PROBABILISTIC EVALUATION OF CRACK BRIDGE PERFORMANCE IN FIBER REINFORCED COMPOSITES

Miroslav Vořechovský*, Václav Sadílek*, Rostislav Rypl**

The pullout response of a short fiber embedded in matrix depends on both the bond between the two materials and on the inclination angle and embedded length of the fiber. Fibers placed and oriented randomly in 3D space bridge matrix cracks with certain inclination angles and embedded lengths. With a pullout law available in analytical form, a statistical description of the force per fiber depending on crack opening can be evaluated for a uniformly loaded crack bridge in a short fiber reinforced composite by integrating the powers of all possible fiber responses multiplied by their probabilities of occurrence. This information is utilized to probabilistically evaluate the crack bridging force by computing the sum of a random number of random contributions; the random number of contributions to be summed is the number of bridging fibers, and the independent random contributions are the single fiber responses.

Keywords: *crack bridge, short fiber reinforced composite, ECC and random sum of random variables*

1. Introduction

Short fiber reinforced composite (FRC) is a relatively new material that extends the variety of applications for cementitious composites. At present, FRC is used e.g. for load-bearing structural elements in combination with continuous reinforcement. A common requirement placed on these composite materials is that in later loading stages they exhibit a ductile response which announces the total failure of a structural element. The fibers modify the macroscopic behavior of the composite and increase its compressive strength, tensile strength, stiffness, fracture toughness, impact resistance, etc. This behavior can be achieved by adding a certain proportion of short fibers to the matrix mixture, mostly in an amount ranging between 0.5–3 vol. %. Experience has shown that such an amount of fibers effectively bridges cracks in the matrix and forces it to develop fine crack patterns rather than a few widely opened cracks. Therefore, a large amount of energy can be dissipated prior to reaching the peak load. The reason is that the occurrence of macro cracks is postponed until later loading stages and the response is ductile as the fibers need a high amount of energy for debonding and pullout [6]. The most common materials used for short fibers are: steel, polyvinyl alcohol (PVA), alkali-resistant glass (AR-glass), aramid, carbon and various other materials.

In previous studies, homogenization approaches were developed [4, 5] after deriving the distribution functions for random orientation and position, of fibers bridging a discrete

* doc. Ing. M. Vořechovský, Ph.D., Ing. V. Sadílek, Institute of Structural Mechanics, Faculty of Civil Engineering, Brno University of Technology, Veveří 95, 602 00 Brno, Czech Republic

** Ing. R. Rypl, Institute of Structural Concrete, Faculty of Civil Engineering, RWTH Aachen University, Mies-van-der-Rohe-Straße 1, 52074 Aachen, Germany

planar crack. These approaches, however, do not take into account the variability of crack bridge performance, which is caused by both the scatter in the response of a single fiber and the scatter in the total number of bridging fibers. Furthermore, the statistical dependency between the distribution of the position and the inclination angle of short fibers was omitted in these works. The randomness of these quantities introduces variability to the crack bridge response and has a significant influence on the reliability of short fiber composites.

In order to statistically evaluate the random force transmitted by fibers bridging a discrete crack in a composite loaded in uniaxial tension, all sources of scatter have to be taken into account. In the following, two groups of sources of scatter in the response are distinguished: (i) the random response of a single fiber caused by the random inclination angle and random fiber position with respect to the crack plane; (ii) the random number of fibers bridging the crack plane. This is the focus of the present paper. The authors derive the statistical moments of force carried by fibers in a crack based on: (a) natural assumptions regarding the distribution of the random positions and orientations of fibers inside the volume of the FRC specimen; (b) the available information about the shape of the composite specimen and the shapes of the fibers and also the volume fraction of the fibers; and (c) the information on the dependence of the pullout force of a single fiber on its position and inclination with respect to the crack plane.

The assumptions in the present work are: (i) possible fiber collisions in the specimen volume are ignored due to the small volume fraction of the fibers, and therefore the position and inclination of the fibers are considered to be unaffected by other fibers; (ii) the fiber orientation is statistically homogeneous within the specimen volume, i.e. it does not depend on the flow direction of the casting process as described e.g. in [8] and [12]; (iii) local fiber clusters, which can appear during the specimen's production, are not taken into account.

The geometry of fibers (all having length ℓ) in a three-dimensional composite specimen (a solid rectangular block of volume $L_\mathrm{x} L_\mathrm{y} L_\mathrm{z}$) can be described by five variables (see Fig. 1a) that uniquely define the fiber center coordinates x, y and z and orientations φ_x and φ_y (i.e. the angles between the fiber longitudinal axis and x-axis and y-axis respectively; see Fig. 1c). These five geometrical descriptors are treated as basic random variables [10, 11].

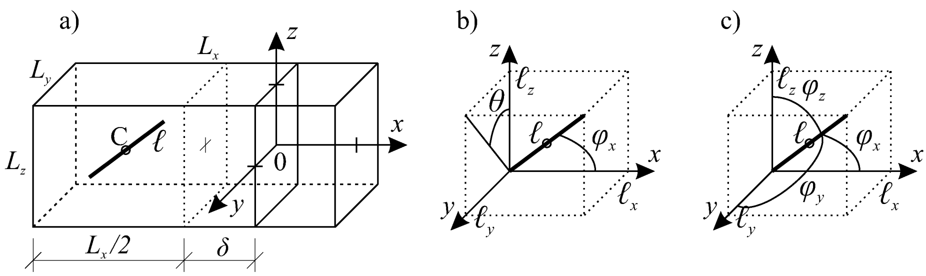

Fig.1: a) position of the coordinate system inside a specimen; b) and c) fiber orientation and position

2. Statistics for the number of fibers intersecting a plane

The overall force transmitted in a discrete crack in FRC by fibers depends significantly on the number of bridging fibers. This number, k, is a random variable. The statistical distribution describing the number of fibers bridging a crack can be calculated as follows:

The probability, p, of one fiber, randomly placed and oriented inside the specimen domain, being intersected by a (planar) crack plane is evaluated. The total number of fibers, n, inside the specimen and the probability p are used as entries for the Binomial distribution that gives the random number k of intersected fibers.

The probability p of a fiber intersecting a plane is obtained as a portion of the total unit probability of all possible fiber positions. This portion is calculated as an integral over all possible configurations with a suitable indicator function that signals the presence of a fiber within the intersecting plane. The derivation of the formulas for the probability $p \in \langle 0, 1\rangle$ has been presented in [10, 11]. For example, in the simple case of fibers' center positions homogeneously distributed throughout the whole specimen volume, and with all possible fiber orientations being equally probable, the probability of a fiber being intersected by a crack plane placed in the center of a specimen of length L equals:

$$p\,(\ell, L) = \frac{1}{2}\,\frac{\ell}{L}\ . \tag{1}$$

The above-mentioned papers also consider the case when the fibers must be wholly inside the specimen volume. This assumption modifies the joint distribution function of the five basic random variables.

The total number of fibers inside the composite, n, can be calculated from the fiber volume fraction v_{f}, the volume of a single fiber V_{f} and the total specimen volume V_{t}:

$$v_{\mathrm{f}} = \frac{n\,V_{\mathrm{f}}}{V_{\mathrm{t}}} = \frac{n\,A_{\mathrm{f}}\,\ell}{A_{\mathrm{c}}\,L} \quad \Rightarrow \quad n = v_{\mathrm{f}}\frac{A_{\mathrm{c}}\,L}{A_{\mathrm{f}}\,\ell} \tag{2}$$

where A_{f} is the cross-sectional area of a single fiber and A_{c} is the cross-sectional area of the specimen, $L_{\mathrm{y}} \times L_{\mathrm{z}}$.

The process of fibers being intersected by a plane in a specimen containing n fibers can be modeled as n independent Bernoulli trials with one fiber, each trial having a probability of success equal to p. The number of successes, k, follows the binomial distribution $\mathrm{Bi}(n, p)$. Therefore, the probability mass function for the random number k reads:

$$p_k = \binom{n}{k}\,p^k\,(1-p)^{(n-k)}\ . \tag{3}$$

The mean value and variance of the number of fibers intersecting a crack plane reads:

$$\begin{aligned} E[k] &= \mu_k = n\,p\ ,\\ D[k] &= \sigma_k^2 = n\,p\,(1-p)\ . \end{aligned} \tag{4}$$

The binomial distribution can be approximated by the Poisson distribution with the parameter $\lambda = n\,p$. Asymptotically, the distribution of k converges by means of the Moivre-Laplace limit theorem to the Gaussian distribution with the mean and variance given by Eq. (4) as the number of Bernoulli trials grows larger.

3. Random force of a fiber pulled out from a matrix

Assume a composite with a single fiber, randomly placed and oriented. This section shows how to calculate the random force carried by such a fiber when a planar crack occurs in the matrix. To demonstrate the application of the proposed probabilistic approach, a simple analytical model for the bridging force is formulated in this section. The statistical analysis of the force carried by a bridging fiber must be performed over all fibers that cross the crack plane, such as the one illustrated in Fig. 2 left. In other words, the embedded length ℓ_e must be positive. The embedded length is the shorter length of a bridging fiber found either to the left or right of the crack plane. The embedded length is calculated, given the fiber center coordinate x (normal distance from the crack plane) and orientation φ_x, as:

$$\ell_e = \max\left(0, \frac{\ell}{2} - \frac{|x|}{\cos\varphi_x}\right) . \tag{5}$$

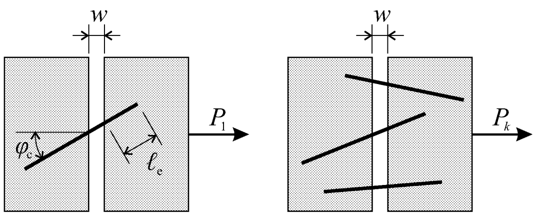

Fig.2: Left: one fiber bridging a crack plane; Right: multiple fibers bridging a crack plane (w – crack width, ℓ_e – embedded length, P_1 – pullout force of a single fiber, P_k – pullout force of k fibers)

In order to consider only fibers that contribute to the force in a crack bridge, it is convenient to find the joint density function of a pair of new random variables: the angle φ_c of fibers intersecting the crack plane and the corresponding embedded length ℓ_e (note that $\ell_e \le \ell/2$). For a statistically homogeneous distribution of fibers inside a specimen this joint density is just a product of the marginal densities of the two newly introduced random variables as derived in [11]:

$$f_{\ell_e,\varphi_c}(\ell_e,\varphi_c) = \begin{cases} \dfrac{2}{\ell}\sin(2\,\varphi_c) & \text{for} \quad \ell_e = \langle 0, \frac{\ell}{2}\rangle\ , \quad \varphi_c = \langle 0, \frac{\pi}{2}\rangle\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{6}$$

The simplified model for the pullout force used here assumes an ideally plastic constant bond law in all material points of the fiber-matrix interface. For a detailed derivation see [3]. Here, only the resulting formulas are given. In addition to [3], the parameter $\varphi_x = \varphi_c$, which stands for the inclination angle between the fiber's longitudinal axis and the crack plane normal, is included in the formula. The force $P_1(w)$ projected on the normal to the crack plane within the *debonding stage* (the ascending branch in Fig. 3 center) in terms of a pair of random variables, ℓ_e and φ_c, reads:

$$P_1^{\text{deb}}(w,\varphi_c) = \sqrt{E\,A_f\,\tau\,w}\,\exp(f\,\varphi_c) \quad \text{for} \quad 0 < w < w_{\text{pul}} \tag{7}$$

and the *pullout stage* is approximated by a constant force

$$P_1^{\text{pul}}(w,\ell_e,\varphi_c) = \tau\,\ell_e\,\exp(f\,\varphi_c) \quad \text{for} \quad w \ge w_{\text{pul}}\ . \tag{8}$$

In terms of random variables x and φ_x, the debonding force reads:

$$P_1^{\text{deb}}(w,\varphi_x) = \sqrt{E\,A_f\,\tau\,w}\ \exp(f\,\varphi_x) \quad \text{for} \quad 0 < w < w_{\text{pul}} \tag{9}$$

and the pullout force is

$$P_1^{\rm pul}(w,x,\varphi_{\rm x}) = \tau\,\ell_{\rm e}(x,\varphi_{\rm x})\exp(f\,\varphi_{\rm x}) \quad \text{for} \quad w \geq w_{\rm pul} \tag{10}$$

where E and $A_{\rm f}$ are the fiber's modulus of elasticity and cross-sectional area, respectively, $\tau = 2\,\pi\,r\,\tau_{\rm fr}$ is the constant frictional force per unit length defined as the frictional stress per unit area $\tau_{\rm fr}$ multiplied by the fiber's perimeter, f is the snubbing coefficient, w is the crack width and r is fiber radius. The two branches of P_1 intersect at the crack width $w_{\rm pul} = \tau\,\ell_{\rm e}^2/(E A_{\rm f})$.

The term $\exp(f\,\varphi)$ summarizes all influences of the inclination angle on the pullout force [5]. Figure 3 shows the bond law (on the left hand side) and the corresponding pullout response (center) of a short steel fiber with diameter $d = 0.3$ mm and length $\ell = 17$ mm; it is embedded in a cementitious matrix with $\ell_{\rm e} = \ell/2 = 8.5$ mm and $\varphi = 0$ rad.

For a random inclination angle and fiber position, the mean bridging force of a single fiber and its corresponding variance, respectively, read (function of crack opening w) :

$$\mathrm{E}[P_1(w)] = \mu_{P_1} = \int\limits_{\varphi_{\rm c}=0}^{\pi/2} \int\limits_{\ell_{\rm e}=0}^{\ell/2} P_1(w;\ell_{\rm e},\varphi_{\rm c})\, f_{\ell_{\rm e},\varphi_{\rm c}}(\ell_{\rm e},\varphi_{\rm c})\,{\rm d}\ell_{\rm e}\,{\rm d}\varphi_{\rm c} \tag{11}$$

$$\mathrm{D}[P_1(w)] = \sigma^2_{P_1} = \int\limits_{\varphi_{\rm c}=0}^{\pi/2} \int\limits_{\ell_{\rm e}=0}^{\ell/2} [P_1(w;\ell_{\rm e},\varphi_{\rm c}) - \mu_{P_1}]^2\, f_{\ell_{\rm e},\varphi_{\rm c}}(\ell_{\rm e},\varphi_{\rm c})\,{\rm d}\ell_{\rm e}\,{\rm d}\varphi_{\rm c}\ . \tag{12}$$

The same results can be obtained by integrating over the domain of the original random variables x and $\varphi_{\rm x}$, which have the joint distribution function $f_{{\rm x}\varphi_{\rm x}}(x,\varphi_{\rm x})$ derived in [11], and by using the conditional probability density (dividing the joint probability density function by the probability p of a fiber intersecting the crack plane as shown above). The integration region exhausts all possible fiber coordinates x and orientations $\varphi_{\rm x}$:

$$\mathrm{E}[P_1(w)] = \mu_{P_1} = \int\limits_{\varphi_{\rm x}=0}^{\pi/2} \int\limits_{x=-L_{\rm x}/2}^{L_{\rm x}/2} P_1(w;x,\varphi_{\rm x})\, \frac{f_{{\rm x}\varphi_{\rm x}}(x,\varphi_{\rm x})}{p}\, \mathbf{1}_{\rm A}(x)\,{\rm d}x\,{\rm d}\varphi_{\rm x}\ , \tag{13}$$

$$\mathrm{D}[P_1(w)] = \sigma^2_{P_1} = \int\limits_{\varphi_x=0}^{\pi/2} \int\limits_{x=-L_{\rm x}/2}^{L_{\rm x}/2} [P_1(w;x,\varphi_{\rm x}) - \mu_{P_1}]^2\, \frac{f_{{\rm x}\varphi_{\rm x}}(x,\varphi_{\rm x})}{p}\, \mathbf{1}_{\rm A}(x)\,{\rm d}x\,{\rm d}\varphi_{\rm x}\ , \tag{14}$$

where $\mathbf{1}_{\rm A}(x)$ is the indicator function (defined in [10, 11]) that indicates the membership of an element in a subset A of X, which has the value 1 for all elements of A (intersection) and the value 0 for all elements of X not in A. The following indicator function can be used :

$$\mathbf{1}_{\rm A}(x) = H\left(\frac{1}{2}\,\ell\,\cos(\varphi_{\rm x}) - |x|\right) \tag{15}$$

where H is the Heaviside (unit step) function. Fig. 3 right shows the mean force $P_1(w)$ and the minus and plus one standard deviation bands for it. The term 'double-sided pullout' means that until the fiber completely debonds along the shorter of its embedded lengths,

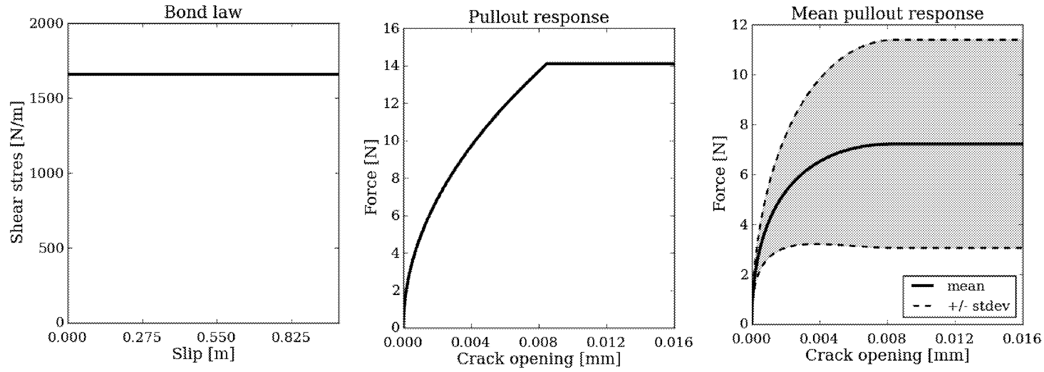

Fig.3: Left: bond law; Center: double-sided pullout of a short fiber $\ell_e = \ell/2 = 8.5$ mm and $\varphi = 0$ rad; Right: mean response of a single fiber bridging a crack with ℓ_e and φ_c as random variables; computed with the following parameters: $d = 0.3$ mm, $E = 200$ GPa, $\tau_{fr} = 1.76$ MPa, $\tau = 1.659$ kN/m, $f = 0.03$

debonding is assumed to propagate symmetrically at both sides of the crack bridge. After that, only the shorter embedded length is pulled out of the matrix.

4. The mean value and variance of a crack-bridging force

The expression for a random bridging force carried by a single fiber will now be exploited for the calculation of the total force, P_k, carried by all fibers within a single crack bridge. Both the number k of fibers bridging a crack and the contributions P_1 of single fibers are random variables. Due to the parallel coupling of the fibers in a crack, the total force P_k carried by all bridging fibers is the sum of k contributions of independent random forces P_1 :

$$P_k(w) = \sum_{i=1}^{k} P_{1,i}(w) \ . \tag{16}$$

In other words, to calculate the total force, a random number of random contributions, which are independent and identically distributed, have to be summed. Note that it would be a mistake to calculate the total force as a product of the random variables k and P_1. By virtue of the central limit theorem, the distribution of the sum P_k tends to the Gaussian distribution [7, 1]. The only parameters to be determined are the mean value and the variance of P_k. To obtain these, the law of total expectation (the tower rule) and the law of total variance (variance decomposition formula) must be applied:

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}[\mathrm{E}(X|Y)] \ , \\ \mathrm{D}[X] &= \mathrm{E}[\mathrm{D}(X|Y)] + \mathrm{D}[\mathrm{E}(X|Y)] \ . \end{aligned} \tag{17}$$

In the present application of the theorems, $X = P_k$ and $Y = k$:

$$\mathrm{E}\left[\sum_{i=1}^{k} P_{1,i}\right] = \mathrm{E}\left[\mathrm{E}\left(\sum_{i=1}^{k} P_{1,i}|k\right)\right] = \mathrm{E}\left[\sum_{i=1}^{k} P_{1,i}\right] = \mathrm{E}[k \cdot \mathrm{E}(P_1)] \ , \tag{18}$$

$$\begin{aligned} \mathrm{D}\left[\sum_{i=1}^{k} P_{1,i}\right] &= \mathrm{E}\left[\mathrm{D}\left(\sum_{i=1}^{k} P_{1,i}|k\right)\right] + \mathrm{D}\left[\mathrm{E}\left(\sum_{i=1}^{k} P_{1,i}|k\right)\right] = \\ &= \mathrm{E}\left[k \cdot \mathrm{D}\left(\sum_{i=1}^{k} P_{1,i}\right)\right] + \mathrm{D}[k \cdot \mathrm{E}(P_1)] \ . \end{aligned} \tag{19}$$

By exploiting the independence of k from P_1, the final result can be written as:

$$\begin{aligned} \mathrm{E}[P_k(w)] &= \mathrm{E}[k]\,\mathrm{E}[P_1(w)]\ , \\ \mathrm{D}[P_k(w)] &= \mathrm{E}[k]\,\mathrm{D}[P_1(w)] + \mathrm{D}[k]\,[\mathrm{E}[P_1(w)]]^2\ . \end{aligned} \tag{20}$$

The coefficient of variation of the total force reads:

$$\mathrm{CoV}_{P_k} = \frac{\sqrt{\mathrm{D}[P_k]}}{\mathrm{E}[P_k]}\ . \tag{21}$$

The authors note that it would *not* be correct to evaluate the composite performance as the homogenized (average) response of a single crack bridge because the variability of P_k cannot be neglected. This variance has influence especially when multiple cracking occurs, which is the desired case e.g. for ECC. The overall composite strength is determined by the weakest crack bridge in the series of cracks. By using the weakest-link model and the associated extreme value theory of independent identically distributed variables P_k, the composite with N parallel cracks (serially couplet crack bridges) has a random strength equal to:

$$P = F_k^{-1}\left(1 - \sqrt[N]{1 - p_\mathrm{f}}\right) \tag{22}$$

where p_f is the probability and F_k is the cumulative distribution function of the strength of one crack bridge (normal distribution with the mean and variance in Eq. (20)). To obtain the median strength of the whole composite, for example, p_f would be equal to 0.5. The authors remark that the variability of the force in a single crack bridge is completely disregarded in previous works [4, 5], the authors of which consider the total force P_k deterministically as a k-multiple of P_1, where k is the average number of fibers intersected by a crack plane and P_1 is considered in terms of its mean value. The following section illustrates that the variability in k and also the variability in P_1 and P_k cannot be ignored.

5. Numerical example

In order to demonstrate the above results on real-world data, a numerical example is presented. We consider a $10 \times 4 \times 4$ cm composite specimen, cut from a larger volume of FRC. This manner of specimen preparation allows the assumption of a uniform distribution of fibers to be satisfied. For specimens manufactured in such a way that one can expect some wall effects, the distributions of angles and fiber centers would have to be modified accordingly. The specimen is loaded in uniaxial tension along the length of $L_\mathrm{x} = 0.1$ m. The cross-sectional area equals $A_\mathrm{c} = 1.6\times10^{-3}\,\mathrm{m}^2$. The specimen is reinforced by steel fibers of length $\ell = 0.017$ m and diameter $d = 0.3$ mm. The cross-sectional area of a single fiber equals $A_\mathrm{f} = 7.069\times10^{-8}\,\mathrm{m}^2$. The considered volume fraction of fibers (Eq. 2) is $v_\mathrm{f} = 1.5\,\%$. The modulus of elasticity of the fibers is $E = 200$ GPa and $\tau_\mathrm{fr} = 1.76$ MPa, see Fig. 3.

For these input parameters, the total number of fibers in the composite specimen is $n = 1997$ (Eq. 2) and the probability that one fiber intersects a plane perpendicular to the longitudinal axis is $p = p(\ell; L) = 0.085$ (Eq. 1). The mean value of the number k of fibers intersecting a crack plane is $\mathrm{E}[k] = 169.745$ and the variance is $\mathrm{D}[k] = 155.32$ (Eq. 4). Hence, the standard deviation equals $\mathrm{Std}(k) = 12.46$ and the coefficient of variation $\mathrm{CoV}(k) = 7.34\,\%$.

The mean value of the peak force (pull out stage) carried by one fiber crossing a crack plane is $\mathrm{E}[P_1] = 7.218\,\mathrm{N}$ and the variance is $\mathrm{D}[P_1] = 17.375\,\mathrm{N}^2$ (Eq. 11). The standard deviation then equals $\mathrm{Std}(P_1) = 4.16\,\mathrm{N}$ and the coefficient of variation $\mathrm{CoV}(P_1) = 57.7\,\%$. Note that no consideration is given to the fact that a certain proportion of the fibers that are close to the contour of the cross section may not contribute fully to the pullout force.

By the application of Eq. (20), the mean value of the total force carried by fibers in one crack bridge is $\mathrm{E}[P_k] = 1225.25\,\mathrm{N}$; the variance is $\mathrm{D}[P_k] = 11041.63\,\mathrm{N}^2$. Therefore, the standard deviation equals $\mathrm{Std}(P_k) = 105.08\,\mathrm{N}$ and the coefficient of variation is $\mathrm{CoV}(P_k) = 8.58\,\%$.

The authors remark that it would also be erroneous to ignore the variance originating from the randomness in the number of bridging fibers k (yet still considering the variance of P_1). To illustrate that error, we will now show how the standard deviation (and coefficient of variation) is underestimated by such a simplification. The standard deviation of P_k would be calculated as $\mathrm{Std}(P_1)\sqrt{k} = 54.31\,\mathrm{N}$ [2, 9] and the coefficient of variation would then drop to 4.43 %, which is approximately one half of the correct value.

a)

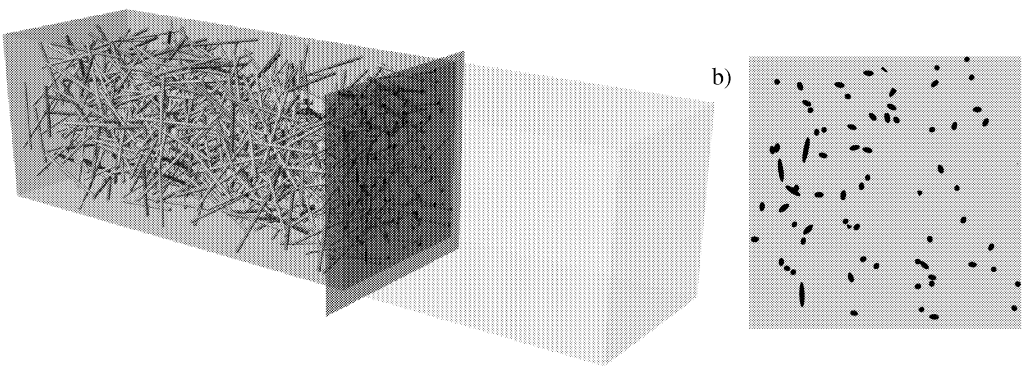

Fig.4: a) fibers randomly distributed inside a 3D volume and cut by a plane; b) view of the cutting plane (crack)

6. Conclusions

This paper studies the random strength of composites reinforced by randomly oriented short fibers. It is assumed that the position and orientation of fibers are random and homogeneous in the composite specimens. By exploiting the information about the volume fraction of fibers, the geometry of the fibers and the specimen, and the material (interface) properties, an evaluation of (i) the random force in one fiber bridging a crack, and (ii) the random total force of the crack bridge is obtained. These data are further used in the estimation of the tensile strength of the whole composite with multiple cracks in series.

Acknowledgement

This research was conducted with the financial support of the Czech Science Foundation under project no. 13-19416J. The support is gratefully acknowledged. Additionally, the work was supported by project no. FAST-S-13-1889.

References

[1] Blum J.R., Hanson D.L., Rosenblatt J.I.: On the central limit theorem for the sum of a random number of independent random variables, Probability Theory and Related Fields 1 (4), 389–393, 1963

[2] Chudoba R., Vořechovský M., Konrad M.: Stochastic modeling of multifilament yarns I: Random properties within the cross-section and size effect, International Journal of Solids and Structures (Elsevier) 43 (3–4), 413–434, 2006

[3] Konrad M.: Effect of multifilament yarn crack bridging on uniaxial behavior of textile reinforced concrete, Ph.D. thesis, RWTH Aachen, 2008

[4] Li V.C., Wang Y., Backer S.: A micromechanical model of tensionsoftening and bridging toughening of short random fiber reinforced brittle matrix composites, Journal of the Mechanics and Physics of Solids 39 (5), 607–625, 1991

[5] Lin Z., Kanda T., Li V.C.: On interface property characterization and performance of fiber-reinforced cementitious composites, Journal of Concrete Science and Engineering 1, 173–184, 1999

[6] Pfyl T.: Tragverhalten von stahlfaserbeton, Ph.D. thesis, ETH Zürich, PhD thesis ETH Nr. 15005, 2003

[7] Rényi A.: On the central limit theorem for the sum of a random number of independent random variables, Acta Mathematica Academiae Scientiarum Hungaricae 11 (1–2), 97–102, 1960

[8] Stähli P., Custer R., van Mier J.G.M.: On flow properties, fibre distribution, fibre orientation and flexural behaviour of FRC, Materials and Structures 41 (1), 189–196, 2007

[9] Vořechovský M., Chudoba R.: Stochastic modeling of multi-filament yarns: II. Random properties over the length and size effect, International Journal of Solids and Structures (Elsevier) 43 (3–4), 435–458, 2006

[10] Vořechovský M., Sadílek V.: Náhodný počet vláken protínajících rovinu trhliny ve vláknobetonu, In: Ledererová J., Svoboda M., Drotterová J. (Eds.), Ecology and new building materials and products, XV. International Conference of the Research Institute of Building Materials, held in Telč, Czech Republic, Research Institute of Building Materials, Telč, Czech Republic, pp. 233–236, full papers on CD-ROM, 2011

[11] Vořechovský M., Sadílek V.: Probabilistic determination of the number of fibers bridging a crack in short fiber reinforced composites, Bulletin of Applied Mechanics 7 (25), 21–25, 2011 URL http://bulletin-am.cz/index.php/vam/article/view/190

[12] Zhou K., Lin J.: Three dimensional fiber orientation distribution in two dimensional flows, Fibers and Polymers 9 (1), 39–47, 2008

Received in editor's office: July 7, 2011
Approved for publishing: October 10, 2012