ANALITICAL SIMULATION OF APPLICATION OF FFT BASED SPECTRAL METHOD OF FATIGUE CYCLE COUNTING FOR MULTIAXIAL STRESS ON EXAMPLE OF PULSE EXCITED BEAM

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Vibration of engineering structures results in time variable displacements functions. For realistic structures modes of vibration are determined by exciting frequency and the natural frequencies of the structure. Hence, the components of stress tensor are time function too. In the presented paper the general idea of method of cycle-counting based on the FFT analysis for engineering structures is discussed and analytically verified for the case vibrating beam excited by series pulse-type force with no damping.

Keywords : *fatigue, FFT analysis, vibration, cycle counting*

1. Introduction

Vibrations of engineering structures generate time variable displacements functions. Therefore the components of stress tensor are time function. Geometry of structures and applied boundary conditions determine the modal characteristic of the system. For complex structures relatively high modal density is often observed. The other important parameter influencing the solution is the type of external excitation, especially its spectrum. For pulse-type excitation, the relatively broad frequency band is observed. The second important role plays values of natural frequencies of the system. Thus often in the transient case, the time history of displacements and stress components have irregular form with respect to time. It means that the vibrations have the form of quasi periodic vibrations, but not random type (if not chaotic ones).

The known in literature method of fatigue life can generally takes the following form :

- Analysis in time domain superposition of harmonically variable stress components (e.g. Dowling [1]).
- Cycle counting of irregular stress functions in time domain (e.g. rain-flow method [1]).
- Spectral method in frequency domain based on PSD analysis of stress (e.g. Nieslony, Macha [5]).
- Spectral method based on PSD analysis simultaneously with kurtosis analysis – Fatigue Damage Spectrum (Van Baren J. & P. [6]).
- Spectral method in frequency domain based on FFT analysis of stress (Kozien, Szybiński [4], Kozień, Smolarski [3]).

The general idea of the last method is the direct application of stress spectrum for cycle counting function and was proposed by Kozien in 2011, as a new attempt. Since it, is verified

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in some practical cases, and due to obtained results modified in proposed counting formulas. In the paper [4] the application of the method for harmonically excited beam with uniaxial stress is discussed. In the paper [3] its application for pulse-excited beam with uniaxial stress is discussed. In the presented paper the application of the method for the case of multiaxial stress is presented for the case of pulse-excited beam with no internal damping.

2. FFT based method of cycle counting

For realistic engineering structures, the problem of finding the equivalent uniaxial cycles of stress arises. Moreover, due to the fact that the fatigue curves are investigated for some characteristic cycles forms, the uniaxial stress should be transformed to the required one, usually of completely reversed type.

The mechanism of plastic deformation and mechanism of fatigue failure are not the same. Therefore, there are no simple reasons for application of the plastic criterion approach for fatigue. However, due to the fact of non existence the other useful estimator, often the same concept of equivalent stress is applied in fatigue analysis to transform amplitudes of the multiaxial stresses into the uniaxial case. The most common formula is the application of von Mises equivalent stress to determine the equivalent stress amplitude σ_{a} (1).

$$\sigma_{\mathrm{a}} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_{1\mathrm{a}} - \sigma_{2\mathrm{a}})^2 + (\sigma_{1\mathrm{a}} - \sigma_{3\mathrm{a}})^2 + (\sigma_{2\mathrm{a}} - \sigma_{2\mathrm{a}})^2}\ . \tag{1}$$

The effective mean stress like the equivalent stress amplitude can be estimated by applying the definition of equivalent von Misses stress in the above form. But for such a case the different effect of mean tension versus mean compression in relation to fatigue is not included. It has been taken into account in Sines' empirical expression (2) for equivalent mean stress, where a is an empirical factor. The equation was simplified by Fuchs and Stephens, by putting the factor a as equal to unity [2]. The new formula is sometimes named as the Sines equivalent stress.

$$\sigma_{\mathrm{m}} = a\,(\sigma_{1\mathrm{m}} + \sigma_{2\mathrm{m}} + \sigma_{3\mathrm{m}})\ . \tag{2}$$

The experimental S-N curve is usually constructed for the one of the characteristic stress cycles in time domain. The most popular is the completely reversed stress cycle. Based on the Goodman hypothesis, for any harmonically varying cycle of stresses it is possible to determine the equivalent completely reversed stress in the form (3), where σ_{u} – ultimate stress [1].

$$\sigma_{\mathrm{ar}} = \frac{\sigma_{\mathrm{a}}}{1 - \dfrac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{u}}}} \quad \text{for} \quad \sigma_{\mathrm{m}} > 0\ , \qquad \sigma_{\mathrm{ar}} = \sigma_{\mathrm{a}} \quad \text{for} \quad \sigma_{\mathrm{m}} \le 0\ . \tag{3}$$

The response of excited vibrating mechanical system is described by set of frequencies, for which the stresses can vary in time. These frequencies are the natural frequencies of the system. The spectrum of loading/stress has in general an irregular form. The fatigue life in spectrum loading is a function of accumulated effects in analyzed part, at various stress levels, throughout its performance, that is, the function of cumulative damage [1]. Such formulated problem is not easy to describe, and there are many known in literature methods of analysis. The most popular is the Palmgren-Miner assessment. For completely irregular histories of loads/stresses with time, the problem of identification of cycles arises. The most commonly used method is the rain flow counting.

In dynamic process the stress components are functions variable of time. If excitation is a pulse type, the response depends only on the natural frequencies of the system. Therefore in frequency domain the stress function has a discrete representation. It means that with every frequency existing in spectrum, the suitable value of stress amplitude is connected. Therefore if the FFT transform of the stress function is known, and it has the form of peak-like form, it seems that the function can be represented as a sum of suitable harmonic functions. So, it is the attempt to a form of the proposed FFT based cycle-counting for considered time period. If the components are in phase it can be amplified for some time points. For simplicity and safety, it is assumed further that the components are in phase. If there are any other types of static-type stress they can be applied in the form of the mean stress in formula (3).

Let us assume that in the FFT amplitude characteristic, there are N non-zeroes (or sufficiently high for fatigue analysis) bars for frequencies $f^{(i)}$ $(i = 1, \ldots, N)$ with stress amplitude values $\sigma_{\mathrm{a}j}^{(i)}$ suitably for stress time-function – written in Voigt's notation – $\sigma_j(t)$ $(j = 1, 2, 3, 4, 5, 6)$. Then the following set of the parameters are defined:

- number of cycles $n^{(i)}$ $(i = 1, \ldots, N)$ in relation to the lowest frequency $f^{(1)}$ and connected parameter of the longest period $T^{(i)}$ $(n^{(1)} = 1)$,
- equivalent stress amplitude value $\sigma_{\mathrm{a}}^{(i)}$ $(i = 1, \ldots, N)$, determined based on formula (1) for each frequency,
- mean stress value $\sigma_{\mathrm{ms}}^{(i)}$ $(i = 1, \ldots, N)$, determined based on formula (2) for each frequency,
- effective mean stress $\sigma_{\mathrm{m}}^{(i)}$ $(i = 1, \ldots, N)$, which is defined in the following rule: $\sigma_{\mathrm{m}}^{(1)} = 0$, $\sigma_{\mathrm{m}}^{(2)} = \sigma_{\mathrm{ms}}^{(1)}$, $\sigma_{\mathrm{m}}^{(3)} = \sigma_{\mathrm{ms}}^{(2)} + \sigma_{\mathrm{ms}}^{(3)}$, ...
- equivalent completely reversed stress value $\sigma_{\mathrm{ar}}^{(i)}$ $(i = 1, \ldots, N)$, determined based on formula (3) for every frequency.

Finally for the lowest frequency f_1 (the longest period T_1) is obtained the following pairs of parameters (number of cycles, stress amplitude) – $(n^{(i)}, \sigma_{\mathrm{ar}}^{(i)})$ This is the base for further fatigue analysis, based on the Palmgren-Miner cumulative rule concept [1, 2].

3. Pulse excited vibrations of beam

Let us consider excited vibration by the point force of a pulse type in time domain of the simply supported Bernoulli-Euler beam without damping (see Fig. 1). The cross section area of I-type, makes possible to determine the multiaxial stress case ($\sigma_1(t) = \sigma_{\mathrm{x}}(t)$, $\sigma_2(t) = \tau_{\mathrm{xz}}(t)$, $\sigma_3(t) = \tau_{\mathrm{xy}}(t)$). The most important are the two components ($\sigma_1(t) = \sigma_{\mathrm{x}}(t)$, $\sigma_2(t) = \tau_{\mathrm{xz}}(t)$). Equation of motion of a beam has the form (4) and the boundary conditions the form (5). Analytical solution of the problem can be written in form (6), where: P – pulse point excitation [Ns], L – length of beam [m], ρ – material density [kg/m^3], A – area of cross-section [m^2], E – Young modulus [N/m^2], ω_n – n-th natural angular frequency [rad/s], $\delta(\bullet)$ – Dirac delta distribution.

$$\frac{\partial^4 w(x,t)}{\partial x^4} + \frac{\rho\, A}{E\, I}\, \frac{\partial^2 w(x,t)}{\partial t^2} = \frac{P\, \delta(x-c)\, \delta(t)}{E\, I} \;, \tag{4}$$

$$w(0,t) = 0 \;, \qquad M_{\mathrm{g}}(0,t) = 0 \;, \qquad w(L,t) = 0 \;, \qquad M_{\mathrm{g}}(L,t) = 0 \;, \tag{5}$$

$$w(x,t) = \frac{2}{l}\, \frac{P}{\rho\, A} \sum_{n=1}^{\infty} \frac{1}{\omega_n}\, \sin\!\Big(\frac{\pi\, n}{l}\, c\Big)\, \sin\!\Big(\frac{\pi\, n}{l}\, x\Big)\, \sin(\omega_n\, t) \;. \tag{6}$$

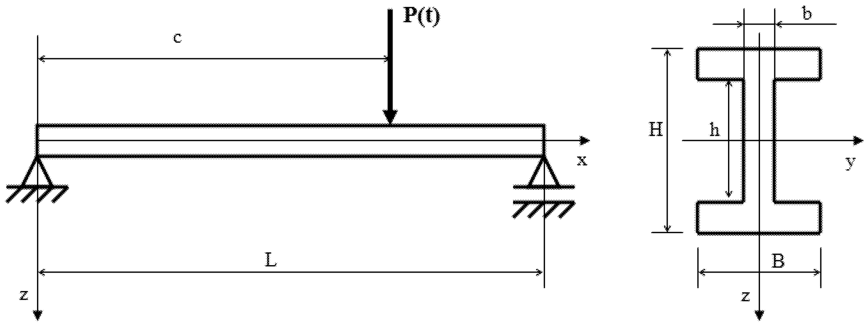

Fig.1: Geometry of a beam, boundary conditions and external force

The normal stress $\sigma_{\mathrm{x}}(x,t)$ can be obtained based on the formulas (6), (7), (8) where z is the distance from the neutral axis of a beam, and the shear stress $\tau_{\mathrm{xz}}(x,t)$ can be obtained based on the formulas (6), (9), (10) all written on the Bernoulli-Euler assumptions. For chosen control point (cross-section $x = x_0$, distance from neutral axis $z = z_0$) and the normal and shear stress function are only a time function. Calculating the FFT of their, the suitable values ($n^{(i)}$, $\sigma_{\mathrm{a}}^{(i)}$) can be obtained. Based on it the pair ($n^{(i)}$, $\sigma_{\mathrm{ar}}^{(i)}$) may be calculated ($i = 1,\dots,N$).

$$M_{\mathrm{g}}(x,t) = -E\,I\,\frac{\partial^2 w(x,t)}{\partial x^2}\ , \tag{7}$$

$$\sigma_{\mathrm{x}}(x,z,t) = \frac{M_{\mathrm{g}}(x,t)}{I}\,z\ , \tag{8}$$

$$T(x,t) = -E\,I\,\frac{\partial^3 w(x,t)}{\partial x^3}\ , \tag{9}$$

$$\tau_{\mathrm{xz}}(x,z,t) = \frac{T(x,t)}{8\,b\,I}\left[b\,(h^2 - 4\,z^2) + B\,(H^2 - h^2)\right]\ . \tag{10}$$

4. Numerical example

As an example of application of the discussed method the beam made of steel 45 (see Fig. 2 for S-N curve; $Z_{\mathrm{G}} = 280\,[\mathrm{MPa}]$ denotes the fatigue limit for bending tests) with geometry shown in Fig. 1 is considered. For the first case the single force of pulse type of the value $P = 1.25\,[\mathrm{Ns}]$ is put in the middle of the beam ($c = 0.5\,[\mathrm{m}]$). It is a control cross-section too ($x_0 = 0.5\,[\mathrm{m}]$). The only odd modes ($n = 1,3,5,\dots$) are identified for considered case because the eigenmodes functions for the beam take zeroes values for the cross-section in the middle of plate for even modes ($n = 2,4,6,\dots$). The second case is connected with control cross-section and place of force action with the same amplitude $P = 1.25\,[\mathrm{Ns}]$, with the values of $x_0 = c = 0.3\,[\mathrm{m}]$). All modes are active for this point. For non-zeroes shear stress, the control point distanced $0.008\,[\mathrm{m}]$ ($z_0 = 0.008\,[\mathrm{m}]$) from the middle axis is chosen. It should be pointed, that higher values of normal stress are obtained on the external surface ($z_0 = 0.01\,[\mathrm{m}]$), but zeroes shear stress exist in this point. Hence it is correct point for fatigue estimation, but not good for validation of the method. Values of geometrical and material parameters are: $L = 1\,[\mathrm{m}]$, $B = 0.01\,[\mathrm{m}]$, $b = 0.001\,[\mathrm{m}]$, $H = 0.02\,[\mathrm{m}]$, $h = 0.016\,[\mathrm{m}]$, $\rho = 7800\,[\mathrm{kg/m^3}]$, $E = 2.1\mathrm{E}{+}11\,[\mathrm{N/m^2}]$, $\sigma_{\mathrm{u}} = 625\,[\mathrm{MPa}]$. Due to assumption (ideal structure with no internal and external damping), vibrations are undamped.

Stress function $\sigma_{\mathrm{x}}(t) = \sigma_{\mathrm{x}}(x_0, z_0, t)$ and $\tau_{\mathrm{xz}}(t) = \tau_{\mathrm{xz}}(x_0, z_0, t)$ are obtained for chosen control points based on the formulas (6), (8) and (10). They are shown in Fig. 3 and Fig. 5 suitably, for the second case. In Fig. 4 and Fig. 6 they are shown the FFT amplitudes of considered time functions.

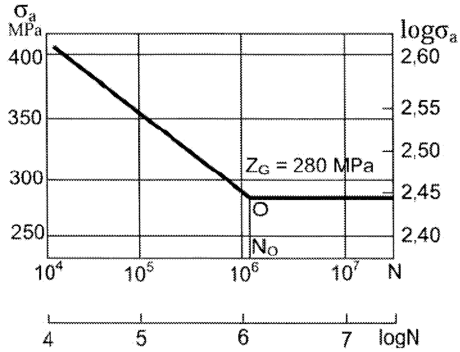

Fig.2: S-N curve for steel 45

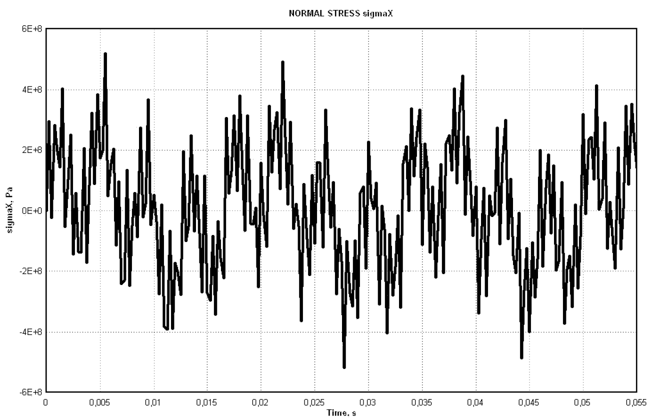

Fig.3: Variation of normal stress $\sigma_{\mathrm{x}}(t)$ in time for the second case ($x_0 = c = 0.3\,[m]$)

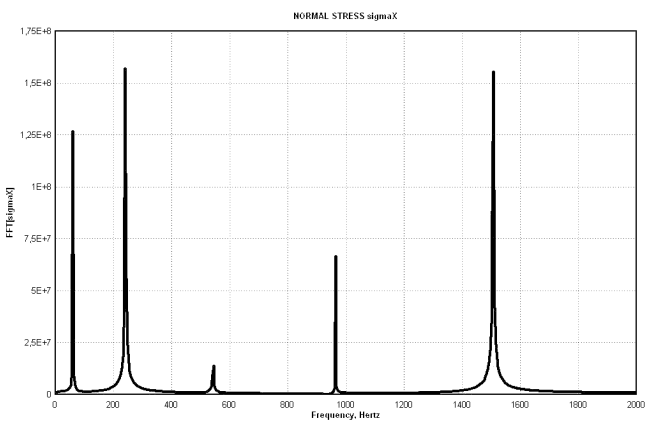

Fig.4: FFT amplitude of normal stress σ_{x} for the second case ($x_0 = c = 0.3\,[m]$)

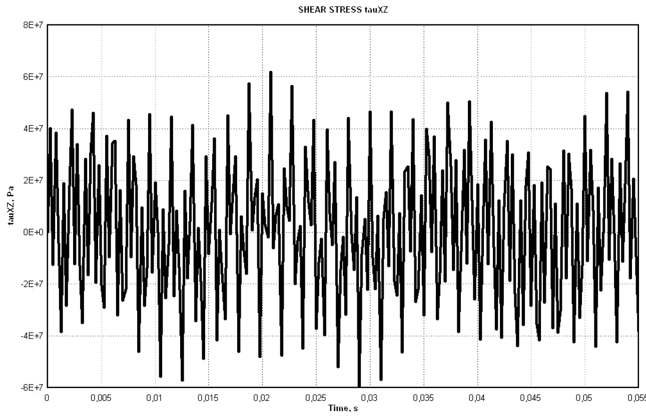

Fig.5: Variation of shear stress $\tau_{xz}(t)$ in time for the second case ($x_0 = c = 0.3$ [m])

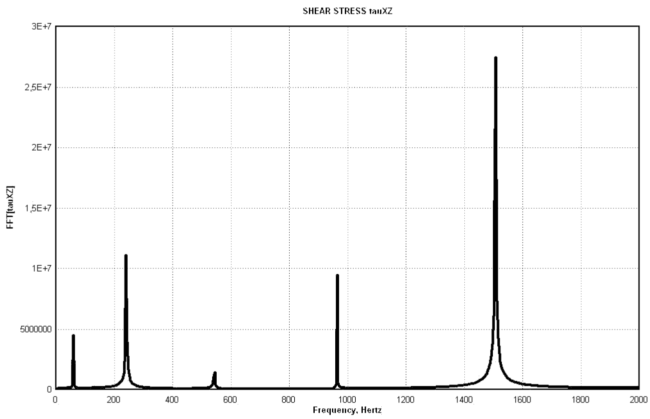

Fig.6: FFT amplitude of shear stress τ_{xz} for the second case ($x_0 = c = 0.3$ [m])

Based on the obtained FFT function, values of required parameters may be calculated. Results for the chosen points and cases are given in Tab. 1 and Tab. 2. Then from S-N curve for steel 45 (Fig. 2), maximal number of cycles ($N^{(i)}$) can be obtained for each mode and based on the Palmgren-Miner rule of cumulative damage, life time can be estimated. It is not the aim of the paper.

i	1	3	5
$f^{(i)}$ [Hz]	60	542	1507
$n^{(i)}$	1	9	25
$\sigma_{a1}^{(i)} = \sigma_{ax}^{(i)}$ [MPa]	193	145	156
$\sigma_{a2}^{(i)} = \tau_{axz}^{(i)}$ [MPa]	7	15	28
$\sigma_{m}^{(i)}$ [MPa]	0	193	338
$\sigma_{a}^{(i)}$ [MPa]	193	147	163
$\sigma_{ar}^{(i)}$ [MPa]	193	213	356
$N^{(i)}$	∞	∞	1.5E+5

Tab. 1: Estimated parameters after cycle counting by FFT spectral method for the first case ($x_0 = c = 0.5$ [m])

I	1	2	3	4	5
$f^{(i)}$ [Hz]	60	241	542	964	1507
$n^{(i)}$	1	4	9	16	25
$\sigma_{\mathrm{a1}}^{(i)} = \sigma_{\mathrm{ax}}^{(i)}$ [MPa]	127	157	14	67	156
$\sigma_{\mathrm{a2}}^{(i)} = \tau_{\mathrm{axz}}^{(i)}$ [MPa]	5	11	2	9	28
$\sigma_{\mathrm{m}}^{(i)}$ [MPa]	0	127	284	298	365
$\sigma_{\mathrm{a}}^{(i)}$ [MPa]	127	158	14	69	163
$\sigma_{\mathrm{ar}}^{(i)}$ [MPa]	127	199	26	132	393
$N^{(i)}$	∞	∞	∞	∞	3.5E+4

Tab. 2: Estimated parameters after cycle counting by FFT spectral method for the second case ($x_0 = c = 0.3$ [m])

5. Conclusions

The proposed and tested method of estimation seems to be easy in application for engineering structures and its assumptions give conservative assessment of fatigue life. This is due to assumed method of estimation of the equivalent mean stress for each mode. Moreover the phase shift between stress is not assumed in the method. In practice it reduces the equivalent stress.

The method can be easy applied together with structural dynamic finite element analysis for different kind of excitations (e.g. force, thermal, magnetic). The application of finite element methods in dynamics is limited to low frequency analysis. It means to the cases with low modal density, when the modes are separated. The application of the proposed method is in practice limited to the same cases of limited numbers of separated modes. For broad spectrum analysis the other method, e.g the rain-flow one or the fatigue damage spectrum one or thus directly based on PSD function gives better results.

The method must be generalized for damped vibrations e.g. by application of the Short Time Fourier Transform (STFT) or the Wavelet Transform (WT), which make possible to analyze time-history of the spectrum. The other problem is description of the stress component function shifted in phase, by special method of superposition. This topics are in-progress.

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Received in editor's office: April 15, 2012
Approved for publishing: February 15, 2013