LINEAR AERO-ELASTIC MODELS OF A PRISMATIC BEAM IN A CROSS-FLOW USING DOUBLE DEGREE OF FREEDOM SYSTEM

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Double degree of freedom (DDOF) linear systems are frequently used to model the aero-elastic response of slender prismatic systems until the first critical state is reached. Relevant mathematical models appearing in literature differ in principle by way of composition of aero-elastic forces. This criterion enables to sort them roughly in three groups: (i) neutral models – aero-elastic forces are introduced as suitable constants independent from excitation frequency and time; (ii) flutter derivatives – they respect the frequency dependence of aero-elastic forces; (iii) indicial functions – they are defined as kernels of convolution integrals formulating aero-elastic forces as functions of time. The paper tries to put all three groups together on one common basis to demonstrate their linkage and to eliminate gaps in mathematical formulations between them. This approach allows formulate more sophisticated models combining main aspects of all groups in question keeping the DDOF basis. These models correspond by far better to results of wind tunnel and full scale measurements.

Keywords: *flutter derivatives, indicial functions, non-symmetric linear systems, dynamic stability*

1. Introduction

Slender prismatic structures exhibited to strong dynamic wind effects (bridge decks, towers, chimneys, etc.) are frequently analyzed using a double degree of freedom (DDOF) linear model working with heaving and torsional components of a cross-section response, see e.g. [2]. This aero-elastic model is often adequate to study the system response until the first critical state is reached. Relevant mathematical models appearing in literature differ in principle by way of composition of aero-elastic forces. This criterion enables to sort them roughly in three groups. The first group can be possibly called neutral models – aero-elastic forces are introduced as suitable constants independent from excitation frequency and time. The second one involves flutter derivatives – they respect the frequency dependence of aero-elastic forces, see [3]. Finally the third is working with indicial functions – they are defined as kernels of convolution integrals formulating aero-elastic forces as functions of time, see [4], [5]. Second and third groups have been developing separately from each other and seem to be isolated until now. Moreover they mostly suffer from various gaps in mathematical formulations and further treatment. The paper tries to put all three groups together on one common basis and to demonstrate linkage of them. This approach allows formulate more sophisticated models combining main aspects of all groups keeping the DDOF basis. These models correspond by far better to results of wind channel and full scale measurements and seem to be very promising for the future investigation and practical applications.

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For purposes of this study the bridge girder is considered as axially symmetric or almost symmetric with possible response components in heave u (vertical direction) and pitch φ (rotation around S point). An outline can be seen in Fig. 1.

In principle all types of above models have been investigating many years. Each of them has its advantages and shortcomings. However most of them suffer very often from mathematical gaps preventing their generalization and synthesis on formal basis in order to identify some special phenomena remaining hidden when dealing with heuristic approaches only. Let us characterize now briefly the groups of models mentioned above in forthcoming parts.

2. Neutral models

Neutral models are relatively the most simple and enable to provide many results analytically in a form of closed solution. These models have been extensively studied for instance in [1].

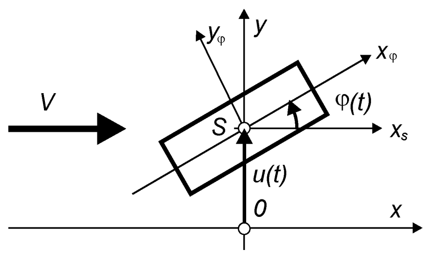

Fig.1: Schematic DDOF model of a bridge symmetric cross-section under wind loading

Although there exist many versions of a basic formulation, in principle the most general model of neutral type can be expressed in the form :

$$\begin{aligned} &\ddot{u} + b_{\mathrm{m}} \cdot \dot{u} - h\, q \cdot \dot{\varphi} + \omega_{\mathrm{u}}^2 \cdot u - p \cdot \varphi = 0 \ , \\ &\ddot{\varphi} + q \cdot \dot{u} + b_{\mathrm{I}} \cdot \dot{\varphi} + g\, p \cdot u + \omega_{\varphi}^2 \cdot \varphi = 0 \end{aligned} \tag{1}$$

where we have denoted : ω_{u}^2, ω_{φ}^2 [s^{-2}] – square of the total eigenfrequencies in relevant components including stiffness and aero-elastic components; b_{m}, b_{I} [s^{-1}] – total damping parameters including internal structural damping and aero-elastic contribution; q [$(\mathrm{ms})^{-1}$] or p [$\mathrm{m.s}^{-2}$] – gyroscopic or non-conservative forces of aero-elastic origin respectively; g [m^{-2}], h [m^2] – auxiliary constants serving for dimensional compatibility of the above equations (they can be regarded as certain characteristics of the cross-section). Parameters q, p in general don't include any static components which follow from elastic properties of the system itself, they consist only of aero-elastic terms vanishing for zero velocity of the air stream. So for stream velocity $V = 0$, the system (1) degenerates in two independent equations.

Application of the Laplace transform on infinite time interval suppresses any influence of initial conditions supposing that the system response is fully stationary. This step is analogous with the Fourier transform adopting an equivalence $\omega = -\mathrm{i}\,\lambda$. It enables to express the system (1) in the frequency domain, which reads in the matrix form as follows :

$$\begin{bmatrix} \lambda^2 + \lambda\, b_{\mathrm{m}} + \omega_{\mathrm{u}}^2 & -\lambda\, h\, q - p \\ \lambda q + g\, p & \lambda^2 + \lambda\, b_{\mathrm{I}} + \omega_{\varphi}^2 \end{bmatrix} \begin{bmatrix} U \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ . \tag{2}$$

The determinant of the system matrix of Eq. (2) is the characteristic equation of the system (1) and represents the main tool for stability investigation. It reads:

$$\begin{aligned} D = \lambda^4 + \lambda^3\,(b_\mathrm{m} + b_\mathrm{I}) + \lambda^2\,(\omega_\mathrm{u}^2 + \omega_\varphi^2 + b_\mathrm{m}\,b_\mathrm{I} + h\,q^2) + \\ + \lambda\big(\omega_\mathrm{u}^2\,b_\mathrm{I} + \omega_\varphi^2\,b_\mathrm{m} + (1 + g\,h)\,p\,q\big) + \omega_\mathrm{u}^2\,\omega_\varphi^2 + g\,p^2 = 0 \end{aligned} \tag{3}$$

The resulting characteristic equation represents annuled polynomial of the fourth order ($n = 4$) with roots λ_1, λ_2, λ_3 and λ_4. The trivial solution of system (1) is stable only if a real part of all four roots is negative. In other words, stability limits are given by conditions:

$$\mathrm{Re}(\lambda_i) = 0\ , \quad i \in (1, \dots, 4)\ . \tag{4}$$

Consequently, the trivial solution of system (1) is stable in a domain representing an intersection of sub-domains $\mathrm{Re}(\lambda_i) < 0$, $i \in (1, \dots, 4)$.

The system (1) and the characteristic equation (3) can provide a lot of information regarding motion stability, critical velocities V_crit, system response on stability limits, etc. Consequently, it enables to predict flutter/divergence onset velocity as well as to estimate their shapes in a particular case. However aero-elastic coefficients in Eqs (1) are introduced as constants corresponding to certain conditions ruling around the cross-section. Anyway, these coefficients are functions of the stream velocity V and of the frequency ω which represents a frequency of the model vibration in components u, φ due to cross-flow. This frequency is unknown a priori and should be looked up solving equation (3) as the frequency of the post-critical response. Therefore some iterative process should follow balancing these effects in order to harmonize velocity V with velocity V_crit. It is obvious that the neutral models considers the aero-elastic forces as parameters only despite being dependent from V. Despite these shortcomings the applicability of neutral models is quite wide if the variability of the aero-elastic terms is approximately linear. Otherwise one of more sophisticated models should be used, as we will see in next two parts.

Strategy of the stability investigation can be based on Routh-Hurwitz inspection of Eq. (3). The detailed analysis and relevant results can be found e.g. in [6], [7]. The most important types of aero-elastic stability loss (flutter and divergence) and their possible interactions are there given together with the conditions of their existence.

3. Models with flutter derivatives

Flutter derivatives have been introduced many years ago, see for instance [8] and more recently [9]. Their various aspects have been investigated extensively for a long time in the aircraft, civil and other branches of engineering. They have been introduced as functions in the frequency domain related to a particular cross-section without any link with other system parameters (inertia, elastic stiffness, internal damping). Nevertheless they can be understood as a certain extension of the damping and stiffness matrices. Flutter derivatives can be interpreted as amplitudes of the heaving forces Q or the pitching moments M which are needed to reach a unit amplitude of one response component under harmonic external kinematic excitation, while remaining components are kept zero in the same time (excitation frequency ω, stream velocity in wind tunnel V). Thus the flutter derivatives are the functions of the excitation frequency and the stream velocity which are combined in an argument $\kappa = B\,\omega/(2\pi\,V)$. So the basic relations between kinematic and force components read:

	$\dot{u}$	u	$\dot{\varphi}$	φ
Q :	$H_1(\kappa)$	$H_4(\kappa)$	$H_2(\kappa)$	$H_3(\kappa)$
M :	$A_1(\kappa)$	$A_4(\kappa)$	$A_2(\kappa)$	$A_3(\kappa)$
Q :	$A_{11}(\kappa)$	$A_{12}(\kappa)$	$A_{13}(\kappa)$	$A_{14}(\kappa)$
M :	$A_{21}(\kappa)$	$A_{22}(\kappa)$	$A_{23}(\kappa)$	$A_{24}(\kappa)$

$$; \qquad \kappa = \frac{B\,\omega}{2\pi\cdot V} \;\Rightarrow \;\Rightarrow\; \kappa = \frac{\omega}{\eta}\,, \quad \eta = \frac{2\pi\cdot V}{B}\,, \tag{5}$$

where following notation has been introduced: $H_i(\kappa)$ or $A_i(\kappa)$ – amplitudes of flutter derivatives corresponding to heaving forces Q or pitching moments M due to individual sets of unit kinematic harmonic excitations of a proper cross-section in an aerodynamic tunnel (notation and indexing corresponds to literature referenced); $A_{ij}(\kappa)$ – alternative notification of flutter derivatives assigned with respect to the table (5); κ – argument ('dimensionless' frequency) depending on excitation frequency ω and a stream velocity V; B – geometric characteristic of the cross-section [m].

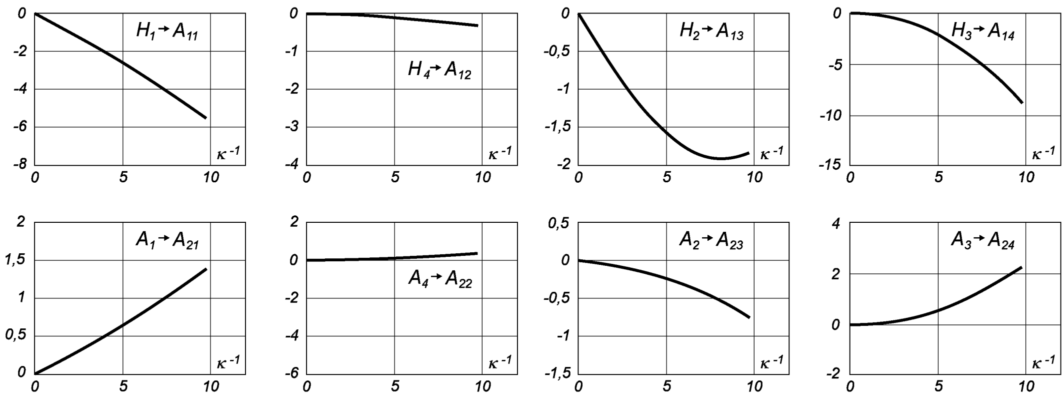

Fig.2: Outline of flutter derivatives; rectangular cross-section, ratio 1:5; position of A_{11}–A_{24} pictures correspond with table in Eq. (5)

The definition itself of flutter derivatives apparently implicates that they can serve only to develop a linear mathematical model as their application is based on the superposition principle. Flutter derivatives can be incorporate into the governing equations of type (1) only if these equations are expressed in the frequency domain. Hence system (1) should be written in a form of the two-way Laplace transform (integration $t \in (-\infty, +\infty)$) to unify the basis of both parts, for instance: $U(\lambda) = \int_{-\infty}^{\infty} u(t)\, \exp(-\lambda\, t)\cdot \mathrm{d}t$. Transformation exists if the system is stable and therefore influence of initial conditions disappear with increasing time. It means, however, that only steady state problems with explicit frequency $-\mathrm{i}\,\lambda$ can be investigated. Finally we write the complete system in the frequency domain, so that it has a character of an algebraic (unknowns U, Φ) rather then differential system:

$$\begin{matrix} Q: \\ \\ M: \\ \end{matrix} \begin{bmatrix} \lambda^2 + \lambda\, b_{\mathrm{m,c}}\,(1+\kappa\, A_{11}) + \omega_{\mathrm{u,c}}^2\,(1+\kappa^2 A_{12})\,; \\ \qquad -\lambda\, h\, q_{\mathrm{c}}\,(1+\kappa\, A_{13}) - p_{\mathrm{c}}\,(1+\kappa^2\, A_{14}) \\ \lambda\, q_{\mathrm{c}}\,(1+\kappa\, A_{21}) + g\, p_{\mathrm{c}}\,(1+\kappa^2\, A_{22})\,; \\ \qquad \lambda^2 + \lambda\, b_{\mathrm{I,c}}\,(1+\kappa\, A_{23}) + \omega_{\varphi,\mathrm{c}}^2\,(1+\kappa^2\, A_{24}) \end{bmatrix} \cdot \begin{bmatrix} U \\ \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}\,, \tag{6}$$

where subscript c means a parameter related to the structure only (static air pressure is not included). The shape of flutter derivatives for the rectangular cross-section as they are plotted in Fig. 2 is commonly accepted. Let us go briefly through individual graphs in this figure. It can be observed that functions A_{ij} related with $\dot{u}$, $\dot{\varphi}$ are odd, while those related to u, φ are even with respect to the vertical axis. Indeed this fact can be shown also theoretically using Theodorsen functions, see [8]. Looking through Fig. 2, it is obvious that the courses of individual A_{ij} are not 'dramatic'. Hence with respect to the interval length needed on the $1/\kappa$ axis, only the first and the second terms of Taylor expansion seem to be satisfactory to characterize A_{ij} in equations (6). Thus for instance:

$$A_{11} \approx a_{11}\,\frac{1}{\kappa} + b_{11}\,\frac{1}{\kappa^3}\;, \qquad A_{12} \approx a_{12}\,\frac{1}{\kappa^2} + b_{12}\,\frac{1}{\kappa^4}\;, \qquad \text{etc.} \tag{7}$$

where a_{ij} and b_{ij} are relevant dimensionless coefficients of the Taylor expansion. Moreover, function values of A_{12} and A_{22} are markedly small. Indeed, dealing with a symmetrical cross-section and supposing perfectly uniform stream velocity in a wind tunnel, functions A_{12} and A_{22} should vanish identically. Their non-zero values presented in Fig. 2 are most probably the results of imperfections ruling in experiments. Despite this fact, these terms have been included to keep theoretical consistency of relevant matrices (many papers omit those and work with six derivatives only).

Coefficients $(1 + a_{ij})$ are ratios of parameters with/without influence of air surrounding the cross-section in the still air. So for instance $1 + a_{11} = b_{\rm m}/b_{\rm m,c}$ or $1 + a_{12} = \omega_{\rm u}^2/\omega_{\rm u,c}^2$. In normal models Eqs (1) are implicitly introduced parameters including the influence of surrounding air, e.g. respecting dimensionless coefficients $1 + a_{ij}$ while coefficients b_{ij} are put equal zero. It is apparent, see Fig. 2, that a_{ij} can be positive or negative. Therefore the application of a_{ij} can result in an increase or a decrease of effective system parameters due to the aero-elastic effects.

Coefficients b_{ij} quantify the influence of $(V/\lambda)^2$ argument. Putting expressions (7) into (6) and being aware that $\omega^2 = -\lambda^2$, one obtains with respect to (5):

$$\begin{array}{l} Q: \\ \\ M: \end{array} \begin{bmatrix} \lambda^2 + \lambda\, b_{\rm m}\,(1 - \frac{\eta^2}{\lambda^2}\,\beta_{11}) + \omega_u^2\,(1 - \frac{\eta^2}{\lambda^2}\,\beta_{12})\,; \\ \qquad -\lambda\, h\, q(1 - \frac{\eta^2}{\lambda^2}\,\beta_{13}) - p\,(1 - \frac{\eta^2}{\lambda^2}\,\beta_{14}) \\ \lambda\, q\,(1 - \frac{\eta^2}{\lambda^2}\,\beta_{21}) + g\, p\,(1 - \frac{\eta^2}{\lambda^2}\,\beta_{22})\,; \\ \qquad \lambda^2 + \lambda\, b_{\rm I}\,(1 - \frac{\eta^2}{\lambda^2}\,\beta_{23}) + \omega_\varphi^2\,(1 - \frac{\eta^2}{\lambda^2}\,\beta_{24}) \end{bmatrix} \cdot \begin{bmatrix} U \\ \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}\;, \tag{8}$$

where for instance $b_{\rm m} = b_{\rm m,c}\,(1 + a_{11})$, $\beta_{11} = b_{11}/(1 + a_{11})$, etc.

In order to inspect the primary form of the differential system, let us make an inverse transform of Eqs (8) back to the time domain. After tedious manipulation, the differential system including the influence of the flutter derivatives can be written as follows:

$$\begin{aligned} &\ddot{u} + b_{\rm m}\Big(\dot{u} - \eta^2\,\beta_{11}\int_{-\infty}^{t} u(\tau)\,{\rm d}\tau\Big) - h\,q\Big(\dot{\varphi} - \eta^2\,\beta_{13}\int_{-\infty}^{t} \varphi(\tau)\,{\rm d}\tau\Big) + \\ &\qquad + \omega_{\rm u}^2\Big(u - \eta^2\,\beta_{12}\int_{-\infty}^{t} (t-\tau)\,u(\tau)\,{\rm d}\tau\Big) - p\Big(-\eta^2\,\beta_{14}\int_{-\infty}^{t} (t-\tau)\,\varphi(\tau)\,{\rm d}\tau\Big) = 0\;, \\ &\ddot{\varphi} + q\Big(\dot{u} - \eta^2\,\beta_{21}\int_{-\infty}^{t} u(\tau)\,{\rm d}\tau\Big) + b_{\rm I}\Big(\dot{\varphi} - \eta^2\,\beta_{23}\int_{-\infty}^{t} \varphi(\tau)\,{\rm d}\tau\Big) + \\ &\qquad + g\,p\Big(\dot{u} - \eta^2\,\beta_{22}\int_{-\infty}^{t} (t-\tau)\,\varphi(\tau)\,{\rm d}\tau\Big) + \omega_\varphi^2\Big(\varphi - \eta^2\,\beta_{24}\int_{-\infty}^{t} (t-\tau)\,\varphi(\tau)\,{\rm d}\tau\Big) = 0 \end{aligned} \tag{9}$$

Both of the systems (8), (9) can be immediately used for further investigation. The system (9) is an expansion of (1). It demonstrates memory properties due to convolution integrals. In principle the integrals could be avoided differentiating twice both equations of the system. However resulting equations of the fourth order are less suitable for further analysis than the form of Eqs (9). Especially stability of the numerical solution of Eqs (9) is far better.

The main tool of dynamic stability analysis follows from the system (8) representing a condition of its zero determinant. This condition determines an existence of non-trivial unknowns U, Φ. The condition has a form of the characteristic equation of the eight degree of the parameter λ:

$$
\begin{aligned}
\lambda^4 \cdot \Big[& \lambda^4 + \lambda^3\,(b_\mathrm{I} + b_\mathrm{m}) + \lambda^2\,(\omega_\mathrm{u}^2 + \omega_\varphi^2 + b_\mathrm{I}\, b_\mathrm{m} + h\, q^2) + \\
& + \lambda(b_I\omega_u^2 + b_m\omega_\varphi^2 + (1+hg)qp) + (\omega_u^2\omega_\varphi^2 + gp^2)\Big] - \\
-\lambda^2\,\eta^2 \cdot \Big[& \lambda^3\,(b_\mathrm{I}\,\beta_{23} + b_\mathrm{m}\,\beta_{11}) + \\
& + \lambda^2\big(\omega_\mathrm{u}^2\,\beta_{12} + \omega_\varphi^2\,\beta_{24} + b_\mathrm{m}\, b_\mathrm{I}\,(\beta_{23} + \beta_{11}) + h\, q^2\,(\beta_{13} + \beta_{21})\big) + \\
& + \lambda\,\Big(b_\mathrm{I}\,\omega_\mathrm{u}^2\,(\beta_{23} + \beta_{12}) + b_\mathrm{m}\,\omega_\varphi^2\,(\beta_{24} + \beta_{11}) + \\
& \qquad + q\, p\,\big(\beta_{14} + \beta_{21} + h\, g\,(\beta_{13} + \beta_{22})\big)\Big) + \\
& + \big(\omega_\mathrm{u}^2\,\omega_\varphi^2\,(\beta_{12} + \beta_{24}) + g\, p^2\,(\beta_{22} + \beta_{14})\big)\Big] + \\
+\,\eta^4 \cdot \Big[& \lambda^2\,(b_\mathrm{m}\, b_\mathrm{I}\,\beta_{11}\,\beta_{23} + h\, q^2\,\beta_{21}\,\beta_{13}) + \\
& + \lambda\,\big(b_\mathrm{I}\,\omega_\mathrm{u}^2\,\beta_{12}\,\beta_{23} + b_\mathrm{m}\,\omega_\varphi^2\,\beta_{11}\,\beta_{24} + q\, p\,(\beta_{21}\,\beta_{14} + h\, g\,\beta_{22}\,\beta_{13})\big) + \\
& + (\omega_\mathrm{u}^2\,\omega_\varphi^2\,\beta_{12}\,\beta_{24} + g\, p^2\,\beta_{22}\,\beta_{14})\Big] = 0\ .
\end{aligned}
\tag{10}
$$

Equation (10) consists of three parts being formulated with respect to degrees of η^2 or degrees of stream velocity V^2. It represents a generalization of Eq. (3). The term in the first brackets (zero degree of η^2) corresponds with the left hand side of Eq. (3). The first part interacts with the first two terms of the second part on the level of λ^5 and λ^4.

Assessing values of β_{ij} as they follow from the particular graphs in Fig. 2, we can adopt approximately: $\beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0$. This assumption simplifies Eq. (10) significantly reducing its degree in λ from eight to six:

$$
\begin{aligned}
\lambda^2 \cdot \Big[& \lambda^4 + \lambda^3\,(b_\mathrm{I} + b_\mathrm{m}) + \lambda^2\,(\omega_\mathrm{u}^2 + \omega_\varphi^2 + b_\mathrm{I}\, b\, m + h\, q^2) + \\
& + \lambda\,\big(b_\mathrm{I}\,\omega_\mathrm{u}^2 + b_\mathrm{m}\,\omega_\varphi^2 + (1 + h\, g)\, q\, p\big) + (\omega_\mathrm{u}^2\,\omega_\varphi^2 + g\, p^2)\Big] - \\
-\,\eta^2 \cdot \Big[& \lambda^3 \cdot b_\mathrm{I}\,\beta_{23} + \lambda^2\,(\omega_\varphi^2\,\beta_{24} + b_\mathrm{m}\, b_\mathrm{I}\,\beta_{23} + h\, q^2\,\beta_{13}) + \\
& + \lambda\,\big(b_\mathrm{I}\,\omega_\mathrm{u}^2\,\beta_{23} + b_\mathrm{m}\,\omega_\varphi^2\,\beta_{24} + q\, p\,(\beta_{14} + h\, g\,\beta_{13})\big) + (\omega_\mathrm{u}^2\,\omega_\varphi^2\,\beta_{24} + g\, p^2\,\beta_{14})\Big] = 0\ .
\end{aligned}
\tag{11}
$$

Although Eq. (11) is approximate only, it is obvious that higher degree of the stream velocity influence on the stability diagram is focused on the rotating component and its velocity: φ,

$\dot{\varphi}$, while influence of heaving component and its velocity corresponds rather with the neutral model in Eqs (1) except multiplicative constants α_{ij} modifying basic system parameters.

4. Models with indicial functions

Another way of modeling being built up in a time domain is based on a strategy of indicial functions. They have been introduced many years ago in the subject area of a wing aero-elasticity, see for instance [10], [11]. Applicability of this tool in a bluff bridge deck aero-elasticity has been demonstrated in several papers, e.g. [12], [13].

Indicial function can be roughly interpreted as function representing a generalized force at time t induced by a generalized unit displacement at time with the delay $\Delta t = t - \tau$. Consequently, the total force should be expressed in a form of a convolution integral. According to this definition the model is linear similarly like the previous one dealing with flutter derivatives.

Let us recall the system Eq. (6) and try to perform the inverse Laplace transform supposing the homogeneous initial conditions. At first a moderate modification respecting that $\kappa = -\mathrm{i}\,\lambda/\eta$ should be done:

$$\begin{aligned}
&\left[\lambda^2 + \lambda\, b_{\mathrm{m,c}} + \omega_{\mathrm{u,c}}^2 - \lambda^2 \left(\mathrm{i}\,\frac{b_{\mathrm{m,c}}}{\eta}\, A_{11}(\lambda) + \frac{\omega_{\mathrm{u,c}}^2}{\eta^2}\, A_{12}(\lambda)\right)\right] \cdot U(\lambda) + \\
&\qquad + \left[-\lambda\, h\, q_{\mathrm{c}} - p_{\mathrm{c}} + \lambda^2 \left(\mathrm{i}\,\frac{h\, q_{\mathrm{c}}}{\eta}\, A_{13}(\lambda) + \frac{p_{\mathrm{c}}}{\eta^2}\, A_{14}(\lambda)\right)\right] \cdot \Phi(\lambda) = 0\ , \\
&\left[\lambda\, q_{\mathrm{c}} + g\, p_{\mathrm{c}} - \lambda^2 \left(\mathrm{i}\,\frac{q_{\mathrm{c}}}{\eta}\, A_{21}(\lambda) + \frac{g\, p_{\mathrm{c}}}{\eta^2}\, A_{22}(\lambda)\right)\right] \cdot U(\lambda) + \\
&\qquad + \left[\lambda^2 + \lambda\, b_{\mathrm{I,c}} + \omega_{\varphi,\mathrm{c}}^2 - \lambda^2 \left(\frac{b_{\mathrm{I,c}}}{\eta}\, A_{23}(\lambda) + \frac{\omega_{\varphi,\mathrm{c}}^2}{\eta^2}\, A_{24}(\lambda)\right)\right] \cdot \Phi(\lambda) = 0\ .
\end{aligned} \tag{12}$$

With reference to properties of convolution integrals and their integral transform, we can express en equivalent of the system (6) in the time domain:

$$\begin{aligned}
&\ddot{u}(t) + b_{\mathrm{m,c}}\, \dot{u}(t) + \omega_{\mathrm{u,c}}^2\, u(t) - \int_{-\infty}^{t} [\Psi_{11}(t-\tau)] \cdot \ddot{u}(\tau) \cdot \mathrm{d}\tau - \\
&\quad - h\, q_{\mathrm{c}} \dot{\varphi}(t) - p_{\mathrm{c}}\, \varphi(t) + \int_{-\infty}^{t} [\Psi_{12}(t-\tau)] \cdot \ddot{\varphi}(\tau) \cdot \mathrm{d}\tau = 0\ , \\
&q_{\mathrm{c}}\, \dot{u}(t) + g\, p_{\mathrm{c}}\, u(t) - \int_{-\infty}^{t} [\Psi_{21}(t-\tau)] \cdot \ddot{u}(\tau) \cdot \mathrm{d}\tau + \\
&\quad + \ddot{\varphi}(t) + b_{\mathrm{I,c}}\, \dot{\varphi}(t) + \omega_{\varphi,c}^2\, \varphi(t) - \int_{-\infty}^{t} [\Psi_{22}(t-\tau)] \cdot \ddot{\varphi}(\tau) \cdot \mathrm{d}\tau = 0\ .
\end{aligned} \tag{13}$$

where following denomination has been introduced:

$$\begin{aligned}
\Psi_{11}(t) &= \mathrm{i}\,\frac{b_{\mathrm{m,c}}}{\eta}\, A_{11}^*(t) + \frac{\omega_{u,c}^2}{\eta^2} A_{12}^*(t)\ , &\quad \Psi_{12}(t) &= \mathrm{i}\,\frac{h\, q_{\mathrm{c}}}{\eta}\, A_{13}^*(t) + \frac{p_{\mathrm{c}}}{\eta^2}\, A_{14}^*(t)\ , \\
\Psi_{21}(t) &= \mathrm{i}\,\frac{q_{\mathrm{c}}}{\eta}\, A_{21}^*(t) + \frac{g\, p_{\mathrm{c}}}{\eta^2}\, A_{22}^*(t)\ , &\quad \Psi_{22}(t) &= \mathrm{i}\,\frac{b_{\mathrm{I,c}}}{\eta}\, A_{23}^*(t) + \frac{\omega_{\varphi,\mathrm{c}}^2}{\eta^2}\, A_{24}^*(t)\ .
\end{aligned} \tag{14}$$

Symbols $A^*_{ij}(t)$ in Eqs (14) mean Laplace originals corresponding with their transforms $A_{ij}(\lambda)$. It should be stressed that the system (13) is valid only for steady state processes, when the initial conditions regarding components $u(t)$, $\varphi(t)$ as well as functions $A^*_{ij}(t)$ can be considered vanishing or not influencing stationary part of processes under study.

Per partes operation applied on integrals leads due to zero initial conditions to modified version of Eqs. (13):

$$
\begin{aligned}
\ddot{u}(t) + b_{\mathrm{m,c}}\,\dot{u}(t) + \omega^2_{\mathrm{u,c}}\,u(t) - \int_{-\infty}^{t} [\dot{\Psi}_{11}(t-\tau)]\cdot\dot{u}(\tau)\cdot \mathrm{d}\tau\, - \\
-\,h\,q_{\mathrm{c}}\,\dot{\varphi}(t) - p_{\mathrm{c}}\,\varphi(t) + \int_{-\infty}^{t} [\dot{\Psi}_{12}(t-\tau)]\cdot\dot{\varphi}(\tau)\cdot \mathrm{d}\tau = 0\ , \\
q_{\mathrm{c}}\,\dot{u}(t) + g\,p_{\mathrm{c}}\,u(t) - \int_{-\infty}^{t} [\dot{\Psi}_{21}(t-\tau)]\cdot\dot{u}(\tau)\cdot \mathrm{d}\tau\, + \\
+\,\ddot{\varphi}(t) + b_{\mathrm{I,c}}\,\dot{\varphi}(t) + \omega^2_{\varphi,\mathrm{c}}\,\varphi(t) - \int_{-\infty}^{t} [\dot{\Psi}_{22}(t-\tau)]\cdot\dot{\varphi}(\tau)\cdot \mathrm{d}\tau = 0\ .
\end{aligned}
\tag{15}
$$

With reference to the literature, functions $\Psi_{ij}(t)$, see Eqs (14) or $\dot{\Psi}_{ij}(t)$ can be considered as indicial functions. They are four independent complex functions in time domain representing a counterpart of eight real flutter derivatives $A_{ij}(\kappa)$ outlined in the previous part, see table in (5).

System (9) as well as system (13) or (15) includes integral terms of the convolution type. Structure of the integrands in these systems implicates memory effects in the flow-structure interaction. Memory effects observed originate from inertia effects of surrounding streaming air, which react with a certain delay on a movement of the body. Nevertheless, the occurrence of the memory terms in the above integro-differential systems comes in principal from experimental measurements. Indeed, their essentials are either in experimentally determined flutter derivatives or in indicial functions.

Effects of aero-elastic interaction cannot be explicitly expressed in models we are dealing with, unless continuous air streaming field is taken into account. Nevertheless such approaches are formulated as FEM models and intended for pure numerical analysis. The aim of models discussed in this paper is different. They are focused on a qualitative analysis of important aero-elastic effects in particular on an investigation of individual stability limit shapes and their interaction considering a large variety of mechanical system and air stream parameters.

Let us compare systems (9) and (13) or (15). System (9) is suitable for a numerical integration and verification of results carried out in the frequency domain. Basically it represents an extension of the neutral model (1). When the flutter derivatives are known for a particular section, shape of respective functions, see Fig. 2, is usually simple. Therefore Taylor coefficients a_{ij}, b_{ij} or β_{ij} can be easily evaluated and immediately used. Frequency interval of the flutter derivatives examined in a wind tunnel is mostly rather limited and particular function values cannot be considered as very exact. So a simple operation preceding the solution itself reveals to be more stable and transparent than other more sophisticated procedures. On the other hand thorough qualitative (analytical) investigation using system (9) is not possible for obvious drawbacks in mathematics.

Regarding systems (13) or (15) indicial functions (14) could be evaluated using Laplace transformation of $A_{ij}(\kappa)$. Nevertheless the integral transformation of $A_{ij}(\kappa)$ on an actual level of knowledge would be unreliable and unstable. However, indicial functions can be identified experimentally in the time domain on a quite a long interval. Then either of the systems in question can be used without hesitation. Subsequent frequency analysis of results obtained can be compared with those obtained using the direct frequency analysis based on the flutter derivatives application.

5. Conclusions

Various types of double degree of freedom (DDOF) linear systems interacting with aero-elastic forces have been investigated and compared. The DDOF system under study describes inherent dynamic features of a slender prismatic beam attacked by a cross wind stream of a constant velocity (long bridge decks, guyed masts, towers, etc.). Relevant mathematical models of aero-elastic forces appearing in literature differ in principle by way of composition of aero-elastic forces. From this point of view following groups have been investigated: (i) neutral models – aero-elastic forces are introduced as suitable constants independent from excitation frequency and time; (ii) flutter derivatives – they respect the frequency dependence of aero-elastic forces; (iii) indicial functions – they are defined as kernels of convolution integrals formulating aero-elastic forces as functions of time. It succeeded to put all three groups together on one common basis to demonstrate their linkage. The platform of qualitative investigation of aero-elastic critical states in a frequency plain has been significantly expanded with respect to the stream velocity. Memory effects ruling in aero-elastic DDOF system have been substantiated and compared in frequency and time domains. The approach presented allows formulate more flexible models combining main aspects of all groups in question keeping the DDOF basis.

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