THE BOUNDARY CONDITIONS FOR THE COMPRESSIBLE GAS FLOW

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The paper deals with a special way of the construction of the boundary conditions for the compressible gas flow. The solution of the Riemann problem is used at the boundary. It can be shown, that the unknown one-side initial condition for this problem can be partially replaced by the suitable complementary condition. Authors work with such complementary conditions (by the preference of pressure, velocity, total quantities, ...) in order to match the physically relevant data. Algorithms were coded and used within the own developed code for the solution of the Euler, NS, and the RANS equations, using the finite volume method. Numerical example shows superior behavior of these boundary conditions in comparison with some other boundary conditions.

Keywords: *compressible gas flow, the Riemann problem, boundary conditions*

1. Introduction

The mathematical equations describing the fundamental conservation laws form a system of partial differential equations (the Euler Equations, the Navier-Stokes Equations, the Navier Stokes Equations with turbulent models). In this work we focus on the numerical solution of these equations in the vicinity of the boundary, which plays an important role in the computational fluid dynamics. We choose the well-known finite volume method to discretize the analytical problem, represented by the system of the equations in generalized (integral) form. The area of the interest is splitted into the elements, and the piecewise constant solution in time is constructed on these elements. The crucial problem of this method lies in the evaluation of the so-called fluxes through the edges/faces of the particular elements. Here we focus on the boundary edges, and we use the analysis of the Riemann problem for the discretization of the fluxes through the boundary edges/faces. The right-hand side initial condition for this local problem can be partially replaced by the suitable complementary condition. We analyze such modified local problems, equipped with various complementary conditions. The own algorithms are constructed and used in the numerical example.

2. The Riemann problem for the split Euler equations

In order to approximate the state values at the particular edges/faces of the mesh (at each time instant), we use the solution of the so-called Riemann problem for the split Euler equations. Using the rotational invariance of the equations describing the fluid flow, the system is expressed in a new Cartesian coordinate system $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ with the origin at

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the center of the gravity of the edge of interest Γ_{ij} and with the new axis $\tilde{x}_1$ in the direction of the normal of the edge. Then the derivatives with respect to $\tilde{x}_2, \tilde{x}_3$ are neglected, and we get the so-called split 3D Euler equations, see [3, page 138] :

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}_1(\mathbf{q})}{\partial \tilde{x}_1} = 0. \tag{1}$$

Here t denotes the time. $\mathbf{q} = \mathbf{q}(\tilde{x}, t) = (\varrho, \varrho\, u, \varrho\, v, \varrho\, w, E)^{\mathrm{T}}$ is the state vector, $\mathbf{f}_1 = = (\varrho\, u, \varrho\, u\, u + p, \varrho\, u\, v, \varrho\, u\, w, (E + p)\, u)^{\mathrm{T}}$ are the inviscid fluxes, $\mathbf{v} = (u, v, w)^{\mathrm{T}}$ denotes the velocity vector in the local coordinate system $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$, ϱ is the density, p the pressure, θ the absolute temperature, $E = \varrho\, e + \varrho\, \mathbf{v}^2/2$ the total energy. For the specific internal energy $e = c_{\mathrm{v}}\, \theta$ we assume the caloric equation of state $e = p/(\varrho\,(\gamma - 1))$, c_{v} is the specific heat at constant volume, $\gamma > 1$ is called the *Poisson adiabatic constant*. The system (1) is considered for $(\tilde{x}_1, t)$ in the set $Q_\infty = (-\infty, \infty) \times (0, +\infty)$.

Let us suppose, that the state values (density, velocity, pressure) are known in the close vicinity of the edge Γ_{ij} at the time instant t_k. These two states form the initial condition for the problem (1).

$$\mathbf{q}(\tilde{x}_1, 0) = \mathbf{q}_{\mathrm{L}} = \mathbb{Q}\, \mathbf{w}_i^k\ , \qquad \tilde{x}_1 < 0\ , \tag{2}$$

$$\mathbf{q}(\tilde{x}_1, 0) = \mathbf{q}_{\mathrm{R}} = \mathbb{Q}\, \mathbf{w}_j^k\ , \qquad \tilde{x}_1 > 0\ . \tag{3}$$

This problem (1), (2), (3) is the so-called Riemann problem for the *split* Euler equations. The solution of this problem at time axis is the desired solution (density, velocity, pressure) at the edge Γ_{ij}, and can be later used within the finite volume method.

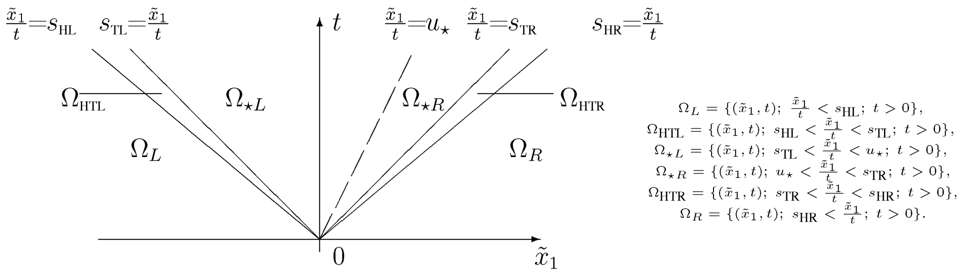

Fig.1: Structure of the solution of the Riemann problem (1), (2), (3)

It is a characteristic feature of the hyperbolic equations, that there is a possible raise of discontinuities in solutions, even in the case when the initial conditions are smooth, see [2, page 390], therefore by solution we mean the so-called *entropy weak solution* to this problem. The analysis to the solution of this problem can be found in many books, i.e. [3], [2], [1]. The general theorem on the solvability of the Riemann problem can be found in [3, page 88]. Here we summarize, that the problem has a unique solution for certain choice of the initial conditions. This solution can be written for $t > 0$ in the similarity form $\mathbf{q}(\tilde{x}_1, t) = \tilde{\mathbf{q}}(\tilde{x}_1/t)$, where $\tilde{\mathbf{q}}(\tilde{x}_1/t) : \mathbb{R} \to \mathbb{R}^3$ ([3, page 82]). The solution is piecewise **smooth** and its general form can be seen in Fig. 1, where the system of half lines is drawn. These half lines define regions, where $\mathbf{q}$ is either constant or given by a smooth function. Let us define the open sets called **wedges** Ω_L, Ω_{HTL}, $\Omega_{\star\mathrm{L}}$, $\Omega_{\star\mathrm{R}}$, Ω_{HTR}, Ω_{R}, see Fig. 1. We will refer to the set Ω_{HTL} as to the left wave, and the set Ω_{HTR} will be called the right wave. The solution in Ω_L, $\Omega_{\star\mathrm{L}}$, $\Omega_{\star\mathrm{R}}$, Ω_{R} is constant (see e.g. [3, page 128], while in Ω_{HTL} and in Ω_{HTR} it is continuous. Let us denote $\mathbf{q}|_{\Omega_{\mathrm{L}}} = \mathbf{q}_{\mathrm{L}}$, $\mathbf{q}|_{\Omega_{\star\mathrm{L}}} = \mathbf{q}_{\star\mathrm{L}}$, $\mathbf{q}|_{\Omega_{\star\mathrm{R}}} = \mathbf{q}_{\star\mathrm{R}}$, $\mathbf{q}|_{\Omega_{\mathrm{R}}} = \mathbf{q}_{\mathrm{R}}$, $\mathbf{q}|_{\Omega_{\mathrm{HTL}}} = \mathbf{q}_{\mathrm{HTL}}$, $\mathbf{q}|_{\Omega_{\mathrm{HTR}}} = \mathbf{q}_{\mathrm{HTR}}$. The exact solution of the Riemann problem has three waves in general, illustrated in Fig. 1. The wedges Ω_{L} and $\Omega_{\star\mathrm{L}}$ are separated by the left wave (either 1-*shock wave*, or 1-*rarefaction wave*). There is a *contact discontinuity* between the regions $\Omega_{\star\mathrm{L}}$ and $\Omega_{\star\mathrm{R}}$. Wedges $\Omega_{\star\mathrm{R}}$ and Ω_{R} are separated by the right wave (either 3-*shock wave*, or 3-*rarefaction wave*). The solution for the primitive variables can be described as follows:

$$(\varrho,u,v,w,p)|_{\Omega_{\mathrm{L}}} = (\varrho_{\mathrm{L}},u_{\mathrm{L}},v_{\mathrm{L}},w_{\mathrm{L}},p_{\mathrm{L}})\ , \qquad (\varrho,u,v,w,p)|_{\Omega_{\star R}} = (\varrho_{\star R},u_\star,v_R,w_R,p_\star)\ ,$$
$$(\varrho,u,v,w,p)|_{\Omega_{\star\mathrm{L}}} = (\varrho_{\star\mathrm{L}},u_\star,v_{\mathrm{L}},w_{\mathrm{L}},p_\star)\ , \qquad (\varrho,u,v,w,p)|_{\Omega_R} = (\varrho_{\mathrm{R}},u_{\mathrm{R}},v_{\mathrm{R}},w_{\mathrm{R}},p_{\mathrm{R}})\ .$$

The folowing relations for these variables hold:

$$u_\star = u_{\mathrm{L}} + \begin{cases} -(p_\star - p_{\mathrm{L}}) \left(\dfrac{\frac{2}{(\gamma+1)\,\varrho_{\mathrm{L}}}}{p_\star + \frac{\gamma-1}{\gamma+1}\,p_{\mathrm{L}}} \right)^{\frac{1}{2}} , & p_\star > p_{\mathrm{L}} \\ \dfrac{2}{\gamma-1}\,a_{\mathrm{L}} \left[1 - \left(\dfrac{p_\star}{p_{\mathrm{L}}}\right)^{\frac{\gamma-1}{2\gamma}} \right] , & p_\star \le p_{\mathrm{L}} \end{cases} \tag{4}$$

$$\varrho_{\star\mathrm{L}} = \begin{cases} \varrho_{\mathrm{L}}\, \dfrac{\frac{\gamma-1}{\gamma+1}\frac{p_{\mathrm{L}}}{p_\star} + 1}{\frac{p_{\mathrm{L}}}{p_\star} + \frac{\gamma-1}{\gamma+1}} , & p_\star > p_{\mathrm{L}} \\ \varrho_{\mathrm{L}} \left(\dfrac{p_\star}{p_{\mathrm{L}}}\right)^{\frac{1}{\gamma}} , & p_\star \le p_{\mathrm{L}} \end{cases} \tag{5}$$

$$s^1_{\mathrm{TL}} = \begin{cases} u_{\mathrm{L}} - a_{\mathrm{L}} \sqrt{\dfrac{\gamma+1}{2\,\gamma}\,\dfrac{p_\star}{p_{\mathrm{L}}} + \dfrac{\gamma-1}{2\,\gamma}} , & p_\star > p_{\mathrm{L}} \\ u_\star - a_{\mathrm{L}} \left(\dfrac{p_\star}{p_{\mathrm{L}}}\right)^{\frac{\gamma-1}{2\gamma}} , & p_\star \le p_{\mathrm{L}} \end{cases} \tag{6}$$

$$u_\star = u_{\mathrm{R}} + \begin{cases} (p_\star - p_{\mathrm{R}}) \left(\dfrac{\frac{2}{(\gamma+1)\varrho_{\mathrm{R}}}}{p_\star + \frac{\gamma-1}{\gamma+1} p_{\mathrm{R}}} \right)^{\frac{1}{2}} , & p_\star > p_{\mathrm{R}}, \\ -\dfrac{2}{\gamma-1} a_{\mathrm{R}} \left[1 - \left(\dfrac{p_\star}{p_{\mathrm{R}}}\right)^{\frac{\gamma-)}{2\gamma}} \right] , & p_\star \le p_{\mathrm{R}} \end{cases} \tag{7}$$

$$\varrho_{\star\mathrm{R}} = \begin{cases} \varrho_{\mathrm{R}}\, \dfrac{\frac{p_\star}{p_{\mathrm{R}}} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_\star}{p_{\mathrm{R}}} + 1} , & p_\star > p_{\mathrm{R}} \\ \varrho_{\mathrm{R}} \left(\dfrac{p_\star}{p_{\mathrm{R}}}\right)^{\frac{1}{\gamma}} , & p_\star \le p_{\mathrm{R}} \end{cases} \tag{8}$$

$$s^3_{\mathrm{TR}} = \begin{cases} u_{\mathrm{R}} + a_{\mathrm{R}} \sqrt{\dfrac{\gamma+1}{2\,\gamma}\dfrac{p_\star}{p_{\mathrm{R}}} + \dfrac{\gamma-1}{2\,\gamma}} , & p_\star > p_{\mathrm{R}} \\ u_\star + a_{\mathrm{R}} \left(\dfrac{p_\star}{p_{\mathrm{R}}}\right)^{\frac{\gamma-1}{2\gamma}} , & p_\star \le p_{\mathrm{R}} \end{cases} \tag{9}$$

Here $a_{\mathrm{L}} = \sqrt{\gamma\, p_{\mathrm{L}}/\varrho_{\mathrm{L}}}$, $a_{\mathrm{R}} = \sqrt{\gamma\, p_{\mathrm{R}}/\varrho_{\mathrm{R}}}$, and γ denotes the adiabatic constant. Further s^1_{TL} denotes 'unknown left wave speed', s^3_{TR} 'unknown right wave speed'. Note, that the system (4)–(9) is the system of 6 equations for 6 unknowns $p_\star$, $u_\star$, $\varrho_{\star\mathrm{L}}$, $\varrho_{\star\mathrm{R}}$, s^1_{TL},s^3_{TR}. The solution of this system leads to a nonlinear algebraic equation, and one cannot express the analytical solution of this problem in a closed form. The problem has a solution only if the *pressure positivity condition* is satisfied

$$u_\mathrm{R} - u_\mathrm{L} < \frac{2}{\gamma - 1}\,(a_\mathrm{L} + a_\mathrm{R})\ . \tag{10}$$

We will use some of these relations to construct and solve the initial-boundary value problem which will be the original result of our work.

Remarks

- Once the pressure $p_\star$ is known, the solution on the left-hand side of the contact discontinuity depends only on the left-hand side initial condition (2). And similarily, with $p_\star$ known, only the right-hand side initial condition (3) is used to compute the solution on the right-hand side of the contact discontinuity.
- The solution in $\Omega_\mathrm{L} \cup \Omega_\mathrm{HTL} \cup \Omega_{\star\mathrm{L}}$ (across 1 wave)
  There are three unknowns in the region $\Omega_{\star\mathrm{L}}$. It is the density $\varrho_{\star\mathrm{L}}$, the pressure $p_\star$, and the velocity $u_\star$. Also the speed s^1_{TL} of the left wave determining the position of the region Ω_HTL is not known. The solution components in $\Omega_\mathrm{L} \cup \Omega_\mathrm{HTL} \cup \Omega_{\star\mathrm{L}}$ region must satisfy the system of equations (4)–(6). It is a system of three equations for four unknowns. We have to add another equation in order to get the uniquely solvable system in $\Omega_\mathrm{L} \cup \Omega_\mathrm{HTL} \cup \Omega_{\star\mathrm{L}}$.

3. Boundary Conditions

At the boundary edges/faces we work with the problem (1) equipped with only one-side initial condition (2). The problem of the **boundary condition** is to choose the **boundary state**

$$\mathbf{q}(0,t) = \mathbf{q}_\mathrm{B}\ , \quad t > 0\ . \tag{11}$$

such a way that the system $(1),(2),(11)$ is well-posed, i.e. it has a unique solution (entropy weak) in the considered set $\Omega_\mathrm{B} = \{(\tilde{x}_1,t);\tilde{x}_1 \le 0, t > 0\}$. It is possible to show, that by adding properly chosen equations into the system $(1),(2)$ it is possible to reconstruct the boundary state $\mathbf{q}_\mathrm{B}$ such that the system $(1),(2),(11)$ has a unique solution in Ω_B, see [5]. We will refer to these added equations as to **complementary conditions**. Several choices of the complementary conditions will be discussed further.

Preference of pressure

Let us, for example, prefer the given value for the pressure $p_\mathrm{GIVEN} > 0$. This would correspond to the real-world problem, when we deal with the experimentally obtained pressure distribution at the boundary. We add the following **complementary conditions** into the system $(1),(2)$

$$p_\star := p_\mathrm{GIVEN}\ , \quad p_\mathrm{R} = p_\star\ , \quad \varrho_{\star\mathrm{R}} := \varrho_\mathrm{GIVEN}\ , \quad v_\mathrm{R} := v_\mathrm{GIVEN}\ , \quad w_R := w_\mathrm{GIVEN}\ . \tag{12}$$

Here $p_\star$ is the pressure in $\Omega_{\star\mathrm{L}} \cup \Omega_{\star\mathrm{R}}$, see Section 2. The conditions (12) prescribe the given pressure wherever it is possible, solution in Ω_L is governed by the condition (2). We seek the **boundary state** $\mathbf{q}_\mathrm{B}$ as the unique solution of the problem $(1),(2),(12)$ at the half line

$S_{\rm B} = \{(0,t); t > 0\}$. The system (4)–(9), (12) is uniquely solvable. The complete analysis of this problem is shown [5].

Preference of velocity

Let us prefer the given value for the velocity $u_{\rm GIVEN}$ at the boundary. We add the following complementary conditions into the system (1), (2)

$$u_\star := u_{\rm GIVEN}\ , \quad u_{\rm R} = u_\star\ , \quad \varrho_{\star\rm R} := \varrho_{\rm GIVEN}\ , \quad v_{\rm R} := v_{\rm GIVEN}\ , \quad w_{\rm R} := w_{\rm GIVEN}\ . \qquad (13)$$

We seek the boundary state $\mathbf{q}_{\rm B}$ as the unique solution of the problem (1), (2), (13) at the half line $S_{\rm B} = \{(0,t); t > 0\}$. The system (4)–(9), (13) is uniquely solvable. The algorithm for the construction of the primitive variables $\varrho_{\rm B}$, $u_{\rm B}$, $v_{\rm B}$, $w_{\rm B}$, $p_{\rm B}$ at the half-line $S_{\rm B}$ is shown in Fig. 2. The complete analysis of this problem is shown in [5]. This boundary condition can be used for the simulation of the **impermeable wall**, with $u_{\rm GIVEN} = 0$ (zero normal velocity of the boundary).

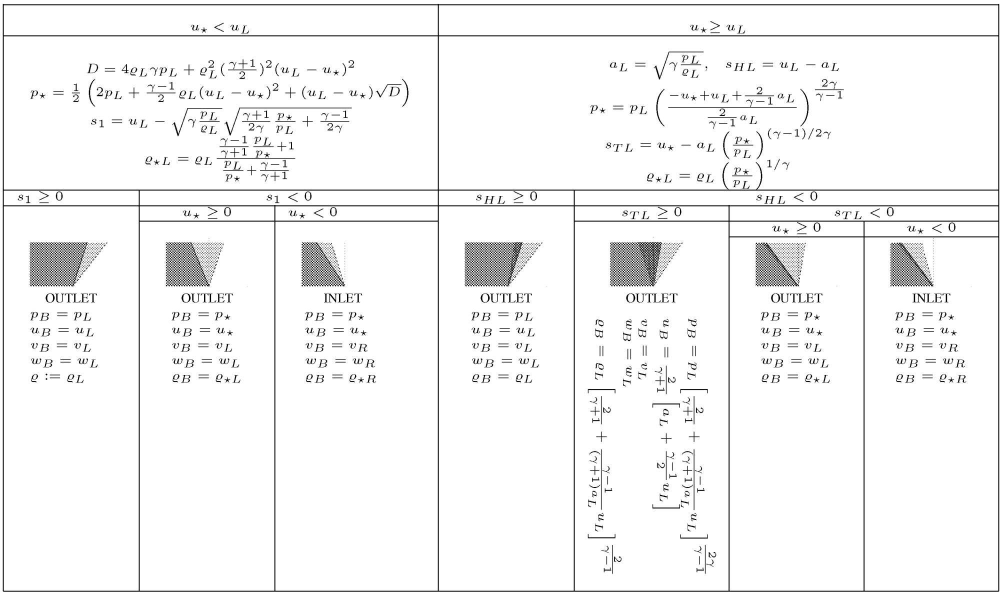

Fig.2: Algorithm for the solution of the problem (1), (2), (13) at the half line $S_{\rm B} = \{(0,t); t > 0\}$; possible situations are illustrated by the pictures showing the region $\Omega_{\rm L} \cup \Omega_{\rm HTL} \cup \Omega_{\star\rm L}$ with the sought boundary state located at the time axis

4. Example

Here we present a computational result of the 2D non-stationary inviscid channel flow at Mach number $M = 0.67$. A body immersed in the flowing fluid establishes a certain wave pattern which evolves in time and eventually exits the channel. At Figure 1. we show, that the fixed (values are fixed at the boundary) and linearized (as described in [3]) boundary conditions do not give the expected result in time. The inlet is located left, outlet right, other boundaries are considered as wall. The fixed boundary conditions give incorrect results near boundaries. The linearized boundary condition reflects the waves into the domain, leading

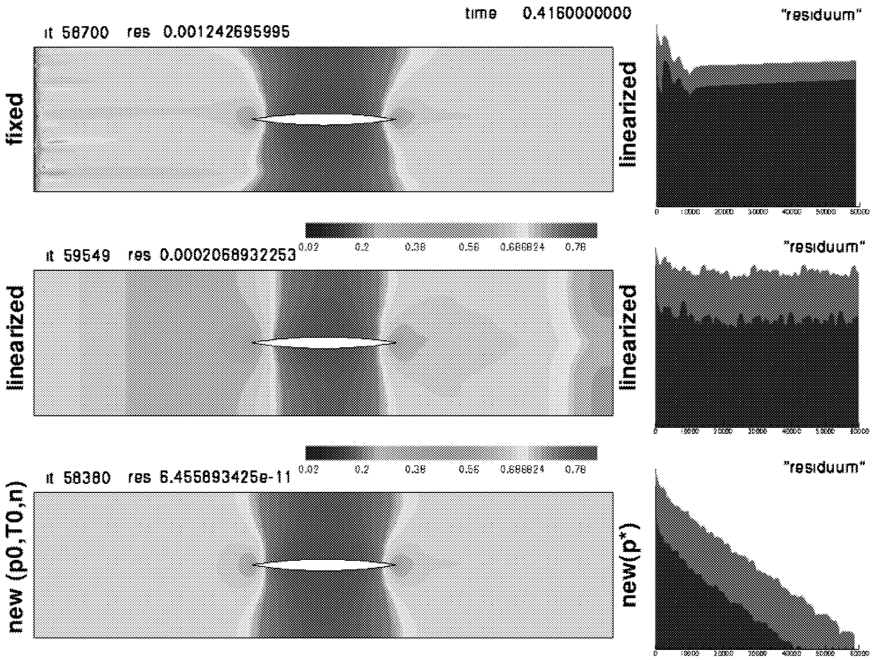

Fig.3: Incompressible flow, body in the channel; comparison of the various boundary conditions

to the oscillations in the solution. The new suggested boundary conditions do not suffer from these drawbacks. The residual behavior (shown right) demonstrates this result.

5. Conclusions

We worked with the system of equations describing the compressible fluid flow. In order to discretize the values at the boundary we solved the modification of the local Riemann problem, with the right-hand side initial-value replaced by the suitable conditions. The algorithms for the solution of the boundary problems were coded and implemented into own-developed software for the solution of the compressible (laminar or turbulent) gas flow (the Euler equations, the Navier-Stokes equations, the Reynolds-Averaged Navier-Stokes equations) in 2D and 3D. The presented numerical example shows superior behaviour of these boundary conditions.

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