CONTRIBUTION TO APPLICATION OF 'PARAMETRIC ANTI-RESONANCE' FOR AUTOPARAMETRIC SYSTEMS

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Dedicated to our friend Ing. Ladislav Půst, DrSc.
on the occasion of his 85th birthday.

This paper deals with the phenomenon of the parametric anti-resonance of autoparametric systems. It is shown that parametric anti-resonance (additional parametric excitation fulfilling special conditions) can be used not only to suppress self-excited or externally excited vibrations but can also be used to stabilize the equilibrium position of the autoparametric system.

Keywords : *autoparametric systems, parametric anti-resonance, equilibrium stabilization*

1. Introduction

The different autoparametric systems are analysed in the book [1]. Parametric anti-resonance is a phenomenon with a vibration suppression effect based on the parametric excitation where the frequency is equal or close to either the difference or sum of eigenfrequencies of the primary system. The former can be realized by using periodic changes of the spring element stiffness (see [2]–[9] and [11], [12]) and the latter appears in some systems where angular displacements are among the coordinates (see [10]). Though the application of this phenomenon was first analysed to suppress the self-excited vibrations, it was proven that it can be used to suppress the parametric resonances, too (see [11]). This contribution deals with a potential utilization of the phenomenon in autoparametric systems which are characterized by a non-linear coupling between the primary and excited autoparametric sub-systems and the excited sub-system is the vibration source of the primary sub-system (see [1]). The aim of this study is to ascertain if the above mentioned phenomenon can be used for a suppression, or at least, a partial decrease of vibrations of the primary sub-system. In some cases, such a system can be used as a tuned mass damper, but in other cases vibrations of this system are reversely spurious and they must be suppressed. We herein restrict to the horizontal harmonic movement of the two-mass pendulum system (Fig. 1) no matter if it is caused by the external, parametric of self-excited excitation.

2. Mathematical model of two-mass pendulum system

The designed system (Fig. 1) consists of two hanged masses M and m. The first pendulum is coupled by one spring to the vertical axis and by the other spring to the other pendulum with the mass m. The stiffness of the interconnection spring k_2 is periodically changed.

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Gravitation is considered. The system is kinematically excited by the periodic function $z(t) = A\cos\omega t$ that describes vibration of very small amplitude.

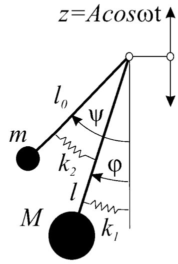

Fig.1: Schema of the two-mass system

The equations of motion are developed from the Lagrange equations of the second order. The kinematic and potential energies are expressed as

$$\begin{aligned} T &= \frac{1}{2}\,M\left[(\dot z + l\,\dot\varphi\,\sin\varphi)^2 + (l\,\dot\varphi\,\cos\varphi)^2\right] + \frac{1}{2}\,m\left[(\dot z + l_0\,\dot\psi\,\sin\psi)^2 + (l_0\,\dot\psi\,\cos\psi)^2\right]\ , \\ U &= \frac{1}{2}\,k_1\,\varphi^2 + \frac{1}{2}\,k_2\,(\psi-\varphi)^2 + M\,l\,g\,(1-\cos\varphi) + m\,l_0\,g\,(1-\cos\psi)\ . \end{aligned} \tag{1}$$

The periodically variable stiffness of the interconnection spring is expressed

$$k_2 = k_{20}\,(1 + \varepsilon\,\cos\nu t)\ . \tag{2}$$

After substituting of (1) and (2) into the Lagrange equations we get

$$\begin{aligned} &\ddot\varphi + \left(\frac{g}{l} - \frac{A}{l}\,\omega^2\,\cos\omega t\right)\sin\varphi - \left(\frac{A}{l}\,\omega\,\sin\omega t\,\cos\varphi\right)\dot\varphi + \\ &\qquad + \frac{k_1}{M\,l^2}\,\varphi + \frac{k_{20}}{M\,l^2}\,(1+\varepsilon\,\cos\nu t)\,(\varphi-\psi) = 0\ , \\ &\ddot\psi + \left(\frac{g}{l_0} - \frac{A}{l_0}\,\omega^2\,\cos\omega t\right)\sin\psi - \left(\frac{A}{l_0}\,\omega\,\sin\omega t\,\cos\psi\right)\dot\psi + \\ &\qquad + \frac{k_{20}}{m\,l_0^2}\,(1+\varepsilon\,\cos\nu t)\,(\psi-\varphi) = 0\ . \end{aligned} \tag{3}$$

The equation (3) can be transformed into the non-dimensional form after the substitution of variables $\sqrt{g/l}\,t = \tau$, $\omega/\sqrt{g/l} = \eta$, $q_1^2 = k_1/(M\,l\,g)$, $q_{20}^2 = k_{20}/(M\,l\,g)$, $\nu/\sqrt{g/l} = \bar\nu$, $m/M = \mu$

$$\begin{aligned} &\varphi'' + \left(1 - \frac{A}{l}\,\eta^2\,\cos\eta t\right)\sin\varphi - \left(\frac{A}{l}\,\eta\,\sin\eta t\,\cos\varphi\right)\varphi' + \\ &\qquad + q_1^2\,\varphi + q_{20}^2\,(1+\varepsilon\,\cos\bar\nu t)\,(\varphi-\psi) = 0\ , \\ &\psi'' + \left(1 - \frac{A}{l_0}\,\eta^2\,\cos\eta t\right)\sin\psi - \left(\frac{A}{l_0}\,\eta\,\sin\eta t\,\cos\psi\right)\psi' + \\ &\qquad + \frac{q_{20}^2}{\mu}\left(\frac{l}{l_0}\right)^2(1+\varepsilon\,\cos\bar\nu t)\,(\psi-\varphi) = 0\ . \end{aligned} \tag{4}$$

We get the final form of the equations by introducing the viscous damping

$$
\begin{aligned}
&\varphi'' + \chi_1 \varphi' + \chi_{12} (\varphi' - \psi') + \left(1 - \frac{A}{l}\eta^2 \cos\eta t\right)\sin\varphi - \left(\frac{A}{l}\eta \sin\eta t \cos\varphi\right)\varphi' + \\
&\qquad + q_1^2 \varphi + q_{20}^2 (1 + \varepsilon \cos\bar{\nu}t)(\varphi - \psi) = 0 \;, \\
&\psi'' + \chi_2 \psi' + \chi_{12} (\psi' - \varphi') + \left(1 - \frac{A}{l_0}\eta^2 \cos\eta t\right)\sin\psi - \left(\frac{A}{l_0}\eta \sin\eta t \cos\psi\right)\psi' + \\
&\qquad + \frac{q_{20}^2}{\mu}\left(\frac{l}{l_0}\right)^2 (1 + \varepsilon \cos\bar{\nu}t)(\psi - \varphi) = 0 \;.
\end{aligned}
\tag{5}
$$

For small vibrations of the masses M and m in the vicinity of the equilibrium position $\varphi = \psi = 0$ and $l = l_0$, eigenfrequencies Ω_1, Ω_2 are determined by the characteristic equation

$$
\begin{vmatrix}
-\Omega^2 + (1 + q_1^2) + q_{20}^2 & -q_{20}^2 \\
-\dfrac{q_{20}^2}{\mu} & -\Omega^2 + 1 + \dfrac{q_{20}^2}{\mu}
\end{vmatrix} = 0 \;. \tag{6}
$$

Denoting the disturbing coordinates as u, v i.e.

$$
\varphi = \varphi_0 + u \;, \qquad \psi = \psi_0 + v \tag{7}
$$

then for $\varphi_0 = \psi_0 = 0$ the equations of the disturbed motion get the form

$$
\begin{aligned}
&u'' + (1 + q_1^2)\, u + q_{20}^2 (u - v) + U = 0 \;, \\
&v'' + 1 + \frac{q_{20}^2}{\mu} (v - u) + V = 0 \;,
\end{aligned}
\tag{8}
$$

where U, V containing only small components have the form

$$
\begin{aligned}
U &= \kappa_1 u' + \kappa_{12} (u' - v') - \frac{A}{l}\left[(\eta^2 \cos\eta\tau)\, u + (\eta \sin\eta\tau)\, u'\right] + \varepsilon\, q_{20}^2 \cos\nu\tau\, (u - v) \;, \\
V &= \kappa_2 v' + \frac{\kappa_{12}}{\mu} (v' - u') - \frac{A}{l}\left[(\eta^2 \cos\eta\tau)\, v + (\eta \sin\eta\tau)\, v'\right] + \varepsilon\, \frac{q_{20}^2}{\mu} \cos\nu\tau\, (v - u) \;.
\end{aligned}
\tag{9}
$$

Using transformation to the quasi-normal form:

$$
\begin{aligned}
u &= x_1 + x_2 \;, \\
v &= a_1 x_1 + a_2 x_2 \;,
\end{aligned}
\tag{10}
$$

where

$$
a_j = \frac{\dfrac{q_{20}^2}{\mu}}{1 + \dfrac{q_{20}^2}{\mu} - \Omega_j^2} \;, \quad j = 1, 2 \;, \quad a_1 \geq 0 \;, \quad a_2 \leq 0 \tag{11}
$$

equations (8) get the form:

$$
x_s + \Omega_s^2 x_s + X_s = 0 \quad (s = 1, 2) \;, \tag{12}
$$

where

$$
X_1 = \frac{1}{a_1 - a_2} (-a_2 U + V) \;, \qquad X_2 = \frac{1}{a_1 - a_2} (a_1 U - V) \;,
$$

where in U and V relations (9) are used. Now, using condition (6) in Appendix, the sufficient stability condition can be formulated for the case $\eta = 2\,\Omega_1$:

$$\frac{P_{12}\,P_{21}}{\Omega_2} - \frac{Q_{11}^2}{\Omega_1} \geq 0 \;, \tag{13}$$

where

$$\begin{aligned} Q_{11} &= -\frac{A}{l}\,\Omega_1^2\,(1-a_2)\;, \\ P_{12} &= -\varepsilon\, q_{20}^2 \left(\frac{1}{\mu} - a_2\right)\;, \\ P_{21} &= -\varepsilon\, q_{20}^2 \left(\frac{1}{\mu} - a_1\right)\;. \end{aligned}$$

For small μ is $P_{12}\,P_{21}$ a positive value and so the condition (13) can be fulfilled.

3. Results of the numerical simulation

The solution of the equations of motion (5) were ascertained by the Runge-Kutta method of the fourth order. The following parameters of the system were chosen: $\mu = 0.2$, $q_1^2 = 1$, $q_{20}^2 = 0.2$, $\chi_1 = \chi_2 = \chi_{12} = 0.01$, $A/l = 0.05$, $\varepsilon = 0.5$. Then the left side of inequality (13) acquires a positive value 0.0541 and therefore the stability condition is fulfilled.

Then, eigenfrequencies computed from the Eq. (6) are $\Omega_1 = 1.28$, $\Omega_2 = 1.6$.

The excitation frequency $\eta = 2\,\Omega_1$ [13] and a variable stiffness frequency, i.e. the so-called frequency of additional parametric excitation $\nu = \Omega_2 - \Omega_1$ [2–5], were chosen for the analysis of the vibration suppression.

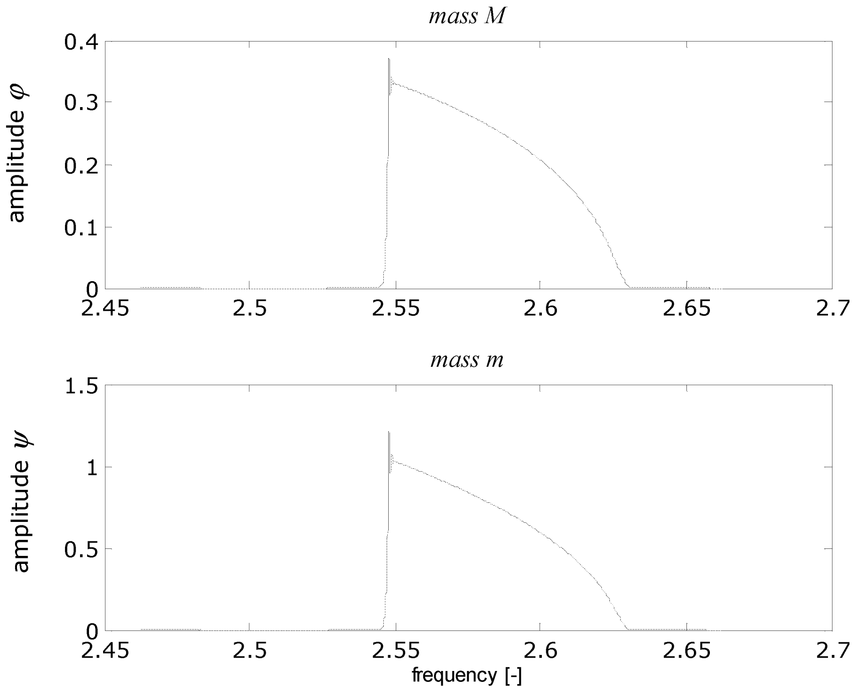

Fig.2: The amplitude-frequency dependences of mass M and m displacements under sweep excitation

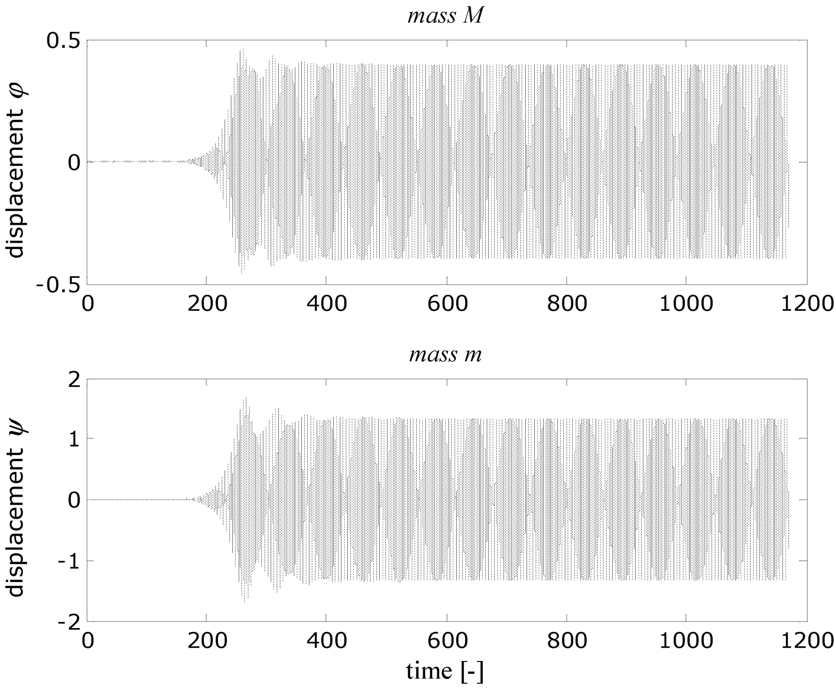

Fig.3: Time characteristics of mass M and m displacement under harmonic excitation ($\eta = 2.56$) and zero additional excitation ($\varepsilon = 0$)

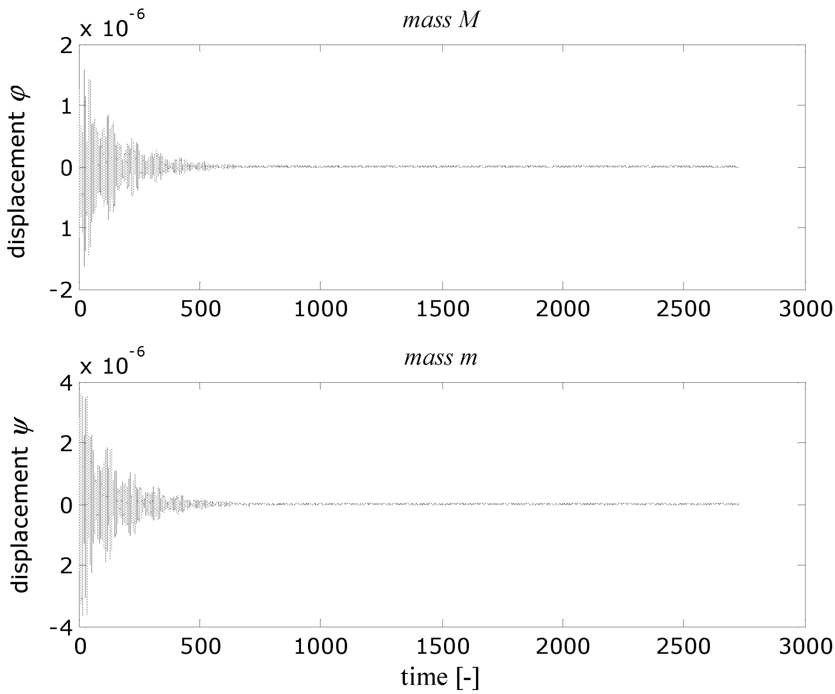

Fig.4: Time characteristics of mass M and m displacement under harmonic excitation ($\eta = 2.56$) and non-zero additional excitation ($\varepsilon = 0.5$)

The amplitude-frequency characteristics of movements of masses M and m under a very small sweep excitation frequency rate ($\dot{\eta} = 1\,\mathrm{e}^{-5}$) in the vicinity of the frequency $\eta = 2\,\Omega_1 = 2.56$ are depicted in the Fig. 2. It is obvious that a very fast increase of amplitudes at the frequency 2.56 is what corresponds to the loss of stability assumption.

The time characteristics of displacements φ, ψ of masses M and m, respectively, at the harmonic excitation $\eta = 2.56$ and the initial conditions $\varphi = \dot{\varphi} = \psi = \dot{\psi} = 1\,\mathrm{e}^{-6}$, are shown for the case of zero additional excitation ($\varepsilon = 0$) and non-zero additional excitation ($\varepsilon = 0.5$) in figures 3 and 4, respectively.

4. Conclusion

This contribution dealt with a potential utilization of the parametric anti-resonance for autoparametric systems. Though the analysis was restricted to the pendulum system which consisted of two masses interconnected with the spring of periodically changing stiffness (the change in frequency given by the difference between the system eigenfrequencies), results confirmed a possibility of the movement stabilization in the equilibrium position. A future analysis will target the autoparametric system with additional mass transmitting external excitation into the system.

Appendix

Let us suppose that both parametric excitations (original and the additional) are harmonic. Such a system after transformation into the quasi-normal form is governed by the following equations:

$$\ddot{x}_s = \Omega_s^2\, x_s + \varepsilon \left\{ \sum_{k=1}^{n} [\Theta_{sk}\, \dot{x}_k + \cos\omega t\, Q_{sk}\, x_k + \cos\eta t\, P_{sk}\, x_k] \right\} = 0 \ , \quad (s = 1, \ldots, n) \ , \tag{A.1}$$

where ε is a small parameter, ω is the frequency of the acting original parametric excitation and η is the frequency of the additional parametric excitation which should suppress the parametric resonance of the original parametric excitation and P_{sk}, Q_{sk} are the coefficients.

Let us suppose that the aim of the additional parametric excitation is to suppress the parametric resonance of the first kind and first order, e.g. at $\omega = 2\,\Omega_1$.

It is necessary to take into account the following facts:

1) Considering the case $\eta\, t = 0$ the resonance at $\omega = 2\,\Omega_1$ is the parametric resonance of the first kind. The trivial solution is unstable, unless the following condition is met (see [1]):

$$\left(\frac{Q_{11}}{2\,\Omega_1}\right)^2 - \Theta_{11}^2 \geq 0 \ . \tag{A.2}$$

For positive damping is Θ_{11} positive and so the condition for suppression this parametric resonance reads:

$$\frac{Q_{11}}{2\,\Omega_1} \leq \Theta_{11} \ . \tag{A.3}$$

2) Considering the case when $\cos\omega t = 0$, $\cos\eta\tau = \cos|\Omega_k - \Omega_1|t$, then the conditions to eliminate the effect of the negative linear damping are (see [14]):

$$\Theta_{11} + \Theta_{kk} \geq 0 \ , \qquad \frac{P_{1k}\, P_{k1}}{4\,\Omega_1\, \Omega_k} + \Theta_{11}\, \Theta_{kk} \geq 0 \ . \tag{A.4}$$

For positive damping the first condition (4) is met and the second is decisive. We can see that the first term in the second condition (4) can represent the additional positive damping when $P_{1k}\,P_{k1}$ is a positive value. For the positive linear damping the conditions for avoiding the parametric resonance at $\omega = 2\,\Omega_1$ is:

$$\frac{P_{1k}\,P_{k1}}{4\,\Omega_1\,\Omega_k} - \left(\frac{Q_{11}}{2\,\Omega_1}\right)^2 + \Theta_{11}^2 + \Theta_{11}\,\Theta_{kk} \geq 0 \; . \tag{A.5}$$

For positive damping the sufficient condition of the parametric resonance suppression at $\omega = 2\,\Omega_1$ reads:

$$\frac{P_{1k}\,P_{k1}}{\Omega_k} - \frac{Q_{11}^2}{\Omega_1} \geq 0 \; . \tag{A.6}$$

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