplifying assumption, since for common magnetic materials B-H relation is non-linear [2], [3]. However, up to the saturation point, the concept of linear permeability can be used.

From the geometry (Fig. 1) follows that $z_{\max} = d_0 - d$, where d_0 is the initial distance between electromagnet and the beam in de-energised state. Then the static equilibrium of the magnetic force $F_{\mathrm{M}}(d, I)$ and the elastic force due to the beam deflection $P = z_{\max}\, k$ is described by :

$$P = (d_0 - d)\, k = F_{\mathrm{M}}(d, I) \ . \tag{3}$$

Let us introduce a dimensionless variable α :

$$\alpha = \frac{z_{\max}}{d_0} = \frac{d_0 - d}{d_0} \ . \tag{4}$$

The quantity α is non-negative and cannot be larger than unity. It can be looked-upon as a non-dimensional relative distance. If $\alpha = 1$ the armature and core would adhere one to another and, subject of ideal smoothness of the adhering surfaces, no air gap would exist.

Introducing α into (3) and using (2), the equilibrium equation is:

$$\alpha\,k = \frac{F_\mathrm{M}}{d_0} = \frac{\mu_0\,S\,N^2}{4\,d_0}\,\frac{I^2}{(d_0-\alpha\,d_0+d_\mathrm{Fe})^2} = \frac{\mu_0\,S\,N^2}{4\,d_0}\,\frac{I^2}{d_0^2\,[(1+\delta_\mathrm{M})-\alpha]^2}\ . \tag{5}$$

A relative measure $\delta_\mathrm{M} = d_\mathrm{Fe}/d_0$ can be introduced, while $\delta_\mathrm{M} < 1$, because $\mu_\mathrm{r} > 1$. In some cases, when $d_\mathrm{Fe} \ll d_0$ (i.e. the reluctance of the air gap is dominant), $\delta_\mathrm{M} \ll 1$ and so δ_M can be neglected [2], [3].

Formula (5) can be, after some algebraic manipulation, re-written:

$$\frac{\alpha(I)}{1+\delta_\mathrm{M}}\,k = \frac{\mu_0\,S\,N^2}{4\,d_0^3}\,\frac{I^2}{(1+\delta_\mathrm{M})^3\left[1-\dfrac{\alpha(I)}{1+\delta_\mathrm{M}}\right]^2}\ , \tag{6a}$$

which calls for introduction of a normalised parameter β: $\beta = \alpha/(1+\delta_\mathrm{M})$. Parameter β relates the *air gap width change* (d_0-d) to the properties of the magnetic circuit δ_M, which are constant for the initial distance d_0. Obviously, $\beta < 1$. The physically feasible limit is $\beta \le 1/(1+\delta_\mathrm{M})$. Then (6a) is modified to:

$$\beta(I)\,k = \frac{\mu_0\,S\,N^2}{4\,(d_0+d_\mathrm{Fe})^3}\,\frac{I^2}{[1-\beta(I)]^2} = K_\mathrm{M}\,\frac{I^2}{[1-\beta(I)]^2}\ . \tag{6b}$$

3. Solution of the equilibrium equation

The (6b) can be solved for variable $\beta(I)$ by an approximate approach or in the exact way, applying analytical or numerical tools:

i. The denominator of the right hand side of (6b) can be approximated by a McLaurin's series:

$$\beta\,k = K_\mathrm{M}\,I^2\,\{1+2\,\beta+3\,\beta^2+\dots\}\ . \tag{7}$$

Just the first two terms of the expansion are considered, i.e. the linear approximation is used. After some algebra the formula for approximate calculation of β emerges. The approximate value of β will be in further denoted as β':

$$\beta' = \frac{K_\mathrm{M}\,I^2}{k-2\,K_\mathrm{M}\,I^2}\ . \tag{8}$$

ii. The exact solution stems from the cubic equation obtained by rewriting (6b):

$$\beta\,[1-\beta]^2 = \frac{K_\mathrm{M}}{k}\,I^2\ , \tag{9a}$$

i.e.:

$$\beta^3-2\,\beta^2+\beta-\frac{K_\mathrm{M}}{k}\,I^2 = 0\ . \tag{9b}$$

The solution of (9b) calls for the use of Cardano's formulas for evaluation of cubic equations or rely on numerical solvers of algebraic equations, embedded in simulation programming environment, e.g. MATLAB®. The solution leads to three different complex roots [5], [6]. In analogy to the quadratic equation there is a cubic discriminator D_3, furnishing for $D_3 > 0$ three real roots. This is the case here. By further analysis, two pairs of special real solutions of this cubic equation were found:

– a pair for $\beta = 0$ and $\beta = 1$, which is a result for $I = 0$;

– a pair for $\beta = 1/3$ and $\beta = 4/3$, which results if I attains a specific threshold value I_{crit}:

$$I_{\mathrm{crit}}^2 = \frac{4}{27}\,\frac{k}{K_{\mathrm{M}}}\ . \tag{10}$$

The threshold current I_{crit} is determined by the beam stiffness k and the magnetic circuit properties K_{M}. The value $\beta = 4/3$ corresponds to the *triple real root* at $D_3 = 0$. For $I > I_{\mathrm{crit}}$ (when $D_3 < 0$) there is only a single real root and two complex conjugate roots.

Let us introduce a generalized variable q_{N}, which is physically the current I normalized by the value of threshold current, $q_{\mathrm{N}} = I/I_{\mathrm{crit}} \le 1$. Then (8) and (9b) can be re-formulated and simplified:

$$\beta' = \frac{1}{\dfrac{27}{4}\,\dfrac{1}{q_{\mathrm{N}}^2} - 2}\ , \tag{11a}$$

$$\beta^3 - 2\,\beta^2 + \beta - \frac{4}{27}\,q_{\mathrm{N}}^2 = 0\ . \tag{11b}$$

For calculation of the exact solutions of β the MATLAB® function ‘roots’ was used, returning a complex three element vector for each q_{N} value. Then the roots were ordered in ascending order and plotted in the form of line graphs (Fig. 2). Note, that this is not a plot of a function, because for any positive value of $q_{\mathrm{N}} < 1$ three different values are possible. The course of the approximate solution β', expression (11a), is plotted as a thin line.

Physically feasible values of numerical solution of the cubic equation (11b) are bound to the interval $[0, \beta \le 1/(1 + \delta_{\mathrm{M}})]$ (white area in Fig. 2); hence the solution in the grey area has no physical meaning (the beam would have to move within the electromagnetic core!). The dashed course is not physically realistic either, because this would assume that the elastic beam was buckled prior to energising the field. The physically plausible course is the lowest curve, starting at zero and reaching for $q_{\mathrm{N}} = 1$ the value of $\beta = 1/3$. For the value $q_{\mathrm{N}} = 1$ two different solutions do exist: $\beta = 1/3$ and $\beta = 4/3$. This can by interpreted as the limit of stability: *at the threshold current I_{crit} the beam buckles* from the value of $\beta = 1/3$ to $\beta = 1/(1 + \delta_{\mathrm{M}})$, as denoted by the thick vertical line and the armature is fully attracted to the core. For $I > I_{\mathrm{crit}}$ the beam would be permanently attracted to the core. If the current reverts from a value of $I > I_{\mathrm{crit}}$ the beam would follow the same trajectory, i.e. as soon as the value of q_{N} drops below unity the beam would attain (after extinction of a transient phenomenon) a position corresponding to the $\beta = 1/3$. The trajectory is once more highlighted in Fig. 3. From the approximate solution no limiting current and no full attraction can be implied! In practice the remanent magnetism may play certain role, changing the beam transient behaviour in the vicinity of q_{N}.

Note, that when $q_{\mathrm{N}} < 0.80$ there is no marked difference between the exact solution and the approximate solution (Fig. 2). For a specific case of initial air-gap width $d_0 = 1.0$ mm and magnetic circuit properties corresponding to $\delta_{\mathrm{M}} = 0.15$ for $q_{\mathrm{N}} = 0.80$ the difference between the exact solution $\beta = 0.123$ and the approximate solution $\beta' = 0.117$ is $-5\,\%$, i.e. still technically acceptable.

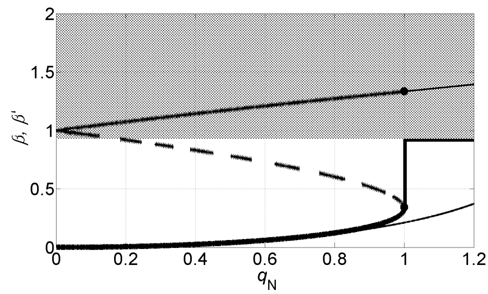

Fig.2: The solution of the cubic equation (11b) in the generalised coordinates and of the approximate solution (11a) (thin); the non-realistic solutions are in the grey area; note the accentuated values for $\beta = 1/3$ and $\beta = 4/3$

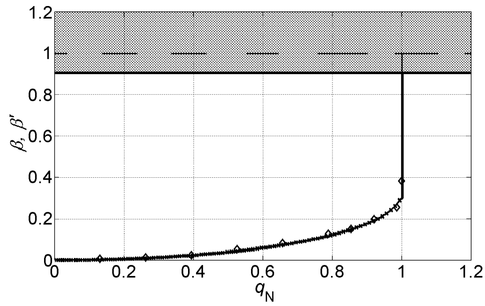

Fig.3: The beam sag trajectory for the electromagnetic case (solid) and the electrostatic case (dashed); diamonds indicates the experimental results

The value of q_N is crucial for discrimination between bending and attraction of the elastic beam to the electromagnet core. Moreover, the maximal displacement due to bending prior to transition to the buckled state is in normalised coordinates $\beta = 1/3$, i.e.

$$d_{\mathrm{lim}} = \frac{d_0}{3} \,(2 - \delta_{\mathrm{M}}) \ . \tag{12}$$

Some selected experimental results are presented in Fig. 3, too [7]. The static deflection of clamped-clamped aluminium beam of known stiffness, with an electromagnet located in the middle was measured for various initial distances d_0 and current values I. From Fig. 3 it can be seen that numerical results agree well with the experimental data. This shows that within the range of analysed distances d_0 the model approximations (constant beam cross-section, linear properties of the core and armature magnetic material, etc.) are acceptable.

4. The electrostatic case

In analogy to Fig. 1 an electrostatic case can be designed: namely a clamped-clamped beam with plane parallel electrodes of surface area S in the beam centre, between which a potential difference (DC voltage U) exist. While neglecting the fringing effects on the electrodes circumference, the electrostatic attraction force is given as [1], [2] :

$$F_{\mathrm{E}}(d,U) = \frac{1}{2}\,\varepsilon_0\,\varepsilon_{\mathrm{r}}\,\frac{S_{\mathrm{E}}\,U^2}{d^2} = C(d)\,\frac{U^2}{2\,d} \ , \tag{13}$$

where ε_0 is the permittivity of vacuum and ε_r is the relative permittivity of the gaseous medium between the electrodes, which can be assumed to be approximately unity. The electrodes form a plate capacitor of capacitance $C(d)$.

If there is no potential difference U between the electrodes a static equilibrium is maintained at a distance d_0 between the electrodes. The capacity of the so formed capacitor C_0 can be calculated, or rather measured. When there is a potential difference the electrodes are attracted and the beam is bent due to the influence of the electrostatic force (corresponding to the force P in (1)) leading to a new equilibrium position, described by a similar force balance equation to (3) [1], [2]. However, no equivalent of the magnetic circuit does exist and there is just a single air gap of width d. After introducing the variable α and some algebraic manipulations the equilibrium equation can be re-formulated as follows:

$$\alpha\,k = \frac{F_{\mathrm{E}}(d,U)}{d_0} = \frac{\varepsilon_0\,S_{\mathrm{E}}}{2\,d_0}\,\frac{U^2}{d_0^2\,[1-\alpha]^2} = \frac{C_0}{2}\,\frac{U^2}{d_0^2\,[1-\alpha]^2}\ . \tag{14}$$

Expression (14) is dual to (5) if the variables for the magnetostatic case are substituted by the relevant variables for the electrostatic case, noting that $\delta_{\mathrm{M}} = 0$. Hence, in analogy with the above case a normalised variable $q_{\mathrm{N}} = U/U_{\mathrm{crit}}$ can be introduced, while for this case $\beta \equiv \alpha < 1$. Then (6b) is to be re-formulated for the electrostatic case:

$$\beta\,k = \frac{C_0}{2\,d_0^2}\,\frac{U^2}{(1-\beta)^2} = K_{\mathrm{E}}\,\frac{U^2}{(1-\beta)^2}\ . \tag{15}$$

So, for the electrostatic case the course of Fig. 3 holds, too; however, the limit is at $\beta \equiv \alpha = 1$, denoted by the dashed horizontal line. Expression (12) is then modified to become $d_{\mathrm{crit}} = 2/3\,d_0$, i.e. in other words the beam with electrodes and imposed DC voltage $U < U_{\mathrm{crit}}$ can be bent utmost by 1/3 of the original, de-energised inter-electrodes distance d_0. The critical voltage U_{crit} is:

$$U_{\mathrm{crit}}^2 = \frac{8}{27}\,\frac{d_0^2}{C_0}\,k\ . \tag{16}$$

If the limiting voltage U_{crit} is exceeded the beam is buckled and the electrodes short-circuited, which may cause harm to associated circuit. This deliberation would be relevant, e.g. to MEMS sensors and to powered capacitive sensors.

5. Conclusions

The threshold value of the current I, I_{crit} is crucial for discrimination between bending and complete attraction to the electromagnet poles of the elastic slender beam fixed on both ends. If a current larger than I_{crit} energises the solenoid the beam is permanently attracted to the electromagnet. This finding is supported by experimental results, presented in [7]. From the experimental results, the agreement with the model and thus with all simplifications used, is obvious. The extent of beam bending is given by formula (12).

The linearised form of the magnetic force in the force equilibrium expression (8) or (11a) can be used up to 80 % of the threshold current with an error not exceeding −5 %. Analogous considerations are relevant for the electrostatic case, occurring, i.e. in MEMS sensors. Here the critical voltage U_{crit} is crucial. The beam can be bent utmost by 1/3 of the original de-energised distance between the electrodes.

Acknowledgement

This contribution is a result of the project No. 2/0075/10 of the Slovak VEGA Grant Agency for Science and of the InterReg IV A – Silent Spaces Project who's one Partner is the Aalborg University.

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Received in editor's office: May 30, 2011
Approved for publishing: August 19, 2011