NONLINEAR VIBRATION OF 2D VISCOELASTIC PLATE SUBJECTED TO TANGENTIAL FOLLOWER FORCE

Armand Robinson Mouafo Teifouet*

In this paper, the problem of nonlinear viscoelastic rectangular thin plate subjected to tangential follower force is examined. The nonlinear strain-displacement relation is used to express non-linearity. After obtaining the equilibrium equation of the system in Laplace domain and performing the Laplace inverse transformation, the nonlinear differential equation of plate constituted by Kelvin-Voigt model and subjected to tangential follower force in time domain is obtained. Multi-scales method is firstly used to solve the governing equation, and the influence of the initial amplitude on the nonlinear to linear frequency ratio is studied. Secondly, the differential quadrature method (DQM) is employed to confirm the obtained results.

Keywords: *nonlinear vibration, multi-scales method, differential quadrature method*

1. Introduction

In order to reduce the weight of structures, thin plates are currently used in many industrial domains such as aeronautics, automotive design or offshore structures. These structures are sometimes forced to vibrate with large amplitude when subjected to tangential follower forces, creating significant geometric nonlinearities. These nonlinearities usually create instability phenomenon. To reduce this instability, it is important to consider the viscoelasticity behavior of the thin rectangular plate. The main difficulty of such problem is firstly the nonlinearity consideration which creates the complex phenomenon as bifurcations and secondly the damping which creates the complex eigenvalues.

It should be noted that, nonlinear oscillation of structure has been studied for many years as observed in the work of Chu et al. [1], and Nayfey et al. [2]. Specifically, many papers have been published in the domain of nonlinear vibration of plates such as in the work of Amabili et al. [3] and Amabili [4]. Many numerical methods are sometimes used to confirm the results given by the analytical method, like differential quadrature [5], as can be seen in the work of Tang et al. [6], Yusheng et al. [7], Hu et al. [8], Chen et al. [9], Chen et al. [10] and Zhong et al. [11]. Considering that the viscoelastic property of the material is very important, it is desirable to investigate nonlinear vibration of viscoelastic plate. Amount of research investigation have been performed on the nonlinear vibration of viscoelastic rectangular plate subjected to external forces. Kim et al. [12] studied the nonlinear vibration of viscoelastic laminated composite plates. They investigated the geometric nonlinearity on the dissipative effect of the material, by parameterizing the former as amplitude-thickness ratio and the latter as relaxation parameters. Aboudi [13] did the postbuckling analysis of viscoelastic laminated plates using higher-order theory. The results based on different theories of plates

* A. R. Mouafo Teifouet, Laboratoire de Mécanique et de Modélisation des Systèmes Physiques L2MSP, Department of Physics, Faculty of Sciences, University of Dschang, P.O.Box 67 Dschang, Cameroon

were compared. G. Cederbaum [14] studied the parametric excitation of viscoelastic plate. The excitation considered was periodic in-plane load and the material behaviour was given in terms of Boltzmann superposition principle. The influence of static and dynamic part of loads on the stability region was investigated. Cederbaum et al. [15] examined the dynamic instability of shear deformable viscoelastic laminated plates. The harmonic in-plane excitation is used and the dynamic stability of viscoelastic plate is discussed using the Lyapunov exponent concept. Touati et al. [16] employed the Liapunov exponent to study numerically the influence of the various parameter involved on stability of nonlinear viscoelastic plates, Aboudi et al. [17] studied the dynamic stability analysis of viscoelastic plates by Lyapunov exponents. Recently in 2005, Chen et al. [18] studied the instability of nonlinear viscoelastic plates, they used the Leadermann nonlinear constitutive relation of viscoelasticity, to derive a nonlinear integro-differential equation by Galerkin method. They employed the averaging method to establish the condition of instability. Numerical results were compared with the analytical one. Esmailzadeh et al. [19] studied the nonlinear oscillation of viscoelastic rectangular plates by assuming the Kelvin-Voigt constitutive model. Daya et al. [20] used the finite element method to study the nonlinear vibration of sandwich viscoelastic plate. They used the industrial test and existing result to compare the obtained results. Touze et al. [21] examined the nonlinear normal modes for damped geometrically nonlinear system and made the application to reduced-order modeling of harmonically forced structures. Abdoum et al. [22] examined the forced harmonic response of viscoelastic structures by an asymptotic numerical method in 2009.

Although many papers have been published in the domain of viscoelastic plate subjected to external excitation, the works which take into consideration 3D constitutive viscoelastic relation are only few. The aim of this paper is to use the 3D constitutive relation and the non-linear strain-displacement relation to establish the nonlinear differential equation of viscoelastic rectangular plate subjected to tangential follower force. The multi-scale method is firstly used to solve the obtained non-linear equation. Secondly we examine the effect of time on nonlinear to linear frequency ratio. The effect of initial amplitude on frequency ratio is also examined for different aspect ratio. Thirdly, the differential quadrature method is used to confirm analytical results.

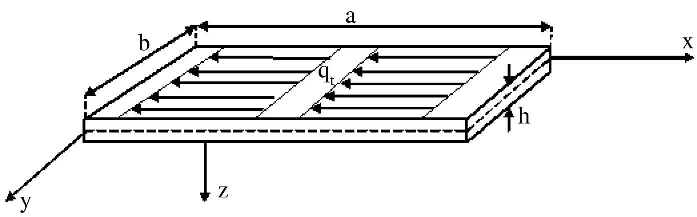

Fig.1: Schematic representation of viscoelastic plate subjected to tangential follower force q_t

2. Problem formulation

The system under investigation as presented in Fig. 1 consists of viscoelastic rectangular plate, subjected to tangential follower force q_t, in x direction. This plate has length a, width b, and thickness h in the x, y, and z directions, respectively. The material density is ρ. The general 3D viscoelastic differential constitutive relation is as follows [23].

$$\begin{aligned} P'\,s_{ij} &= Q'\,e_{ij}\ , \\ P''\,\sigma_{ii} &= Q''\,\varepsilon_{ii}\ . \end{aligned} \tag{1}$$

The Laplace transformation of equation (1) is:

$$\begin{aligned} \bar{P}'\,s_{ij} &= \bar{Q}'\,\bar{e}_{ij}\ , \\ \bar{P}''\,\sigma_{ii} &= \bar{Q}''\,\bar{\varepsilon}_{ii}\ , \end{aligned} \tag{2}$$

where s_{ij}, and e_{ij} are deviatoric tensors of stress and strain, σ_{ii} and ε_{ii} are spherical tensors of stress and strain, and the operators

$$P' = \sum_{k=0}^{l} p'_k\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\ ,\quad Q' = \sum_{k=0}^{r} q'_k\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\ ,\quad P'' = \sum_{k=0}^{l_1} p'_k\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\ ,\quad Q'' = \sum_{k=0}^{r_1} q'_k\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\ ,$$
$$p'_k\ ,\quad q'_k\ ,\quad p'_k\ ,\quad q'_k$$

depending on the properties of the material, and the bar on every function denotes the Laplace transformation. For the plane stress problem, the constitutive equations of linear viscoelastic material in the Laplace domain [24] are

$$\begin{aligned} \bar{P}'\,(\bar{P}'\,\bar{Q}'' + 2\,\bar{Q}'\,\bar{P}'')\,\bar{\sigma}_\mathrm{x} &= \bar{Q}'\,(2\,\bar{P}'\,\bar{Q}'' + \bar{Q}'\,\bar{P}'')\,\bar{\varepsilon}_\mathrm{x} + \bar{Q}'\,(\bar{P}'\,\bar{Q}'' - \bar{Q}'\,\bar{P}'')\,\bar{\varepsilon}_\mathrm{y}\ , \\ \bar{P}'\,(\bar{P}'\,\bar{Q}'' + 2\,\bar{Q}'\,\bar{P}'')\,\bar{\sigma}_\mathrm{y} &= \bar{Q}'\,(\bar{P}'\,\bar{Q}'' - \bar{Q}'\,\bar{P}'')\,\bar{\varepsilon}_\mathrm{x} + \bar{Q}'\,(2\,\bar{P}'\,\bar{Q}'' + \bar{Q}'\,\bar{P}'')\,\bar{\varepsilon}_\mathrm{y}\ , \\ \bar{P}'\,\bar{\tau}_\mathrm{xy} &= \bar{Q}'\,\bar{\varepsilon}_\mathrm{xy}\ , \end{aligned} \tag{3}$$

where $\bar{\sigma}_\mathrm{x}$, $\bar{\sigma}_\mathrm{y}$, $\bar{\tau}_\mathrm{xy}$, $\bar{\varepsilon}_\mathrm{x}$, $\bar{\varepsilon}_\mathrm{y}$, $\bar{\varepsilon}_\mathrm{xy}$ are the Laplace transforms of σ_x, σ_y, τ_xy, ε_x, ε_y, ε_xy respectively, $\bar{P}'$, $\bar{Q}'$, $\bar{P}''$, $\bar{Q}''$ are the Laplace transforms of differential operators of P', Q', P'', Q'', respectively. Introducing operators

$$\begin{aligned} \bar{P}_0 &= \bar{P}'\,(\bar{P}'\,\bar{Q}'' + 2\,\bar{Q}'\,\bar{P}'')\ , \\ \bar{Q}_0 &= \bar{Q}'\,(2\,\bar{P}'\,\bar{Q}'' + \bar{Q}'\,\bar{P}'')\ , \\ \bar{Q}_1 &= \bar{Q}'\,(\bar{P}'\,\bar{Q}'' - \bar{Q}'\,\bar{P}'')\ . \end{aligned} \tag{4}$$

Equation (3) can be simplified as

$$\begin{aligned} \bar{P}_0\,\bar{\sigma}_\mathrm{x} &= \bar{Q}_0\,\bar{\varepsilon}_\mathrm{x} + \bar{Q}_1\,\bar{\varepsilon}_\mathrm{y}\ , \\ \bar{P}_0\,\bar{\sigma}_\mathrm{y} &= \bar{Q}_1\,\bar{\varepsilon}_\mathrm{x} + \bar{Q}_0\,\bar{\varepsilon}_\mathrm{y}\ , \\ \bar{P}'\,\bar{\tau}_\mathrm{xy} &= \bar{Q}'\,\bar{\varepsilon}_\mathrm{xy}\ . \end{aligned} \tag{5}$$

The bending and M_x, M_y, twisting moment M_xy, M_yx per unit length of the plate are

$$\begin{aligned} M_\mathrm{x} &= \int_{\frac{-h}{2}}^{\frac{h}{2}} z\,\sigma_\mathrm{x}\,\mathrm{d}z\ , \qquad M_\mathrm{y} = \int_{\frac{-h}{2}}^{\frac{h}{2}} z\,\sigma_\mathrm{y}\,\mathrm{d}z\ , \\ M_\mathrm{xy} &= \int_{\frac{-h}{2}}^{\frac{h}{2}} z\,\tau_\mathrm{xy}\,\mathrm{d}z\ , \qquad M_\mathrm{yx} = \int_{\frac{-h}{2}}^{\frac{h}{2}} z\,\tau_\mathrm{yx}\,\mathrm{d}z\ . \end{aligned} \tag{6}$$

The membrane stress resultants per unit length N_{x}, N_{y} and N_{xy}, are respectively

$$
\begin{gathered}
N_{\mathrm{x}} = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{\mathrm{x}}\,\mathrm{d}z\ , \qquad N_{\mathrm{y}} = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{\mathrm{y}}\,\mathrm{d}z\ , \\
N_{\mathrm{xy}} = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} \tau_{\mathrm{xy}}\,\mathrm{d}z\ , \qquad N_{\mathrm{yx}} = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} \tau_{\mathrm{yx}}\,\mathrm{d}z\ .
\end{gathered} \tag{7}
$$

Applying the operators $\bar{P}_0$ and $\bar{P}'$ to the Laplace transformation results of equations (6) and (7) we have respectively for moment and membrane stress per unit length

$$
\begin{gathered}
\bar{P}_0(\bar{M}_{\mathrm{x}}) = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} z\,\bar{P}_0(\bar{\sigma}_{\mathrm{x}})\,\mathrm{d}z\ , \qquad \bar{P}_0(\bar{M}_{\mathrm{y}}) = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} z\,\bar{P}_0(\bar{\sigma}_{\mathrm{y}})\,\mathrm{d}z\ , \\
\bar{P}'(\bar{M}_{\mathrm{xy}}) = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} z\,\bar{P}'(\bar{\tau}_{\mathrm{xy}})\,\mathrm{d}z
\end{gathered} \tag{8}
$$

and

$$
\begin{gathered}
\bar{P}_0(\bar{N}_{\mathrm{x}}) = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} \bar{P}_0(\bar{\sigma}_{\mathrm{x}})\,\mathrm{d}z\ , \qquad \bar{P}_0(\bar{N}_{\mathrm{y}}) = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} \bar{P}_0(\bar{\sigma}_{\mathrm{y}})\,\mathrm{d}z\ , \\
\bar{P}'(\bar{N}_{\mathrm{xy}}) = \int\limits_{\frac{-h}{2}}^{\frac{h}{2}} \bar{P}'(\bar{\tau}_{\mathrm{xy}})\,\mathrm{d}z\ .
\end{gathered} \tag{9}
$$

The strain-displacement relation in this paper is for classical thin plate including the nonlinearity due to the midline strething. This relation is:

$$
\begin{gathered}
\varepsilon_{\mathrm{x}} = -z\,\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\ , \qquad \varepsilon_{\mathrm{y}} = -z\,\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\ , \\
\varepsilon_{\mathrm{xy}} = \frac{\gamma_{\mathrm{xy}}}{2} = -z\,\frac{\partial^2 w}{\partial x\,\partial y} + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\ .
\end{gathered} \tag{10}
$$

After substituting equation (5) into equation (8) and using the strain-displacement relation (10) we obtain the relations between moment and the Laplace transformation of deflection:

$$
\begin{aligned}
\bar{P}_0(\bar{M}_{\mathrm{x}}) &= -\frac{h^3}{12}\left[\bar{Q}_0\,\frac{\partial^2 \bar{w}}{\partial x^2} + \bar{Q}_1\,\frac{\partial^2 \bar{w}}{\partial y^2}\right]\ , \\
\bar{P}_0(\bar{M}_{\mathrm{y}}) &= -\frac{h^3}{12}\left[\bar{Q}_0\,\frac{\partial^2 \bar{w}}{\partial y^2} + \bar{Q}_1\,\frac{\partial^2 \bar{w}}{\partial x^2}\right]\ , \\
\bar{P}'(\bar{M}_{\mathrm{xy}}) &= -\frac{h^3}{12}\,\bar{Q}'\,\frac{\partial^2 \bar{w}}{\partial x\,\partial y}\ .
\end{aligned} \tag{11}
$$

After substituting equation (7) into equation (9) and using The strain-displacement relation (11), we obtain the relations between membrane stress and the Laplace transformation of deflection:

$$
\begin{aligned}
\bar{P}_0(\bar{N}_\mathrm{x}) &= \frac{h}{2}\left[\bar{Q}_0\left(\frac{\partial\bar{w}}{\partial x}\right)^2+\bar{Q}_1\left(\frac{\partial\bar{w}}{\partial y}\right)^2\right] ,\\
\bar{P}_0(\bar{N}_\mathrm{y}) &= \frac{h}{2}\left[\bar{Q}_1\left(\frac{\partial\bar{w}}{\partial y}\right)^2+\bar{Q}_0\left(\frac{\partial\bar{w}}{\partial x}\right)^2\right] ,\\
\bar{P}'(\bar{N}_\mathrm{xy}) &= \frac{h}{2}\,\bar{Q}'\,\frac{\partial\bar{w}}{\partial x}\,\frac{\partial\bar{w}}{\partial y} .
\end{aligned}
\tag{12}
$$

The equilibrium equation of thin plate, subjected to tangential follower force q_t is:

$$
\begin{aligned}
&\frac{\partial^2 M_\mathrm{x}}{\partial x^2}+2\,\frac{\partial^2 M_\mathrm{xy}}{\partial x\partial y}+\frac{\partial^2 M_\mathrm{y}}{\partial y^2}+N_\mathrm{x}\,\frac{\partial^2 w}{\partial x^2}+2\,N_\mathrm{xy}\,\frac{\partial^2 w}{\partial x\partial y}+N_\mathrm{y}\,\frac{\partial^2 w}{\partial x^2}-\\
&\qquad -q_\mathrm{t}\,(a-x)\,\frac{\partial^2 w}{\partial x^2}-\rho\,h\,\frac{\partial^2 w}{\partial t^2}=0 .
\end{aligned}
\tag{13}
$$

Applying $\bar{P}_0\,\bar{P}'$ to Laplace transformation of equation (13), and assuming that the partial derivative is continuous, the resulting equation can be rewritten as:

$$
\begin{aligned}
&\bar{P}'\,\frac{\partial^2\left(\bar{P}_0\,\bar{M}_\mathrm{x}\right)}{\partial x^2}+2\,\bar{P}_0\,\frac{\partial^2\left(\bar{P}'\bar{M}_\mathrm{xy}\right)}{\partial x\,\partial y}+\bar{P}'\,\frac{\partial^2\left(\bar{P}_0\,\bar{M}_\mathrm{y}\right)}{\partial y^2}+\bar{P}'(\bar{P}_0\,\bar{N}_\mathrm{x})\,\frac{\partial^2\bar{w}}{\partial x^2}+\\
&+2\,\bar{P}_0(\bar{P}'\bar{N}_\mathrm{xy})\,\frac{\partial^2\bar{w}}{\partial x\,\partial y}+\bar{P}'(\bar{P}_0\,\bar{N}_\mathrm{y})\,\frac{\partial^2\bar{w}}{\partial y^2}-\bar{P}_0\,\bar{P}' q_\mathrm{t}\,(a-x)\,\frac{\partial^2 w}{\partial x^2}-\bar{P}_0\,\bar{P}'\rho\,h\,\frac{\partial^2\bar{w}}{\partial t^2}=0 .
\end{aligned}
\tag{14}
$$

Assuming that the material of the plate obeys elastic behavior in dilatation and Kelvin-Voigt law of distortion, the constitutive equations are:

$$
\begin{aligned}
s_{ij} &= 2\,G\,e_{ij}+2\,\eta\,\dot{e}_{ij} ,\\
\sigma_{ii} &= 3\,K\,\varepsilon_{ii} ,
\end{aligned}
\tag{15}
$$

where K, η, G are bulk elastic modulus, viscoelastic coefficient, and shear elastic modulus respectively. s_{ij} and σ_{ij} are respectively deviatoric tensor of stress and strain. s_{ii} and σ_{ii} are spherical tensor of strain and stress respectively. Performing the Laplace transformation on equation (15) and comparing the result with equation (2) we get

$$
\begin{aligned}
\bar{P}' &= 1 , &\qquad \bar{Q}' &= 2\,G+2\,\eta\,\frac{\partial}{\partial t} ,\\
\bar{P}'' &= 1 , &\qquad \bar{Q}'' &= 3\,K .
\end{aligned}
\tag{16}
$$

After substituting equations (11) and (12) into equation (15), considering equation (16), and carrying out the inverse Laplace transformation, a nonlinear differential equation of viscoelastic rectangular plate subjected to tangential follower force in time domain is:

$$
\begin{aligned}
&\frac{h^3}{12}\left(A_3+A_4\,\frac{\partial}{\partial t}+A_5\,\frac{\partial^2}{\partial t^2}\right)\nabla^4 w-\\
&\quad -\frac{h}{2}\left(A_3+A_4\,\frac{\partial}{\partial t}+A_5\,\frac{\partial^2}{\partial t^2}\right)\left[\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2}+\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial y^2}\right]-\\
&\quad -\frac{h}{2}\left(A_6+A_7\,\frac{\partial}{\partial t}+A_8\,\frac{\partial^2}{\partial t^2}\right)\left[\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial x^2}+\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial y^2}\right]-\\
&\quad -h\left(A_9+A_{10}\,\frac{\partial}{\partial t}+A_{11}\,\frac{\partial^2}{\partial t^2}\right)\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\,\frac{\partial^2 w}{\partial x\,\partial y}+\\
&\quad +q_\mathrm{t}\,(a-x)\left(A_1+A_2\,\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial x^2}+\left(A_1+A_2\,\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial t^2}=0
\end{aligned}
\tag{17}
$$

with

$$A_1 = 3\,K + 4\,G\ , \quad A_2 = 4\,\eta\ , \quad A_3 = 3\,G\,(6\,K + 2\,G)\ , \quad A_4 = 8\,G\,\eta + 12\,K\,\eta\ ,$$
$$A_5 = 4\,\eta^2\ , \quad A_6 = 2\,G\,(3\,K - 2\,G)\ , \quad A_7 = 6\,K\,\eta - 8\,G\,\eta\ , \quad A_8 = -4\,\eta^2\ ,$$
$$A_9 = 2\,G\,(3\,K + 4\,G)\ , \quad A_{10} = 6\,K\,\eta + 16\,G\,\eta\ , \quad A_{11} = 8\,\eta^2\ ,$$
$$G = \frac{E}{2\,(1+\nu)}\ , \quad K = \frac{E}{3\,(1-2\nu)}\ , \quad \nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\,\lambda^2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \lambda^4\,\frac{\partial^4 w}{\partial y^4}\ .$$

The boundary condition of viscoelastic plate subjected to tangential follower force is four edges simply supported given by:

$$\begin{aligned} x = 0, a: &\quad w = \frac{\partial^2 w}{\partial x^2} = 0\ ,\\ y = 0, b: &\quad w = \frac{\partial^2 w}{\partial y^2} = 0\ . \end{aligned} \tag{18}$$

Introducing the dimensionless variables

$$\begin{gathered} X = \frac{x}{a}\ , \quad Y = \frac{y}{b}\ , \quad \bar{W} = \frac{w}{\sqrt{k\,\epsilon\,h}}\ , \quad \lambda = \frac{a}{b}\ , \quad q = \frac{12\,q_{\mathrm{t}}\,a^3\,(1-\nu^2)}{E\,h^3}\ ,\\ \tau = \frac{t\,h}{a^2}\sqrt{\frac{E}{12\,\rho\,(1-\nu^2)}}\ , \quad H = \frac{h}{a^2}\sqrt{\frac{E}{12\,\rho\,(1-\nu^2)}}\,\frac{\eta}{E}\ . \end{gathered} \tag{19}$$

Equation (17) can be rewritten with dimensionless variables as:

$$\begin{aligned} &\left(1 + \alpha_{43}\,H\,\frac{\partial}{\partial\tau} + \alpha_{53}\,H^2\,\frac{\partial^2}{\partial\tau^2}\right)\nabla^4\bar{W} + \left(1 + \alpha_{21}\,H\,\frac{\partial}{\partial\tau}\right)\left\{q\,(1-X)\,\frac{\partial^2\bar{W}}{\partial X^2} + \frac{\partial^2\bar{W}}{\partial\tau^2}\right\} -\\ &\quad - 6\,k\,\epsilon\left(1 + \alpha_{43}\,H\,\frac{\partial}{\partial\tau} + \alpha_{53}\,H^2\,\frac{\partial^2}{\partial\tau^2}\right)\left[\left(\frac{\partial\bar{W}}{\partial X}\right)^2\frac{\partial^2\bar{W}}{\partial X^2} + \lambda^4\left(\frac{\partial\bar{W}}{\partial Y}\right)^2\frac{\partial^2\bar{W}}{\partial Y^2}\right] -\\ &\quad - 6\,\nu\,\lambda^2\,k\,\epsilon\left(1 + \alpha_{76}\,H\,\frac{\partial}{\partial\tau} + \alpha_{86}\,H^2\,\frac{\partial^2}{\partial\tau^2}\right)\left[\left(\frac{\partial\bar{W}}{\partial Y}\right)^2\frac{\partial^2\bar{W}}{\partial X^2} + \left(\frac{\partial\bar{W}}{\partial X}\right)^2\frac{\partial^2\bar{W}}{\partial Y^2}\right] -\\ &\quad - 12\,(1-\nu)\,\lambda^2\,k\,\epsilon\left(1 + \alpha_{109}\,H\,\frac{\partial}{\partial\tau} + \alpha_{119}\,H^2\,\frac{\partial^2}{\partial\tau^2}\right)\frac{\partial\bar{W}}{\partial X}\,\frac{\partial\bar{W}}{\partial Y}\,\frac{\partial^2\bar{W}}{\partial X\partial Y} = 0\ , \end{aligned} \tag{20}$$

where

$$\alpha_{21} = \frac{4\,(1-2\,\nu)\,(1+\nu)}{3\,(1-\nu)}\ , \quad \alpha_{43} = \frac{4\,(2-\nu)\,(1+\nu)}{3}\ , \quad \alpha_{53} = \frac{4\,(1-2\,\nu)\,(1+\nu)^2}{3}\ ,$$
$$\alpha_{76} = \frac{2\,(1+\nu)\,(5\,\nu-1)}{3\,\nu}\ , \quad \alpha_{86} = -\frac{4\,(1-2\,\nu)\,(1+\nu)^2}{3\,\nu}\ ,$$
$$\alpha_{109} = \frac{2\,(5-7\,\nu)\,(1+\nu)}{3\,(1-\nu)}\ , \quad \alpha_{119} = \frac{8\,(1-2\,\nu)\,(1+\nu)^2}{3\,(1-\nu)}\ .$$

with the dimensionless boundary condition:

$$\begin{aligned} X = 0, 1: &\quad W = \frac{\partial^2 W}{\partial X^2} = 0\ ,\\ Y = 0, 1: &\quad W = \frac{\partial^2 W}{\partial Y^2} = 0\ . \end{aligned} \tag{21}$$

3. Multi-scale analysis and analytical investigations

3.1. Multi-scale analysis

The solutions to Eq. (20) can be assumed as:

$$\bar{W} = \bar{W}_0(X,Y,T_0,T_1) + \epsilon\,\bar{W}_1(X,Y,T_0,T_1) + 0(\epsilon^2)\ , \tag{22}$$

where $T_0 = \tau$ and $T_1 = \epsilon\,\tau$ are respectively, the fast and slow time scales in the method of multiple scales. Substitution of Eq. (22) and the following relationship:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial T_0} + \epsilon\,\frac{\partial}{\partial T_1} + 0(\epsilon^2)\ , \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\partial^2}{\partial T_0^2} + 2\,\epsilon\,\frac{\partial^2}{\partial T_0 \partial T_1} + 0(\epsilon^2)\ , \tag{23}$$

into Eq. (20) and then equalization of coefficients of ϵ^0 and ϵ^1 in the resulting Equation leads to:

ϵ^0 :

$$\begin{aligned}&\left(1 + \alpha_{43}\,H\,\frac{\partial}{\partial T_0} + \alpha_{53}\,H^2\,\frac{\partial^2}{\partial T_0^2}\right)\nabla^4 \bar{W}_0 + \\ &\quad + \left(1 + \alpha_{21}\,H\,\frac{\partial}{\partial T_0}\right)\left\{q\,(1-X)\,\frac{\partial^2 \bar{W}_0}{\partial X^2} + \frac{\partial^2 \bar{W}_0}{\partial T_0^2}\right\} = 0\ ,\end{aligned} \tag{24}$$

$$\begin{aligned}X = 0,1: &\quad \bar{W}_0(X,Y) = \frac{\partial^2 \bar{W}_0}{\partial X^2} = 0\ ,\\ Y = 0,1: &\quad \bar{W}_0(X,Y) = \frac{\partial^2 \bar{W}_0}{\partial Y^2} = 0\ ;\end{aligned} \tag{25}$$

ϵ^1 :

$$\begin{aligned}&\left(1 + \alpha_{43}\,H\,\frac{\partial}{\partial T_0} + \alpha_{53}\,H^2\,\frac{\partial^2}{\partial T_0^2}\right)\nabla^4 \bar{W}_1 + \\ &+ \left(1 + \alpha_{21}\,H\,\frac{\partial}{\partial T_0}\right)\left\{q\,(1-X)\,\frac{\partial^2 \bar{W}_1}{\partial X^2} + \frac{\partial^2 \bar{W}_1}{\partial T_0^2}\right\} - \\ &- 6\,k\left(1 + \alpha_{43}\,H\,\frac{\partial}{\partial T_0} + \alpha_{53}\,H^2\,\frac{\partial^2}{\partial T_0^2}\right)\left[\left(\frac{\partial \bar{W}_0}{\partial X}\right)^2 \frac{\partial^2 \bar{W}_0}{\partial X^2} + \lambda^4\left(\frac{\partial \bar{W}_0}{\partial Y}\right)^2 \frac{\partial^2 \bar{W}_0}{\partial Y^2}\right] - \\ &- 6\,\nu\,\lambda^2\,k\left(1 + \alpha_{76}\,H\,\frac{\partial}{\partial T_0} + \alpha_{86}\,H^2\,\frac{\partial^2}{\partial T_0^2}\right)\left[\left(\frac{\partial \bar{W}_0}{\partial Y}\right)^2 \frac{\partial^2 \bar{W}_0}{\partial X^2} + \left(\frac{\partial \bar{W}_0}{\partial X}\right)^2 \frac{\partial^2 \bar{W}_0}{\partial Y^2}\right] - \\ &\quad - 12\,(1-\nu)\,\lambda^2\,k\left(1 + \alpha_{109}\,H\,\frac{\partial}{\partial T_0} + \alpha_{119}\,H^2\,\frac{\partial^2}{\partial T_0^2}\right)\frac{\partial \bar{W}_0}{\partial X}\,\frac{\partial \bar{W}_0}{\partial Y}\,\frac{\partial^2 \bar{W}_0}{\partial X\,\partial Y} + \\ &\qquad + \alpha_{43}\,H\,\frac{\partial(\nabla^4 \bar{W}_0)}{\partial T_1} + 2\,\alpha_{53}\,H^2\,\frac{\partial^2(\nabla^4 \bar{W}_0)}{\partial T_0\,\partial T_1} + \\ &\qquad\quad + \alpha_{21}\,H\,q\,(1-X)\,\frac{\partial^3 \bar{W}_0}{\partial T_1\,\partial X^2} + 3\,\frac{\partial^2 \bar{W}_0}{\partial T_0^2} + 2\,\frac{\partial^2 \bar{W}_0}{\partial T_1\,\partial T_0} = 0\ ,\end{aligned} \tag{26}$$

$$\begin{aligned}X = 0,1: &\quad \bar{W}_1(X,Y) = \frac{\partial^2 \bar{W}_1}{\partial X^2} = 0\ ,\\ Y = 0,1: &\quad \bar{W}_1(X,Y) = \frac{\partial^2 \bar{W}_1}{\partial Y^2} = 0\ ;\end{aligned} \tag{27}$$

To investigate nonlinear free transverse vibration of plate subjected to follower force, the solution of Eq. (24) is assumed to be expressed by:

$$W_0(X,Y,T_0,T_1) = W_{sl}(X,Y)\,A_{sl}(T_1)\,\mathrm{e}^{\mathrm{j}\,\Omega_{0sl}\,T_0} + CC\,, \tag{28}$$

where Ω_{0sl} is the slth frequency of ϵ^0-order system calculated in [24]. $\mathrm{j}^2 = -1$. Substitution of Eq. (28) into Eq. (24) yields:

$$\begin{aligned}&\left(1+\alpha_{43}\,H\,\frac{\partial}{\partial T_0}+\alpha_{53}\,H^2\,\frac{\partial^2}{\partial T_0^2}\right)\nabla^4\bar{W}_1 + \\ &+\left(1+\alpha_{21}\,H\,\frac{\partial}{\partial T_0}\right)\left\{q\,(1-X)\,\frac{\partial^2\bar{W}_1}{\partial X^2}+\frac{\partial^2\bar{W}_1}{\partial T_0^2}\right\} = \\ &= -\left[E_1\,\frac{\partial A_{sl}}{\partial T_1}+k\,G_1\,A_{sl}^2\,\bar{A}_{sl}\right]\mathrm{e}^{\mathrm{j}\,\Omega_{0sl}\,T_0}+CC+NST\,,\end{aligned} \tag{29}$$

where NST stands for non-secular terms,

$$E_1 = 2\,\mathrm{j}\,W_{sl}\,\Omega_{0sl} + (\alpha_{43}\,H + 2\,\mathrm{j}\,\alpha_{53}\,H^2\,\Omega_{0sl})\,\nabla^4 W_{sl} + \alpha_{21}\,H\,q\,(1-X)\,\frac{\partial^2 W_{sl}}{\partial X^2} + 3\,W_{sl}\,\Omega_{0sl}^2 \tag{30}$$

and

$$\begin{aligned}G_1 = &-6\left[1+\mathrm{j}\,\alpha_{43}\,H\,\Omega_{0sl}-\alpha_{53}\,H^2\,\Omega_{0sl}^2\right]\left[\lambda^4\,\frac{\partial W_{sl}}{\partial Y}\left(\frac{\partial W_{sl}}{\partial Y}\,\frac{\partial^2\bar{W}_{sl}}{\partial Y^2}+2\,\frac{\partial\bar{W}_{sl}}{\partial Y}\,\frac{\partial^2 W_{sl}}{\partial Y^2}\right)+\right.\\ &\left.+\frac{\partial W_{sl}}{\partial X}\left(\frac{\partial W_{sl}}{\partial X}\,\frac{\partial^2\bar{W}_{sl}}{\partial X^2}+2\,\frac{\partial\bar{W}_{sl}}{\partial X}\,\frac{\partial^2 W_{sl}}{\partial X^2}\right)\right]-\\ &-6\,\nu\,\lambda^2\left[1+\mathrm{j}\,\alpha_{76}\,H\,\Omega_{0sl}-\alpha_{86}\,H^2\,\Omega_{0sl}^2\right]\left[\frac{\partial W_{sl}}{\partial Y}\left(\frac{\partial W_{sl}}{\partial Y}\,\frac{\partial^2\bar{W}_{sl}}{\partial X^2}+2\,\frac{\partial\bar{W}_{sl}}{\partial Y}\,\frac{\partial^2 W_{sl}}{\partial X^2}\right)+\right.\\ &\left.+\frac{\partial W_{sl}}{\partial X}\left(\frac{\partial W_{sl}}{\partial X}\,\frac{\partial^2\bar{W}_{sl}}{\partial Y^2}+2\,\frac{\partial\bar{W}_{sl}}{\partial X}\,\frac{\partial^2 W_{sl}}{\partial Y^2}\right)\right]-\\ &-12\,(1-\nu)\,\lambda^2\left(1+\mathrm{j}\,\alpha_{109}\,H\,\Omega_{0sl}-\alpha_{119}\,H^2\,\Omega_{0sl}^2\right)\left(\frac{\partial W_{sl}}{\partial X}\,\frac{\partial W_{sl}}{\partial Y}\,\frac{\partial^2\bar{W}_{sl}}{\partial X\,\partial Y}+\right.\\ &\left.+\frac{\partial W_{sl}}{\partial X}\,\frac{\partial\bar{W}_{sl}}{\partial Y}\,\frac{\partial^2 W_{sl}}{\partial X\,\partial Y}+\frac{\partial\bar{W}_{sl}}{\partial X}\,\frac{\partial W_{sl}}{\partial Y}\,\frac{\partial^2 W_{sl}}{\partial X\,\partial Y}\right)\,.\end{aligned} \tag{31}$$

It should be noted that, the solvability condition presented by You-Qi Tang and al. [6] demands the orthogonal relationship.

$$\left\langle E_1\,\frac{\partial A_{sl}}{\partial T_1}+k\,G_1\,A_{sl}^2\,\bar{A}_{sl}\,,\ W_{sl}\right\rangle = 0\,, \tag{32}$$

where the the inner product is defined for complex functions f and g on $[0,1]$ as

$$\langle f,g\rangle = \int_0^1 f\,\bar{g}\,\mathrm{d}x\,. \tag{33}$$

Application of the distributive law of the inner product to Eq. (32) leads to

$$\frac{\partial A_{sl}}{\partial T_1}+k\,g_{11}\,A_{sl}^2\,\bar{A}_{sl} = 0\,, \tag{34}$$

where

$$g_{11} = \frac{\int_0^1 \int_0^1 G_{11}\, \bar{W}_{sl}\, \mathrm{d}x\, \mathrm{d}y}{\int_0^1 \int_0^1 E_1\, \bar{W}_{sl}\, \mathrm{d}x\, \mathrm{d}y} \; . \tag{35}$$

It can be numerically demonstrated that g_{11} is complex number.

Express the solution of Eq. (34) in polar form

$$A_{sl} = \alpha_{sl}(T_1)\, \mathrm{e}^{\mathrm{j}\, \beta_{sl}(T_1)} \; , \tag{36}$$

and substituting Eq. (36) into Eq. (34) yields

$$\begin{aligned} \frac{\partial \alpha_{sl}}{\partial T_1} &= 0 \; , \\ \alpha_{sl}\, \frac{\partial \beta_{sl}}{\partial T_1} &= -k\, g_{11}^{\mathrm{j}}\, \alpha_{sl}^3 \; , \end{aligned} \tag{37}$$

where g_{11}^{j} is imaginary part of g_{11}. The physical solution of Eq. (37) is when $\alpha_{sl} \neq 0$, which gives the following solutions after integration:

$$\alpha_{sl} = \alpha_{sl0} \; , \quad \beta_{sl} = -k\, g_{11}^{\mathrm{j}}\, \alpha_{sl0}^2\, T_1 + \beta_{sl0} \; , \tag{38}$$

where the initial amplitude α_{sl0} and the phase β_{sl0} are constants. Inserting Eq. (37) into Eq. (36) and then inserting the resulting equation into Eq. (28) gives the slth nonlinear frequency

$$\Omega_{sl} = \Omega_{0sl} - \epsilon\, k\, g_{11}^{\mathrm{j}}\, \alpha_{sl0}^2 \; . \tag{39}$$

3.2. Analytical investigations

Fig. 2 shows us the effect of delay time H on the lateral deflexion in the center of viscoelastic plate versus time for first and for second mode. The observation in this figure is that, the amplitude of the vibration decreases with time. For the first mode, the amplitude of vibration vanishes earlier than the second mode: The first mode amplitude, initially $\simeq 0.035$ needs $t \simeq 15$ to become $\simeq 0$. For second mode of vibration, the initial amplitude is $\simeq 5\times10^{-18}$ negligible comparing to the first mode amplitude, and vanishes lately when $t \simeq 250$. This curves, plotted at a given values of system's parameters give two informations: the system looses energy with time and the first mode of vibration is the most affected mode.

Fig. 3 presents the evolution with time t of first mode deflexion for two values delay-time H. The solid line denotes $H = 10^{-5}$ and the dashed line denotes $H = 10^{-3}$. The solid line evolution is with constant amplitude because the viscoelastic plate's behaviour is no far than elastic one. In the dashed line, the effect of viscoelasticity is more perceptible as the amplitude diminishes with a time, due to the energy loss of the system.

In Fig. 4 the curve explains how the nonlinear to linear frequency ratio of nonlinear viscoelastic plate decreases with time while Fig. 5 shows the relationship of the viscoelastic plate subjected to follower force between nonlinear to linear frequency ratio and the initial amplitude of first mode, at different nonlinear coefficients (a): $H = 10^{-5}$ and (b): $H = 10^{-3}$.

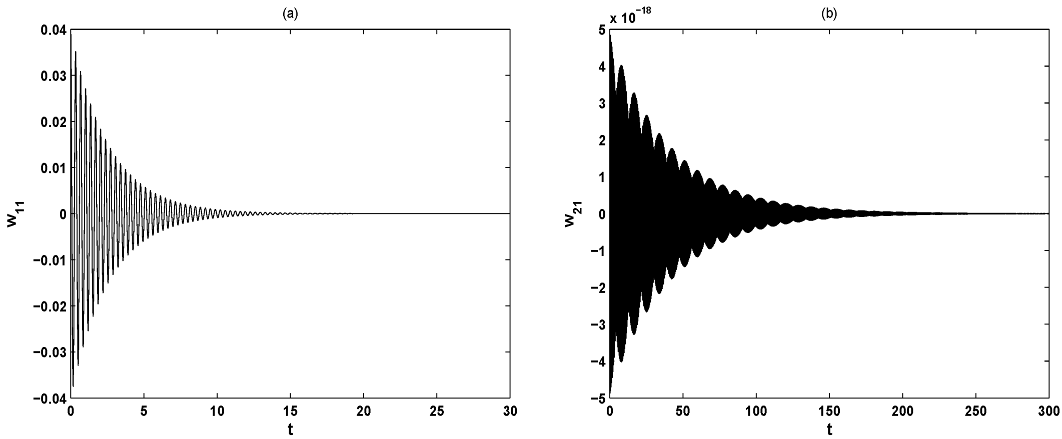

Fig.2: Response at central point of the plate with $k = 1$, $\varepsilon = 1$; *(a) first mode:* $\alpha_{110} = 0.04$, $q = 10$; *(b) second mode:* $\alpha_{210} = 0.04$

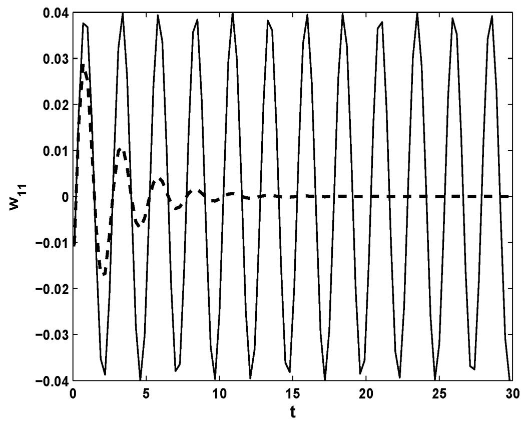

Fig.3: Response at central point of the plate for first mode with $k = 1$, $\varepsilon = 1$, $\alpha_{110} = 0.04$, $\lambda = 1$, $q = 10$, *(—):* $H = 10^{-5}$, $(--)$: $H = 10^{-3}$

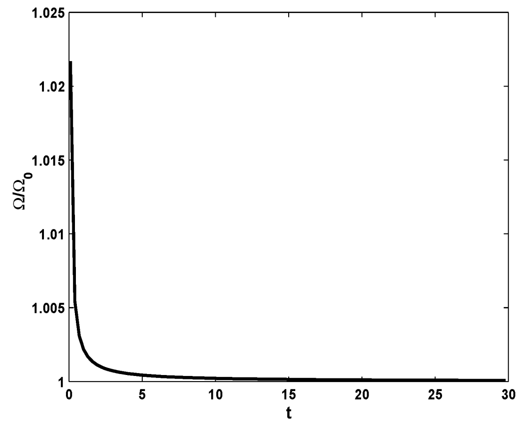

Fig.4: Nonlinear frequency to Linear frequency ratio vs time with $k = 1$, $\varepsilon = 1$, $\alpha_{110} = 0.04$, $\lambda = 1$, $q = 10$, $H = 10^{-3}$

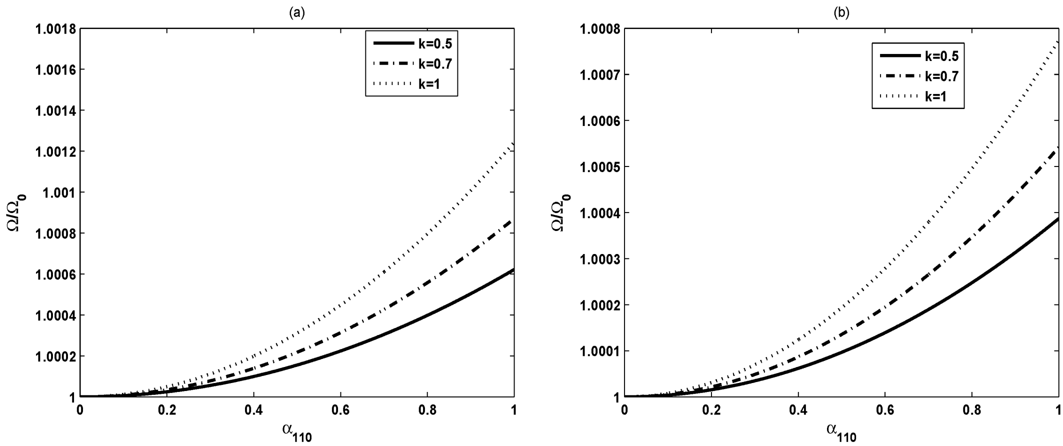

Fig.5: Nonlinear frequency to Linear frequency ratio vs initial amplitude at different nonlinear coefficients for first mode with $\varepsilon = 1$, $\lambda = 1$, $H = 10^{-3}$, $q = 10$; (a) $H = 10^{-5}$ and (b) $H = 10^{-3}$

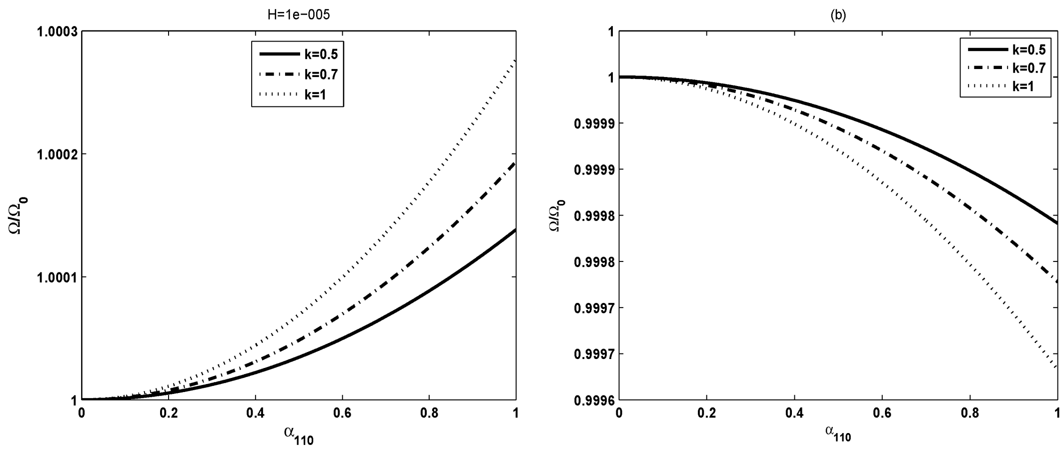

Fig.6: Nonlinear frequency to Linear frequency ratio vs initial amplitude at different nonlinear coefficients for second mode with $\varepsilon = 1$, $\lambda = 1$, $H = 10^{-3}$, $q = 10$; (a) $H = 10^{-5}$ and (b) $H = 10^{-3}$

The larger nonlinear coefficient results in rapid increase of the nonlinear to linear frequency ratio with the initial amplitude. Beside, the effect is less significant with the increase of delay-time. Fig. 6 presents the same relation as in Fig. 5 but for second mode. Here the effect of nonlinear coefficient is less significant than in first mode but when the delay-time increases, we note decreasing of the frequency ratio.

4. Numerical results

4.1. Differential quadrature method

Although many numerical schemes can be used to solve the differential equation (20) with boundary conditions (21), the DQ scheme is one of the most accurate method. Its essence is that, a partial derivative of the function $W(X,Y)$ at any sample point (X_i, Y_j), is considered as a weighted linear sum of the function $W(X_i, Y_j) \equiv W_{i,j}$ chosen on the defined domain of the spacial variables [25]. More precisely, taking the $2D$ rectangular plate under consideration for which the XY variables is delimited as follows: $0 \le X \le 1$ and $0 \le Y \le 1$, N and M the total number of discrete points along X and Y directions, respectively. The r^{th} order partial derivative of $W(X,Y)$ with respect to X, s^{th} order partial derivative of $W(X,Y)$ with respect to Y and $(r+s)^{\text{th}}$ order mixed partial derivative of $W(X,Y)$ with respect to both X and Y, are respectively written at a given point (X_i, Y_j) as [25]:

$$\frac{\partial^r W(X_i,Y_j)}{\partial X^r} = \sum_{k=1}^{N} A_{ik}^{(r)}\, W_{kj}\ , \quad (i=1,2,\dots,N\ ,\ k=1,2,\dots,N-1)\ , \tag{40}$$

$$\frac{\partial^s W(X_i,Y_j)}{\partial Y^s} = \sum_{k=1}^{M} B_{jl}^{(s)}\, W_{il}\ , \quad (j=1,2,\dots,M\ ,\ l=1,2,\dots,M-1)\ , \tag{41}$$

$$\frac{\partial^{r+s} W(X_i,Y_j)}{\partial X^r\, \partial Y^s} = \sum_{k=1}^{N} A_{ik}^{(r)} \sum_{k=1}^{M} B_{jl}^{(s)} W_{kl}\ , \tag{42}$$

where $A_{ik}^{(r)}$, and $B_{jl}^{(s)}$ are the weighting coefficients with [6]:

$$A_{ik}^{(1)} = \begin{cases} \dfrac{\prod_{\mu=1,\,\mu\neq i}^{N}(X_i - X_\mu)}{(X_i - X_k)\prod_{\mu=1,\,\mu\neq k}^{N}(X_k - X_\mu)}\ , & (i,k=1,2,\dots,N\ ,\ i\neq k)\ , \\ \displaystyle\sum_{\mu=1,\,\mu\neq i}^{N} \frac{1}{(X_i - X_\mu)}\ , & (i=1,2,\dots,N\ ,\ i=k)\ . \end{cases} \tag{43}$$

$$B_{jl}^{(1)} = \begin{cases} \dfrac{\prod_{\mu=1,\,\mu\neq j}^{M}(Y_j - Y_\mu)}{(Y_j - Y_l)\prod_{\mu=1,\,\mu\neq l}^{N}(Y_l - Y_\mu)}\ , & (j,l=1,2,\dots,M\ ,\ j\neq l)\ , \\ \displaystyle\sum_{\mu=1,\,\mu\neq j}^{M} \frac{1}{(Y_j - Y_\mu)}\ , & (j=1,2,\dots,M\ ,\ j=l)\ . \end{cases} \tag{44}$$

for $r=s=1$ and

$$A_{ik}^{(r)} = \begin{cases} r\left(A_{ii}^{(r-1)}\, A_{ik}^{(1)} - \dfrac{A_{ik}^{(r-1)}}{X_i - X_k}\right)\ , & (i,k=1,2,\dots,N\ ,\ i\neq k)\ , \\ -\displaystyle\sum_{\mu=1,\,\mu\neq i}^{N} A_{i\mu}^{(r)}\ , & (i=1,2,\dots,N\ ,\ i=k)\ . \end{cases} \tag{45}$$

$$B_{jl}^{(s)} = \begin{cases} s\left(B_{jj}^{(s-1)}\, B_{jl}^{(1)} - \dfrac{B_{jl}^{(s-1)}}{Y_j - Y_l}\right)\ , & (j,l=1,2,\dots,M\ ,\ j\neq l)\ , \\ -\displaystyle\sum_{\mu=1,\,\mu\neq j}^{M} B_{j\mu}^{(s)}\ , & (j=1,2,\dots,M\ ,\ j=l)\ . \end{cases} \tag{46}$$

for $r=2,3,\dots,N-1$, and $s=2,3,\dots,M-1$.

The distribution forms of the grid point are taken following the approach developed in [26] and we use the Coupling Boundary Conditions with General Equation (CBCGE) technic to implement boundary conditions. Accordingly, The form of grid point of SSSS plate is given by:

$$\begin{aligned} Xi &= \frac{1}{2}\left[1-\cos\left(\frac{i-1}{N-1}\,\pi\right)\right] , \quad (i=1,2,\dots,N)\ , \\ Yj &= \frac{1}{2}\left[1-\cos\left(\frac{j-1}{M-1}\pi\right)\right] , \quad (j=1,2,\dots,M)\ . \end{aligned} \tag{47}$$

With the above considerations, the Eq. (20) is transformed into a following discretized form:

$$\begin{aligned}
&\alpha_{21}\,H\,W_{ij}\,\mathrm{j}^3\,\Omega^3 + \\
&+\left[\alpha_{53}\,H^2\left(\sum_{k=1}^{N}A_{ik}^{(4)}\,W_{kj}+2\,\lambda^2\sum_{l=1}^{M}A_{jl}^{(2)}\sum_{i=1}^{N}A_{ik}^{(2)}\,W_{kl}+\lambda^4\sum_{l=1}^{M}A_{jl}^{(4)}\,W_{il}\right)+\mathrm{j}^2\,W_{ij}\right]\mathrm{j}^2\,\Omega^2 + \\
&+\left[\alpha_{43}\,H\,\mathrm{j}\left(\sum_{k=1}^{N}A_{ik}^{(4)}\,W_{kj}+2\,\lambda^2\sum_{l=1}^{M}A_{jl}^{(2)}\sum_{i=1}^{N}A_{ik}^{(2)}\,W_{kl}+\lambda^4\sum_{l=1}^{M}B_{jl}^{(4)}\,W_{il}\right)+\right. \\
&\left.+\,\alpha_{21}\,H\,q\sum_{k=1}^{N}A_{ik}^{(2)}\,W_{kj}\right]\mathrm{j}\,\Omega + \\
&+\left(\sum_{k=1}^{N}A_{ik}^{(4)}\,W_{kj}+2\,\lambda^2\sum_{l=1}^{M}A_{jl}^{(2)}\sum_{i=1}^{N}A_{ik}^{(2)}\,W_{kl}+\lambda^4\sum_{l=1}^{M}A_{jl}^{(4)}\,W_{il}\right)+ \\
&+\,q\,(1-X)\sum_{i=1}^{N}A_{ik}^{(2)}\,W_{kj}\,- \\
&-\,6\,k\,H^2\,\epsilon\left\{\alpha_{53}\left[\left(\sum_{i=1}^{N}A_{ik}^{(1)}\,W_{kj}\right)^2\sum_{i=1}^{N}A_{ik}^{(2)}\,W_{kj}+\lambda^4\left(\sum_{l=1}^{M}B_{jl}^{(1)}\,W_{il}\right)^2\sum_{l=1}^{M}B_{jl}^{(2)}\,W_{il}\right]+\right. \\
&+\,\nu\,\lambda^2\,\alpha_{86}\left[\left(\sum_{l=1}^{M}B_{jl}^{(1)}\,W_{il}\right)^2\sum_{i=1}^{N}A_{ik}^{(2)}\,W_{kj}+\left(\sum_{i=1}^{N}A_{ik}^{(1)}\,W_{kj}\right)^2\sum_{l=1}^{M}B_{jl}^{(2)}\,W_{il}\right]+ \\
&\left.+\,2\,(1-\nu)\,\lambda^2\,\alpha_{119}\left(\sum_{i=1}^{N}A_{ik}^{(1)}\,W_{kj}\right)\left(\sum_{l=1}^{M}B_{jl}^{(1)}\,W_{il}\right)\left(\sum_{l=1}^{M}B_{jl}^{(1)}\sum_{i=1}^{N}A_{ik}^{(1)}\,W_{kl}\right)\right\}\mathrm{j}^2\,\Omega^2\,- \\
&-\,6\,k\,H\,\epsilon\left\{\alpha_{43}\left[\left(\sum_{i=1}^{N}A_{ik}^{(1)}\,W_{kj}\right)^2\sum_{i=1}^{N}A_{ik}^{(2)}\,W_{kj}+\lambda^4\left(\sum_{l=1}^{M}B_{jl}^{(1)}\,W_{il}\right)^2\sum_{l=1}^{M}B_{jl}^{(2)}\,W_{il}\right]+\right. \\
&+\,\nu\,\lambda^2\,\alpha_{76}\left[\left(\sum_{l=1}^{M}B_{jl}^{(1)}\,W_{il}\right)^2\sum_{i=1}^{N}A_{ik}^{(2)}\,W_{kj}+\left(\sum_{i=1}^{N}A_{ik}^{(1)}\,W_{kj}\right)^2\sum_{l=1}^{M}B_{jl}^{(2)}\,W_{il}\right]+ \\
&\left.+\,2\,(1-\nu)\,\lambda^2\,\alpha_{109}\left(\sum_{i=1}^{N}A_{ik}^{(1)}\,W_{kj}\right)\left(\sum_{l=1}^{M}B_{jl}^{(1)}\,W_{il}\right)\left(\sum_{l=1}^{M}B_{jl}^{(1)}\sum_{i=1}^{N}A_{ik}^{(1)}\,W_{kl}\right)\right\}\mathrm{j}\,\Omega\,- \\
&-\,6\,k\,\epsilon\left\{\left[\left(\sum_{i=1}^{N}A_{ik}^{(1)}\,W_{kj}\right)^2\sum_{i=1}^{N}A_{ik}^{(2)}\,W_{kj}+\lambda^4\left(\sum_{l=1}^{M}B_{jl}^{(1)}\,W_{il}\right)^2\sum_{l=1}^{M}B_{jl}^{(2)}\,W_{il}\right]+\right.
\end{aligned}$$

$$+ \nu \lambda^2 \left[\left(\sum_{l=1}^{M} B_{jl}^{(1)} W_{il} \right)^2 \sum_{i=1}^{N} A_{ik}^{(2)} W_{kj} + \left(\sum_{i=1}^{N} A_{ik}^{(1)} W_{kj} \right)^2 \sum_{l=1}^{M} B_{jl}^{(2)} W_{il} \right] +$$

$$+ 2(1-\nu)\lambda^2 \left(\sum_{i=1}^{N} A_{ik}^{(1)} W_{kj} \right) \left(\sum_{l=1}^{M} B_{jl}^{(1)} W_{il} \right) \left(\sum_{l=1}^{M} B_{jl}^{(1)} \sum_{i=1}^{N} A_{ik}^{(1)} W_{kl} \right) \Bigg\} = 0 \ . \quad (48)$$

The differential quadrature form of boundary conditions (21) is:

$$\begin{gathered} W_{1,j} = 0\,,\ W_{N,j} = 0\,,\ W_{i,1} = 0\,,\ W_{i,M} = 0\ , \quad i = 1,2,\ldots,N\,,\ j = 1,2,\ldots,M\ , \\ \sum_{k=1}^{N} A_{ik}^{(2)} W_{kj} = 0\ , \quad i = 1, N\,,\ j = 1,2,\ldots,M\ , \\ \sum_{k=1}^{M} B_{jk}^{(2)} W_{ik} = 0\ , \quad j = 1, M\,,\ i = 1,2,\ldots,N\ . \end{gathered} \quad (49)$$

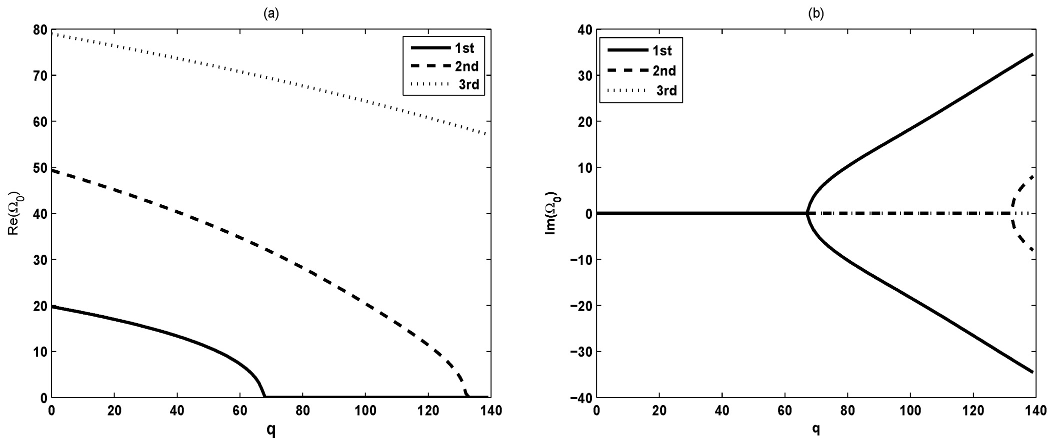

Fig.7: Linear frequency ((a)Real part, (b) Imaginary part) for first three modes vs dimensionless follower force with $\lambda = 1$, $H = 10^{-5}$ and $k = 0$

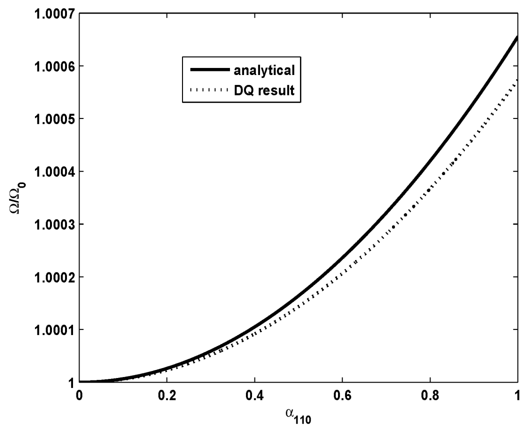

Fig.8: Comparison of analytical and numerical nonlinear to Linear frequency ratio vs initial amplitude for first mode with $\varepsilon = 1$, $\lambda = 1$, $q = 5$, $H = 10^{-5}$

4.2. Numerical results and discussion

In order to verify the results obtained by multi-scale method, we used the differential quadrature method as our numerical scheme. Before applying this method to calculate the nonlinear eigenfrequencies of Equation (20) with boundary conditions (21), we firstly apply this method to calculate the linear eigenfrequencies ($k = 0$) and plot the evolution of first three eigenfrequencies versus the follower force as you can see in Fig. 7. This curve shows the evolution of real and imaginary part of eigenfrequency with follower force when $H = 10^{-5}$, $N = M = 12$. These curves are the same as curves plotted in [24]. The nonlinear eigenfrequency is then calculated, using the nonlinear implementation of DQ method as we can see in [5] and [10], and the iterative scheme developed in [7].

Fig. 8 shows the evolution of nonlinear to linear frequency ratio versus initial amplitude. The comparison of the two curves shows us that, when the initial amplitude increases, we observe a small quantity difference between analytical and DQ results, which means that the two approaches give very similar results.

5. Conclusions

The present paper investigated the nonlinear vibration of 2D rectangular viscoelastic plate subjected to tangential follower force. Based on nonlinear strain-displacement relation, the 3D constitutive viscoelastic relation, the Laplace transformation and Inverse Laplace Transformation, the equation of motion of nonlinear viscoelastic plate subjected to tangential follower force in time domain is obtained. The fully simply-supported plate is considered. Firstly The multi-scale method is used to solve analytically the obtained equation and secondly the differential quadrature method is used to compare the multi-scale results. The main results are as follows : When initial amplitude increases, the nonlinear eigenfrequency increases too dependently on nonlinear coefficient. The damping effect reduce slightly the speed of increase. For the higher mode of vibration, the nonlinear frequency decreases when the delay time becomes high contrarily to the fundamental mode of vibration. The differential quadrature calculation of the nonlinear eigenfrequency is a very accurate, because the obtained results are very close to multi-scale results. The differential quadrature method can then be used to investigate nonlinear vibration of viscoelastic plate.

References

[1] Chu H.-N., Herman G.: Influence of large amplitude on free flexural vibrations of rectangular elastic plates, Journal of Applied Mechanics 1957, 23, 532–540

[2] Nayfeh A.H., Mook D.J.: Nonlinear Oscillation, John Wiley and Sons, New York, 1979, chap. 7, 444–454

[3] Amabili M., Paidoussis M.P.: Review of studies on geometrically nonlinear vibrations and dynamics of circular cylindrical sheels and panels with and without fluid-structure interaction, Applied Mechanical Review 2003, 56, 349–381

[4] Amabili M.: Nonlinear vibrations of rectangular plates with different boundary conditions : Theory and experiment, Computer and Structures 2004, 82, 2587–2605

[5] Chen W., Zhong T.: The study on the nonlinear computations of the DQ and the DC methods, Numerical Methods for Partial Differerential Equation 1997, 13(36), 57–75

[6] Tang Y.-Q., Chen L.-Q.: Nonlinear free transverse vibration of in-plane moving plate: Without and with internal resonances, Journal of Sound and Vibration 2011, 330, 110–126

[7] Yusheng F., Bert C.W.: Application of the quadrature method to flexural vibration analysis of geometrically nonlinear beams, Nonlinear Dynamics 1992, 3, 13–18

[8] Hu D., Chen L.: Nonlinear dynamics of axially accelerating viscoelastic beams based on diffrential quadrature Acta Mechanica Solida Sinica 2009, vol. 22, No. 3, 267–275

[9] Bert Ch.W., Jang S.K., Striz A.G.: Nonlinear bending analysis of orthotropic rectangular plates by the method of differential quadrature, Computational Mechanics, 5(1989) 217–226

[10] Chen W., Shu C., He W., Zhong T.: The DQ solution of geometrically nonlinear bending of orthotropic rectangular plates by using Hadamard and SJT product, Computer and Structures 2000, 74(1), 65–74

[11] Zhong H., Guo Q.: Nonlinear vibration analysis of Timoshenko beams using the differential quadrature method, Nonlinear Dynamics 2003, 32, 223–23

[12] Kim T.-W., Kim J.-H.: Nonlinear vibration of viscoelastic laminated composite plates, International Journal of Solids and Structures 2002, 39, 2857–2870

[13] Aboudi.J.: Postbuckling analysis of viscoelastic lainated plates using higher-order theory, International Journal of Solid and Structures 1991, 27 (14), 1747–1755

[14] Cederbaum G.: Parametric excitation of viscolastic plates, mechanics Based Structures and Machines 1992, 20(1), 37–51

[15] Cederbaum G., Aboudi,J., Elishakoff I.: Dynamic instability of shear deformable viscoelastic laminated plates by Lyapunov exponent, International Journal of Solid and Structures 1992, 28 (3), 317–327

[16] Touati D., Cederbaum G.: Dynamic stability of nonlinear viscoelastic plates, International Journal of Solid and Structures, vol. 31, 1994, 17, 2367–2373

[17] Aboudi J., Cederbaum G.: Dynamic Stability analysis of viscoelastic plates by Lyapunov exponents, Journal of Sound and Vibration 1990, 139(3), 459–467

[18] Chen L.-Q., Cheng Ch.-J.: Instability of nonlinear viscoelastic plates, Applied Mathematics and Computation 2005, 162, 1453–1463

[19] Esmailzadeh E., Jalali M.A.: Nonlinear Oscillation of viscoelastic rectangular plates, Nonlinear Dynamics 1999, 18, 311–319

[20] Daya E.M., Azrar L., Poitier-Ferry M.: Modelisation par elements finis des vibrations nonlieaires des plaques sandwich viscoelastiques, Mecanique et Industries 2005, 6, 13–20

[21] Touze C., Amabili M.: Nonlinear normal modes for damped geometrically nonlinear system: Application to reduced-order moder modelling of harmonically forced structures, Journal of Sound and Vibration 2006, 298, 958–981

[22] Abdoum F., Azrar L., Daya E.M., Poitier-Ferry M.: Forced harmonic response of viscoelastic structures by an asymtotic numerical method, Computer and Structures 2009, 87, 91–100

[23] Flügge W.: Viscoelastcity [M], Berlin:Springer 1975

[24] Wang Z.-M., Zhou Y.-F., Wang Y.: Dynamic stability of non-conservative viscoelastic rectangular plate, Journal of Sound and Vibration 2007, 307, 250–264

[25] Bert C.W., Malik M.: Implementing multiple boundary conditions in the DQ solution of higher-order pde's: Application to free vibration of plates, International Journal for Numerical Method in Engineering 1996, vol 39, 1237–1258

[26] Shu C., Du H.: A generalized approach for implementing general boundary conditions in the GDQ free vibration analysis of plates, International Journal of Solids and Structure 1997, vol. 34, 837–846

Received in editor's office: November 24, 2012
Approved for publishing: May 28, 2013