TORSION OF A BAR WITH HOLES

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The contribution is a continuation of [2] which deals with analytic solution of torsion of a bar with simply connected profile, i.e. profile without holes. In this paper the case of multiply connected profile, i.e. profile with holes, is studied. The stress-strain analysis leads to the Airy stress function Φ. On boundary of each hole the function Φ has prescribed an unknown constant value completed with an integral condition. The mathematical model is also derived from the variational principle.

The second part of the paper contains solutions for the ring profile and for comparison also for incomplete ring profiles including the 'broken' ring profile. The results are compared in tables and pictures.

Keywords: *torsion of non-circular bar, Airy stress function, profile with holes*

1. Introduction

Torsion of the elastic bars is studied in several textbooks, see e.g. [6], but the results are mostly introduced without proofs or circular cross-section only is considered, e.g. in monograph [1]. In this circular case the cross-sections remain planar, but in case of non-circular bar, the real cross-sections are deflected from the planar shape. The equation for a non-circular bar with 'full' profile is derived correctly in [5], but no examples are introduced. Worth of visiting is an older monograph [3] published in 1953 by Anselm Kovář in Czech. It contains solution for many profiles and also a brief history of the torsion theory. Let us mention paper [4] which deals with the torsion problem for profiles with holes even for some nonelastic materials but in variational formulation only.

What is the purpose of studying analytical methods in the present time when any profile can be computed e.g. by FEM or other numerical methods? Although the analytic methods can solve only particular cases, they yield, in addition, also dependence of the solution on the data, e.g. dimensions of the profile, etc. It leads to better understanding of the problem, for further arguments see Introduction of [2].

In [2] the stress-strain analysis of the torsion of a bar with constant profile was carried out. The analysis was directed mostly to the case of a bar with 'full' profile, i.e. the profile without 'holes'. In mathematical terminology the profile without holes is called simple connected domain. In this contribution, which is a continuation of [2], we shall deal with the case of profiles with one or more holes, the so-called multiply connected domain. It brings several interesting difficulties.

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The strain-stress state is described by means of the so-called Airy stress function Φ satisfying the Poisson equation $-\Delta\Phi = 2$ on the profile Ω. Zero stress condition on the bar surface leads to constant value condition on the boundary. Since the boundary $\partial\Omega$ of a multiply connected domain Ω has more separated components, we can choose zero value on the outer boundary Γ_0. On any hole boundary Γ_i the constant value c_i is not determined. By the theory of partial differential equations for any values c_i we obtain different solutions Φ. We need to find additional conditions to ensure uniqueness of the solution Φ.

Modeling of the torsion starts with the deflection function f (in this paper it will be denoted by f instead of usual φ, since the symbol φ will be reserved for the polar coordinates ρ, φ). In the case of multiply connected domain Ω having a function Φ the deflection function f need not exists. The potentiality condition for the deflection function in multiply connected domains yields the desired additional conditions, see Subsection 2.2., Problem (P) – explanation of this phenomena seems to be new.

Variational approach to the problem in Subsection 2.3 leads to minimization of an integral functional $\mathcal{J}(\Phi)$ over a space $\mathcal{S}$ of functions Φ being constant on the holes Ω_i, Problem (V). We derive that its minimum Φ satisfies the Problem (P), see (16), including the additional integral condition on the holes Γ_i.

Having the solution Φ we can complete the stress-strain analysis. The torque M depends by (22) on the cross-section moment J, which can be computed from Φ by (23). The stress is proportional to the gradient $|\nabla\Phi|$, we prove that its maximum can appear only on the boundary. We derive that while in the case of profile without holes the thicker the profile is, the higher the stress is. In the case of the profile with holes the stress behaves in the opposite way, see Subsection 2.4.

The second part is devoted to concrete examples. Solution for the ring profile can be simply derived from the full circle profile. We want to compare it with the 'broken' ring profile which is already profile without hole. It is a special case of the open β-angle segment of the ring profile. Since the exact solution based on Fourier series is by no means simple, we show also an approximative solution. This exact solution was briefly derived already in [3], but study of series convergence and of singular cases of angle $\beta = \pi/2$ and $\beta = 3\pi/2$ is missing there.

We compare the values of approximative and exact solutions of various values of ratio $\lambda = r/R$ of the inner r and outer R radius and various angles β. Further we compare the cases of the 'complete' ring profile and the 'broken' ring profile. Using similar methods we can obtain results for elliptic rings, i.e. the space between two similar ellipses.

The computations were carried out using the MAPLE system of symbolic and numeric computations. The results are visualized in pictures and tables.

2. Theory

To make the paper self-contained we briefly repeat settings of the problem. Its analysis will be carried out with respect to problems appearing in the case of profile with holes, i.e. the case of multiply connected profile of the bar.

2.1. Geometry, strain and stress analysis

As in the previous paper [2] we consider an isotropic homogeneous prismatic bar. The axis of the bar (which is also axis of the torsion) coincides with x-axis, the cross-section Ω is a set in the y, z plane. Contrary to [2] the profile Ω may have holes, it need not be simply connected. We shall extend the notation in the following way, see Fig. 1:

- Ω_0 – a bounded open simply connected set – the profile including space occupied by the holes,
- $\Omega_1, \Omega_2, \ldots, \Omega_k$ – finite number of simply connected disjoint open subsets of Ω_0 – the 'holes' in the profile,
- $\Omega = \Omega_0 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \cdots \cup \overline{\Omega}_k)$ – multiply connected open bounded set in $\mathbf{R}^2$, the actual cross-section of the twisted bar.

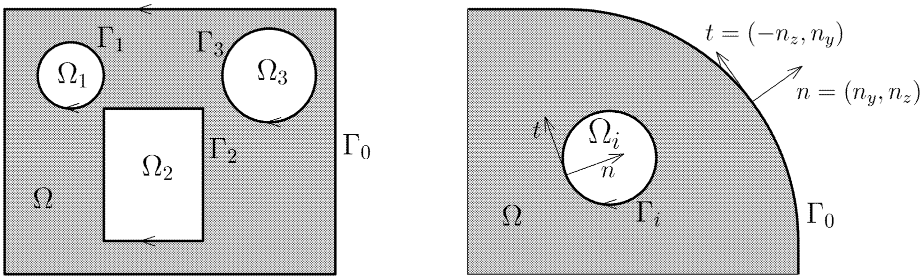

Fig.1: Profiles with holes, notation of tangent and normal vectors

The bar thus occupies the reference volume $(0, \ell) \times \Omega$. Let us denote the boundaries:

- $\Gamma_0 = \partial\Omega_0$ – the outer boundary of the profile Ω,
- $\Gamma_i = \partial\Omega_i$, $i = 1, \ldots, k$ – boundary of the i-th hole Ω_i, thus
- $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_k$.
- The boundary Γ_i is a simple closed curve parameterized by

$$\Gamma_i = \{(y, z)\,|y = \gamma_{\mathrm{y}}(s), z = \gamma_{\mathrm{z}}(s), s \in I_i = \langle a_i, b_i \rangle\} ,$$

  where the piecewise differentiable functions $\gamma_{\mathrm{y}}, \gamma_{\mathrm{z}}$ satisfy $(\gamma'_{\mathrm{y}})^2 + (\gamma'_{\mathrm{z}})^2 = 1$. The curve and its parametrization of Γ_0 is oriented counterclockwise, the other curves $\Gamma_1, \ldots, \Gamma_k$ and their parametrizations are oriented clockwise, see Fig. 1.

All the sets have piecewise smooth boundaries (Lipschitz continuous would be sufficient). We suppose also that all boundaries $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$ are disjoint sets, which implies that each two of them have positive distance. Further we denote

- $t = (t_{\mathrm{y}}, t_{\mathrm{z}}) = (\gamma'_{\mathrm{y}}, \gamma'_{\mathrm{z}})$ – the unit tangent vector to Γ_i.
- $n = (n_{\mathrm{y}}, n_{\mathrm{z}})$ – the unit outer normal vector to Γ_i.

Since the boundaries are piecewise smooth, the vectors n and t are uniquely defined on each Γ_i except for a finite number of isolated points. They are connected by the relation

$$\gamma'_{\mathrm{y}} = t_{\mathrm{y}} = -n_{\mathrm{z}} \qquad \text{and} \qquad \gamma'_{\mathrm{z}} = t_{\mathrm{z}} = n_{\mathrm{y}} \; . \tag{1}$$

Let us remark that the outer normal vector n on the hole boundary Γ_i, $i = 1, \ldots, k$ with respect to the profile Ω is directed inside the hole Ω_i, see Fig. 1.

The bar is fixed at $x = 0$ base, the opposite base $x = \ell$ is twisted by an angle $\ell\,\alpha$. We also adopt the hypothesis that the cross-sections in the y, z-plane rotates as a rigid body, in the case of a non-circular shape the cross-section is not planar, it is deflected in the x-direction. We also suppose that the twist rate α is constant along the whole length of the bar. Thus the problem can be reduced to the two-dimensional one.

The displacements u, v, w in directions x, y, z under these assumptions can be written

$$u = \alpha\, f(y, z)\ , \qquad v = -\,\alpha\, x\, z\ , \qquad w = \alpha\, x\, y\ , \tag{2}$$

where $f(y, z)$ is an unknown function describing the deflection in x direction. It is denoted by f (not φ as in [2]), since φ will be used in polar coordinates. The function f is supposed to be differentiable. Then the corresponding strain (small deformation) tensor $e = \{e_{ij}\}$ is

$$\begin{aligned} e_{\mathrm{xy}} = e_{\mathrm{yx}} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\,\alpha\left(\frac{\partial f}{\partial y} - z\right)\ , \\ e_{\mathrm{xz}} = e_{\mathrm{zx}} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\,\alpha\left(\frac{\partial f}{\partial z} + y\right)\ , \end{aligned} \tag{3}$$

the other components $e_{\mathrm{xx}}, e_{\mathrm{yy}}, e_{\mathrm{zz}}, e_{\mathrm{yz}}$ are zero. Simple computation yields

$$\frac{\partial e_{\mathrm{xz}}}{\partial y} - \frac{\partial e_{\mathrm{xy}}}{\partial z} = \frac{\alpha}{2}\left(\frac{\partial^2 f}{\partial y\,\partial z} + 1 - \frac{\partial^2 f}{\partial z\,\partial y} + 1\right) = \alpha\ . \tag{4}$$

The Hooke's law of linear elasticity with the sheer modulus μ (in literature often denoted by G) yields

$$\tau_{\mathrm{xy}} = 2\,\mu\, e_{\mathrm{xy}} = \alpha\,\mu\left(\frac{\partial f}{\partial y} - z\right)\ , \qquad \tau_{\mathrm{xz}} = 2\,\mu\, e_{\mathrm{xz}} = \alpha\,\mu\left(\frac{\partial f}{\partial z} + y\right)\ , \tag{5}$$

all the other components $\tau_{\mathrm{xx}}, \tau_{\mathrm{yy}}, \tau_{\mathrm{zz}}, \tau_{\mathrm{yz}}$ are zero.

The equilibrium equations $\sum_j \partial_j \tau_{ij} = F_i$ with zero forces F_i reduce to

$$\frac{\partial \tau_{\mathrm{xy}}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0\ , \qquad \frac{\partial \tau_{\mathrm{xy}}}{\partial x} = 0\ , \qquad \frac{\partial \tau_{\mathrm{xz}}}{\partial x} = 0\ . \tag{6}$$

The second and the third equality in (6) imply that τ_{xy} and τ_{xz} are independent of the variable x, the first equality means that the vector field $v = (-\tau_{\mathrm{xz}}, \tau_{\mathrm{xy}})$ is irrotational, i.e.

$$\mathrm{rot}(v) = (\partial_{\mathrm{y}}, \partial_{\mathrm{z}}) \times (-\tau_{\mathrm{xz}}, \tau_{\mathrm{xy}}) = \partial_{\mathrm{y}}\tau_{\mathrm{xy}} + \partial_{\mathrm{z}}\tau_{\mathrm{xz}} = 0\ .$$

Let us recall that for a simply connected domain Ω the irrotational vector field $v = (v_{\mathrm{y}}, v_{\mathrm{z}})$ is potential, i.e. there exists a function $\Phi(y, z)$ such that its gradient yields the vector field: $\nabla\Phi = v$. Thus the equalities in (6) imply existence of a function $\Phi(y, z)$ such that the only nonzero stress components τ_{xy} and τ_{xz} are given by

$$\tau_{\mathrm{xy}} = \alpha\,\mu\,\frac{\partial\Phi}{\partial z}\ , \qquad \tau_{\mathrm{xz}} = -\alpha\,\mu\,\frac{\partial\Phi}{\partial y}\ . \tag{7}$$

Vector $(\tau_{\mathrm{xy}}, \tau_{\mathrm{xz}})$ defined by (7) satisfies all the equilibrium equalities (6).

Let us express the components e_{xy} and e_{xz} using (5) by means of Φ

$$e_{\mathrm{xy}} = \frac{1}{2\,\mu}\,\tau_{\mathrm{xy}} = \frac{\alpha}{2}\frac{\partial\Phi}{\partial z}\;, \qquad e_{\mathrm{xz}} = \frac{1}{2\,\mu}\,\tau_{\mathrm{xz}} = -\frac{\alpha}{2}\frac{\partial\Phi}{\partial y} \tag{8}$$

and insert it into the equation (4). Multiplying by $2/\alpha$ we obtain

$$-\Delta\Phi \equiv -\left[\frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}\right] = 2 \qquad \text{in} \quad \Omega\;. \tag{9}$$

The equation has to be completed by boundary conditions. Since zero surface forces are considered, on the boundary Γ_i the traction vector $T = \tau \cdot n$ must have zero components $(T_{\mathrm{x}}, T_{\mathrm{y}}, T_{\mathrm{z}})$. Due to $n_{\mathrm{x}} = 0$ the components $T_{\mathrm{y}}, T_{\mathrm{z}}$ are zero. Further inserting $\tau_{\mathrm{xy}}, \tau_{\mathrm{xz}}$ from (5) to equality $T_{\mathrm{x}} = \tau_{\mathrm{xx}}\,n_{\mathrm{x}} + \tau_{\mathrm{xy}}\,n_{\mathrm{y}} + \tau_{\mathrm{xz}}\,n_{\mathrm{z}} = 0$ we obtain

$$T_{\mathrm{x}} = \tau_{\mathrm{xy}}\,n_{\mathrm{y}} + \tau_{\mathrm{xz}}\,n_{\mathrm{z}} = \tau_{\mathrm{xy}}\,t_{\mathrm{z}} - \tau_{\mathrm{xz}}\,t_{\mathrm{y}} = \mu\,\alpha\left(\frac{\partial\Phi}{\partial z}\,t_{\mathrm{z}} + \frac{\partial\Phi}{\partial y}\,t_{\mathrm{y}}\right) = \mu\,\alpha\,\frac{\partial\Phi}{\partial t} = 0\;. \tag{10}$$

Therefore Φ is constant along each component Γ_i. In the case of simply connected profile Ω the boundary $\partial\Omega = \Gamma_0$ is connected and we can choose $\Phi = 0$ on Γ_0.

2.2. Case of profile with holes

In this paper we study the profile Ω with holes, i.e. multiply connected domain Ω. In this case the equalities (6) are necessary but not sufficient conditions for existence of the potential $\Phi(y,z)$. The following vector field v serves as an counterexample:

$$v = (v_{\mathrm{y}}, v_{\mathrm{z}}) = \left(\frac{-z}{y^2+z^2}, \frac{y}{y^2+z^2}\right)$$

on a ring shape domain $G = \{[y,z] \in \mathbf{R}^2 \,|\, r^2 < y^2+z^2 < R^2\}$, where $0 < r < 1 < R$. Simple calculation verifies that $\partial_{\mathrm{y}}v_{\mathrm{z}} - \partial_{\mathrm{z}}v_{\mathrm{y}} = 0$, i.e. the vector field v is irrotational in G. But v has no potential on G. Indeed, if there were a potential $\Phi(y,z)$ then each line integral $\int_C (v_{\mathrm{y}}\,\mathrm{d}y + v_{\mathrm{z}}\,\mathrm{d}z)$ over a closed curve C in G would equal to zero. Let C be the unit circle $y = \cos s$, $z = \sin s$, $s \in \langle 0, 2\pi\rangle$. Then $y^2+z^2 = 1$, $\mathrm{d}y = -\sin s\,\mathrm{d}s$, $\mathrm{d}z = \cos s\,\mathrm{d}s$ and

$$\int\limits_C (v_{\mathrm{y}}\,\mathrm{d}y + v_{\mathrm{z}}\,\mathrm{d}z) = \int\limits_0^{2\pi} [(-\sin s)(-\sin s) + \cos s\,\cos s]\,\mathrm{d}s = \int\limits_0^{2\pi} 1\,\mathrm{d}s = 2\pi\;,$$

which does not equal to zero and thus contradicts potentiality.

To ensure that the irrotational vector field $v = (v_{\mathrm{y}}, v_{\mathrm{z}})$ is potential we need to verify that its line integral is path independent, i.e. the line integral of v over any closed curve in Ω equals to zero. Since for each simple connected domain irrotational vector field v is potential, it is sufficient to test only the curves C which encircle each hole. In our case the vector field v can be continuously extended to the closure $\overline{\Omega}$, thus the conditions $\int_{\Gamma_i} (v_{\mathrm{y}}\,\mathrm{d}y + v_{\mathrm{z}}\,\mathrm{d}z) = 0$ ensure existence of the potential Φ. Using $\mathrm{d}y = t_{\mathrm{y}}\,\mathrm{d}s$, $\mathrm{d}z = t_{\mathrm{z}}\,\mathrm{d}s$ and (1), the condition can be rewritten to

$$\int\limits_{\Gamma_i} (v_{\mathrm{y}}\,\mathrm{d}y + v_{\mathrm{z}}\,\mathrm{d}z) = \int\limits_{\Gamma_i} (v_{\mathrm{y}}\,t_{\mathrm{y}} + v_{\mathrm{z}}\,t_{\mathrm{z}})\,\mathrm{d}s = \int\limits_{\Gamma_i} (-v_{\mathrm{y}}\,n_{\mathrm{z}} + v_{\mathrm{z}}\,n_{\mathrm{y}})\,\mathrm{d}s = 0\;, \quad i = 1,\dots,k\;. \tag{11}$$

Let us verify the condition ensuring existence of the potential Φ to $(v_\mathrm{y}, v_\mathrm{z}) = (-\tau_\mathrm{xz}, \tau_\mathrm{xy})$:

$$\int_{\Gamma_i} (v_\mathrm{y}\,\mathrm{d}y + v_\mathrm{z}\,\mathrm{d}z) = \int_{\Gamma_i} (\tau_\mathrm{xz}\, n_\mathrm{z} + \tau_\mathrm{xy}\, n_\mathrm{y})\,\mathrm{d}s \ .$$

But $\tau_\mathrm{xy}\, n_\mathrm{y} + \tau_\mathrm{xz}\, n_\mathrm{z}$ equals to traction T_x which is zero on each part of the boundary Γ_i. Thus the vector field v has the potential $\Phi(y,z)$ such that the stress components can be written in the form (7). The potential satisfies equation (10) and is constant on each part Γ_i of the boundary $\partial\Omega$.

As we already mentioned, in the case of simply connected domain Ω the boundary $\partial\Omega$ has only one component and thus the only constant can be chosen to be zero.

In the case of multiply connected domain Ω we can choose the constant $c_0 = 0$. Then we have $\Phi = 0$ on the outer boundary Γ_0. On the other boundaries there are conditions $\Phi = c_i$ on Γ_i, $i = 1, \ldots, k$ with undetermined constants $c_1, \ldots, c_k$.

According to the theory of elliptic partial differential equations for any choice of the constants $c_1, \ldots, c_k$ we obtain different solutions Φ which yield different stress tensors. Since the solution of the real torsion of a bar should have unique solution, some additional conditions must be added.

Reformulating the problem for the Airy stress function Φ we lost connection to the deflection function f. The boundary value problem for Φ should be completed by a condition that the corresponding stress components $\tau_\mathrm{xy}, \tau_\mathrm{xz}$ admit the deflection function f. From (5) and (7) we have

$$\frac{\partial f}{\partial y}(y,z) = \frac{\partial \Phi}{\partial z}(y,z) + z \ , \qquad \frac{\partial f}{\partial z}(y,z) = -\frac{\partial \Phi}{\partial y}(y,z) - y \ . \tag{12}$$

It is the problem of finding a potential $f(y,z)$ from its differential $\mathrm{d}f = v_\mathrm{y}\,\mathrm{d}y + v_\mathrm{z}\,\mathrm{d}z$, where in this case

$$v_\mathrm{y} = \frac{\partial \Phi}{\partial z}(y,z) + z \ , \qquad v_\mathrm{z} = -\frac{\partial \Phi}{\partial y}(y,z) - y \ . \tag{13}$$

Simple calculation with (9) verifies that the vector field $(v_\mathrm{y}, v_\mathrm{z})$ is irrotational:

$$\operatorname{rot} v = \frac{\partial v_\mathrm{y}}{\partial z} - \frac{\partial v_\mathrm{z}}{\partial y} = \frac{\partial^2 \Phi}{\partial z^2} + 1 + \frac{\partial^2 \Phi}{\partial y^2} + 1 = \Delta\Phi + 2 \ .$$

In our case of multiply connected domain Ω we need to add the condition that each line integral is path independent, which reads (11). Inserting from (13) we obtain

$$I \equiv \int_{\Gamma_i} (-v_\mathrm{y}\, n_\mathrm{z} + v_\mathrm{z}\, n_\mathrm{y})\,\mathrm{d}s = -\int_{\Gamma_i} \left(\frac{\partial \Phi}{\partial z} n_\mathrm{z} + \frac{\partial \Phi}{\partial y} n_\mathrm{y}\right) \mathrm{d}s - \int_{\Gamma_i} (z\, n_\mathrm{z} + y\, n_\mathrm{y})\,\mathrm{d}s = 0 \ .$$

Integrand of the first integral equals to the normal derivative of Φ. The second integral will be transformed using the Gauss-Ostrogradski theorem $\int_{\Gamma_i} v \cdot (-n)\,\mathrm{d}s = \iint_{\Omega_i} \operatorname{div} v\,\mathrm{d}y\,\mathrm{d}z$. In the introduced formula we changed the sign, since in the theorem the normal vector is oriented outward Ω_i but our normal vector n is taken outward of Ω, i.e. inward the hole Ω_i:

$$\int_{\Gamma_i} (z\, n_\mathrm{z} + y\, n_\mathrm{y})\,\mathrm{d}s = -\iint_{\Omega_i} \left(\frac{\partial z}{\partial z} + \frac{\partial y}{\partial y}\right) \mathrm{d}y\,\mathrm{d}z = -\iint_{\Omega_i} (1+1)\,\mathrm{d}y\,\mathrm{d}z = -2\,|\Omega_i| \ ,$$

where $|\Omega_i|$ means the area of the domain Ω_i. Thus with our orientation of the normal vector, the potentiality condition $I = 0$ yields the additional conditions

$$\int\limits_{\Gamma_i} \frac{\partial \Phi}{\partial n}\, \mathrm{d}s = 2\,|\Omega_i| \;, \qquad i = 1, \dots, k \;. \tag{14}$$

As a potential, the deflection function f is given up to an additive constant. To ensure uniqueness we add the zero mean condition

$$\iint\limits_{\Omega} f(y,z)\,\mathrm{d}y\,\mathrm{d}z = 0 \;. \tag{15}$$

We have obtained the following boundary value problem:

Problem (P). *Find the unknown $\Phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $c_1, \dots, c_k$ such that*

$$\begin{aligned} -\Delta\Phi &= 2 \qquad \text{in } \Omega \;, \\ \Phi &= 0 \qquad \text{on } \Gamma_0 \;, \\ \Phi &= c_i \qquad \text{on } \Gamma_i \;, \quad i = 1, \dots, k \;, \\ \int\limits_{\Gamma_i} \frac{\partial \Phi}{\partial n}\, \mathrm{d}s &= 2\,|\Omega_i| \;, \quad i = 1, \dots, k \;. \end{aligned} \tag{16}$$

2.3. Variational formulation of the problem

Let us briefly introduce the variational formulation of the problem (16). The approach in our case looks for the minimizer of an energy functional $\mathcal{J}$ over a set $\mathcal{S}$ of admissible potentials Φ.

An admissible potential Φ is defined on the domain Ω with holes $\Omega_1, \dots, \Omega_k$. On the surface Γ_i it is constant. Thus we can extent the potential Φ by this constant inside the hole Ω_i to $\overline{\Phi}$. According to the boundary conditions we can look for the potentials $\overline{\Phi}$ satisfying

$$\overline{\Phi} = 0 \;\text{ on }\; \Gamma_0 \;, \qquad \overline{\Phi} = c_i \;\text{ in }\; \overline{\Omega}_i \;, \quad i = 1, \dots, k$$

with undetermined constants c_i. In the following we skip the bar in $\overline{\Phi}$ and write only Φ.

Solution of the Poisson equation $-\Delta\Phi = 2$ in Ω minimizes the following functional

$$\mathcal{J}(\Phi) = \iint\limits_{\Omega} \left[\frac{1}{2} \left(\frac{\partial \Phi}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{\partial \Phi}{\partial z} \right)^2 - 2\Phi \right] \mathrm{d}y\,\mathrm{d}z \,.$$

Since the gradient of a constant function on Ω_i is zero on Ω_i, the integral can be extended to the whole Ω_0. The functional contains integrals over the squared gradient thus we shall suppose that the gradient of Φ is square integrable. More precisely we assume that it is from the Sobolev space $H^1(\Omega_0)$ of measurable functions having square integrable generalized derivatives on Ω_0

$$H^1(\Omega_0) = \left\{ \Phi : \Omega_0 \to \mathbf{R} \;\text{ s.t. } \iint\limits_{\Omega_0} \left[\left(\frac{\partial \Phi}{\partial y} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 + \Phi^2 \right] \mathrm{d}y\,\mathrm{d}z < \infty \right\} .$$

To define the set $\mathcal{S}$ of admissible potentials we add the zero boundary condition on Γ_0 and constant condition in the holes Ω_i :

$$\mathcal{S} = \left\{ \Phi \in H^1(\Omega_0) \text{ s.t. } \Phi = 0 \text{ on } \Gamma_0 \text{ and } \Phi = c_i \text{ in } \overline{\Omega}_i,\ i = 1, \dots, k \right\} , \tag{17}$$

where the constants c_i are arbitrary.

Instead of detailed derivation of the variational formulation we set the problem and then we prove that it is equivalent to the original one. The variational formulation reads:

Problem (V). *Find $\Phi \in \mathcal{S}$ such that it minimizes the functional*

$$\mathcal{J}(\Phi) = \iint\limits_{\Omega_0} \left[\frac{1}{2} \left(\frac{\partial \Phi}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{\partial \Phi}{\partial z} \right)^2 - 2\Phi \right] dy\, dz \tag{18}$$

over the set $\mathcal{S}$, i.e. the inequality $\mathcal{J}(\Phi) \le \mathcal{J}(\Psi)$ holds for each $\Psi \in \mathcal{S}$.

Theorem. *Smooth (i.e. differentiable) solution of the Problem (V) solves the Problem (P).*

Proof. The set $\mathcal{S}$ is a linear space. Let the functional $\mathcal{J}$ attain its minimum at $\Phi \in \mathcal{S}$. Then also for each $\Psi \in \mathcal{S}$ the function $\varphi : \mathbf{R} \to \mathbf{R}$ defined by $\varphi(t) = \mathcal{J}(\Phi + t\Psi)$ attains its minimum at $t = 0$, i.e. $\varphi'(0) = 0$. Let us compute $\varphi'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{J}(\Phi + t\Psi)$:

$$\varphi'(t) = \iint\limits_{\Omega_0} \left[\left(\frac{\partial \Phi}{\partial y} + t \frac{\partial \Psi}{\partial y} \right) \frac{\partial \Psi}{\partial y} + \left(\frac{\partial \Phi}{\partial z} + t \frac{\partial \Psi}{\partial z} \right) \frac{\partial \Psi}{\partial z} - 2\,\Psi \right] \mathrm{d}y\, \mathrm{d}z \ .$$

For $t = 0$ the condition $\varphi'(0) = 0$ yields

$$\iint\limits_{\Omega_0} \left[\frac{\partial \Phi}{\partial y} \frac{\partial \Psi}{\partial y} + \frac{\partial \Phi}{\partial z} \frac{\partial \Psi}{\partial z} - 2\,\Psi \right] \mathrm{d}y\, \mathrm{d}z = 0$$

for each $\Psi \in \mathcal{S}$. Since Φ is constant on each Ω_i, its gradient $\nabla\Phi$ is zero on $\overline{\Omega}_i$ and the integral of $\nabla\Phi$ is reduced to the integral over Ω only, integrals of Ψ over Ω_i remain:

$$\iint\limits_{\Omega} \left[\frac{\partial \Phi}{\partial y} \frac{\partial \Psi}{\partial y} + \frac{\partial \Phi}{\partial z} \frac{\partial \Psi}{\partial z} - 2\,\Psi \right] \mathrm{d}y\, \mathrm{d}z - 2 \sum_{i=1}^{k} \iint\limits_{\Omega_i} \Psi\, \mathrm{d}y\, \mathrm{d}z = 0 \ . \tag{19}$$

To obtain the result we shall use of the following lemma:

Test lemma. *Let a continuous function f on a domain Ω satisfy*

$$\iint\limits_{\Omega} f(y,z)\, \psi(y,z)\, dy\, dz = 0 \tag{20}$$

for each 'test function' ψ on Ω from a set containing for each open ball $B \subset \Omega$ a continuous function ψ which is positive in the ball B and zero in $\Omega \setminus B$.

Then the function f is zero in Ω.

To use the Test lemma we have to transform the integral into the form (20). Assuming that Φ is twice differentiable, integration by parts yields

$$\iint\limits_{\Omega} \frac{\partial \Phi}{\partial y} \frac{\partial \Psi}{\partial y}\, \mathrm{d}y\, \mathrm{d}z = \int\limits_{\partial\Omega} \frac{\partial \Phi}{\partial y}\, \Psi\, n_{\mathrm{y}}\, \mathrm{d}s - \iint\limits_{\Omega} \frac{\partial^2 \Phi}{\partial y^2}\, \Psi\, \mathrm{d}y\, \mathrm{d}z \ ,$$

where $n = (n_y, n_z)$ is the unit normal vector outward with respect to Ω, i.e. it is oriented inward to the hole Ω_i, see Fig. 1. The boundary of Ω consists of Γ_0 and curves Γ_i being boundaries of the holes Ω_i. Using $\Psi = 0$ on Γ_0 we obtain

$$\iint_\Omega \frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial y}\,\mathrm{d}y\,\mathrm{d}z = \sum_{i=1}^{k}\int_{\Gamma_i}\frac{\partial \Phi}{\partial y}\,\Psi\, n_y\,\mathrm{d}s - \iint_\Omega \frac{\partial^2 \Phi}{\partial y^2}\,\Psi\,\mathrm{d}y\,\mathrm{d}z\ .$$

The analogous equality holds for the z derivatives. Since $\frac{\partial\Phi}{\partial y} n_y + \frac{\partial\Phi}{\partial z} n_z$ is the normal derivative of Φ, the equality $\varphi'(0) = 0$ can be rewritten in the form

$$\iint_\Omega \left(-\frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Phi}{\partial z^2} - 2\right)\Psi\,\mathrm{d}y\,\mathrm{d}z + \sum_{i=1}^{k}\left[\int_{\Gamma_i}\frac{\partial \Phi}{\partial n}\,\Psi\,\mathrm{d}s - 2\iint_{\Omega_i}\Psi\,\mathrm{d}y\,\mathrm{d}z\right] = 0\ . \tag{21}$$

Choosing Ψ which is nonzero only inside any ball $B \subset \Omega$, the Test lemma yields $-\Delta\Phi - 2 = 0$ inside Ω, which yields the first equation of the Problem (P). Using the obtained equality, the first integral in (21) vanishes. Let us take any function Ψ nonzero, e.g. equals to 1 in a point in Ω_i and zero in the other holes. Then Ψ equals to 1 on $\overline{\Omega}_i$. In this way we obtain

$$\int_{\Gamma_i}\frac{\partial \Phi}{\partial n}\,\Psi\,\mathrm{d}s - 2\iint_{\Omega_i}\Psi\,\mathrm{d}y\,\mathrm{d}z = \int_{\Gamma_i}\frac{\partial \Phi}{\partial n}\,\mathrm{d}s - 2\iint_{\Omega_i}\mathrm{d}y\,\mathrm{d}z = \int_{\Gamma_i}\frac{\partial \Phi}{\partial n}\,\mathrm{d}s - 2\,|\Omega_i| = 0\ ,$$

which implies the last condition of the Problem (P). The remaining two conditions follow directly from the condition $\Phi \in \mathcal{S}$ and the definition of $\mathcal{S}$. Thus the differentiable solution Φ of the Problem (V) is a solution of the Problem (P). $\square$

The variational formulation enables us to prove the following

Theorem. *The problem (V) admits unique solution.*

The proof is based on the following general abstract existence theorem:

Theorem. *Let $\mathcal{S}$ be a non-empty closed subspace of a Hilbert space H and let $\mathcal{J}$ be a continuous coercive strictly convex functional on $\mathcal{S}$. Then the problem*

Find $\Phi \in \mathcal{S}$ minimizing the functional $\mathcal{J}$ over a set $\mathcal{S}$

admits unique solution.

The proof consists of verifying the following steps:

- the set $\mathcal{S}$ is a nonempty closed subspace of the Hilbert space $H^1(\Omega)$. Since its elements are zero on Γ_0, the functional

$$|\Phi| = \left[\iint_\Omega \left(\left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right)\mathrm{d}y\,\mathrm{d}z\right]^{1/2}$$

  is equivalent to the norm $\|\cdot\|$ of $H^1(\Omega)$ on $\mathcal{S}$.
- $\mathcal{J}$ is defined and continuous on $\mathcal{S}$.
- $\mathcal{J}$ is coercive, i.e. $\mathcal{J}(\Phi) \to \infty$ as $\|\Phi\| \to \infty$.

– $\mathcal{J}$ is strictly convex, i.e. for each different $\Phi_0, \Phi_1 \in \mathcal{S}$ and $0 < \lambda < 1$ there is

$$\mathcal{J}((1-\lambda)\,\Phi_0 + \lambda\,\Phi_1) < (1-\lambda)\,\mathcal{J}(\Phi_0) + \lambda\,\mathcal{J}(\Phi_1) \ .$$

2.4. Application of the results in mechanics

Torque. Let us compute the torque M of the twisted bar. It is given by the formula

$$M = \iint\limits_{\Omega} (-\tau_{xy}\,z + \tau_{xz}\,y)\,\mathrm{d}y\,\mathrm{d}z \ .$$

Inserting from (7) we obtain

$$M = -\alpha\,\mu \iint\limits_{\Omega} \left(\frac{\partial \Phi}{\partial z}\,z + \frac{\partial \Phi}{\partial y}\,y \right) \mathrm{d}y\,\mathrm{d}z \ .$$

Integration by parts, using $\Phi = 0$ on Γ_0, $\Phi = c_i$ on Γ_i yields

$$\iint\limits_{\Omega} \frac{\partial \Phi}{\partial y}\,y\,\mathrm{d}y\,\mathrm{d}z = \int\limits_{\partial\Omega} \Phi\,y\,n_{\mathrm{y}}\,\mathrm{d}s - \iint\limits_{\Omega} \Phi\,\frac{\partial y}{\partial y}\,\mathrm{d}y\,\mathrm{d}z = \sum_{i=1}^{k} c_i \int\limits_{\Gamma_i} y\,n_{\mathrm{y}}\,\mathrm{d}s - \iint\limits_{\Omega} \Phi\,\mathrm{d}y\,\mathrm{d}z \ .$$

Due to orientation of the normal n inward to Ω_i we have $\int_{\Gamma_i} y\,n_{\mathrm{y}}\,\mathrm{d}s = -\iint_{\Omega_i} 1\,\mathrm{d}y\,\mathrm{d}z = -|\Omega_i|$ and by analogous calculation for the z-part we get

$$M = 2\,\alpha\,\mu \left(\iint\limits_{\Omega} \Phi\,\mathrm{d}y\,\mathrm{d}z + \sum_{i=1}^{k} c_i\,|\Omega_i| \right) \ .$$

We obtained dependence of the torque M on the twisting rate α

$$M = \alpha\,\mu\,J \ , \tag{22}$$

where the moment of the cross-section J is given by

$$J = 2 \left(\iint\limits_{\Omega} \Phi(y,z)\,\mathrm{d}y\,\mathrm{d}z + \sum_{i=1}^{k} c_i\,|\Omega_i| \right) \ . \tag{23}$$

Maximal stress. The maximum $|T|_{\max}$ of the stress is a very important value in engineering practice. It is often expressed in the form

$$|T|_{\max} = \frac{M}{W} = \frac{\alpha\,\mu\,J}{W} \ , \tag{24}$$

where $M = \alpha\,\mu\,J$. The quantity W is called the twist section modulus. Equality (24) yields definition of the twist section modulus W

$$W = \frac{M}{|T|_{\max}} = \frac{\alpha\,\mu\,J}{|T|_{\max}} \tag{25}$$

and for particular shapes of the profile it can be expressed by means of profile dimensions and a shape constant. To compute W we have to find $|T|_{\max}$.

How to find the maximal stress? The stress force T in direction n is $T = \tau \cdot n = \tau_{xy}\, n_{\mathrm{y}} + \tau_{xz}\, n_{\mathrm{z}}$. Its modulus equals to $|T| = [\tau_{xy}^2 + \tau_{xz}^2]^{1/2}$. Inserting from (7) we obtain

$$|T| = \alpha\,\mu \left[\left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right]^{\frac{1}{2}} = \alpha\,\mu\,|\nabla\Phi| \;, \tag{26}$$

i.e. the modulus of the stress is proportional to the slope of Φ. Since Φ satisfies $-\Delta\Phi = 2$, the stress cannot attains its maximum inside the profile Ω.

We shall prove it by a contradiction. Let Φ attains its maximum slope $|\nabla\Phi| = m > 0$ in a direction n in a point (y_0, z_0) inside Ω. Let us choose a shifted and rotated orthogonal coordinates (ξ, η) with their origin in (y_0, z_0) and ξ oriented in direction n. Thus in the new coordinates the partial derivatives of Φ^* are $\frac{\partial\Phi^*}{\partial\xi}(0,0) = m$ and $\frac{\partial\Phi^*}{\partial\eta}(0,0) = 0$.

Let us look at the second order derivatives. If $\frac{\partial^2\Phi^*}{\partial\xi^2}(0,0)$ was positive, then for a small $\xi > 0$ the value $\frac{\partial\Phi^*}{\partial\xi}(\xi,0)$ would be bigger than m. On the other hand if $\frac{\partial^2\Phi^*}{\partial\xi^2}(0,0) < 0$, then for a small $\xi < 0$ again $\frac{\partial\Phi^*}{\partial\xi}(\xi,0) > m$. Thus $\frac{\partial^2\Phi^*}{\partial\xi^2}(0,0) = 0$. Since the solution $\Phi(y,z)$ in the new coordinates $\Phi^*(\xi,\eta)$ satisfies the same equation $-\Delta\Phi^* = 2$ we obtain $\frac{\partial^2\Phi^*}{\partial\eta^2}(0,0) = -2$. Thus in a neighborhood of the origin the second order Taylor polynomial reads

$$\Phi^*(\xi,\eta) = m\,\xi + \frac{1}{2}\left[2\,c\,\xi\,\eta - 2\,\eta^2\right] = m\,\xi + c\,\xi\,\eta - \eta^2 \;,$$

where $c = \frac{\partial^2\Phi^*}{\partial\xi\,\partial\eta}(0,0)$. Then in the neighborhood $\nabla\Phi^*(\xi,\eta) = (m + c\,\eta, \xi - 2\,\eta)$ – up to the third order terms – and the modulus of gradient

$$|\nabla\Phi^*(\xi,\eta)| = \sqrt{(m + c\,\eta)^2 + (\xi - 2\,\eta)^2}$$

for some small (ξ,η) attains bigger value than m, which is the contradiction.

Let us derive the difference between open (simply connected) and closed (multiply connected) profile.

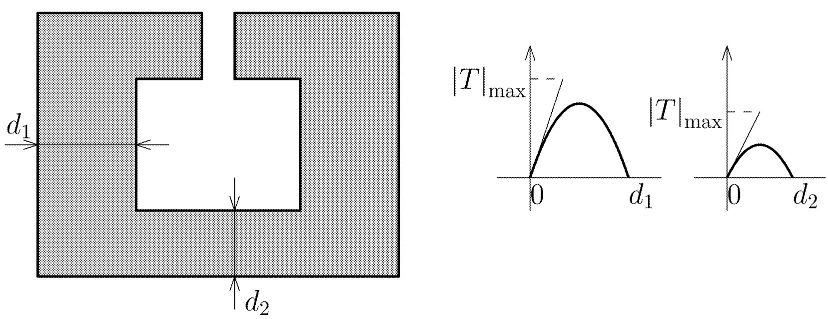

Fig.2: Maximal stress in the open profile

Comparison of maximal stress in open and closed profile. Let us consider a profile being a simply connected domain. Then on all parts of boundary Φ has zero value, see Fig. 2.

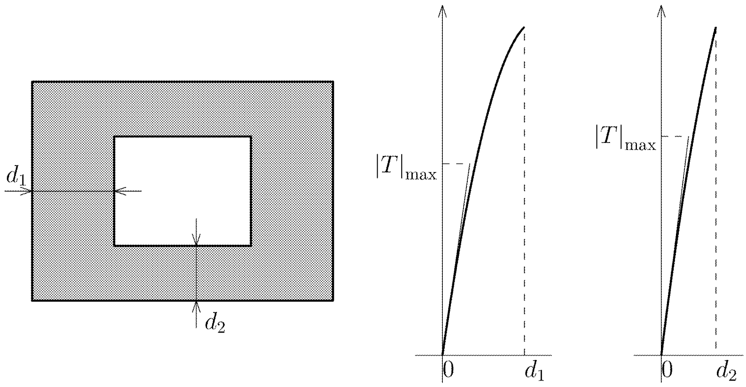

Fig.3: Maximal stress in the closed profile

From the equation $-\Delta\Phi = 2$ one can estimate, that cross section of Φ will be approximately a parabola $\Phi(t) = t\,(d - t)$, where d is a thickness of the profile. The stress will be proportional to the derivative $\Phi'(t) = d - 2\,t$ which attains its maximum d. Thus the stress maximum will be bigger in thicker places of the profile than in narrower places, i.e. when $d_1 > d_2$.

The situation is different in the case of a closed profile with a hole, see Fig. 3. On the outer part of the profile boundary the value of Φ is zero and on the inner part it is a positive value h. In this case the cross section of Φ on a segment is approximately a parabola $\Phi(t) = (h/d + d)\,t - t^2$, where $h > d^2$. The derivative $\Phi'(t) = h/d + d - 2\,t$ attains its maximum $h/d + d$ in $t = 0$. Thus bigger diameter d means smaller stress.

2.5. Summary of the results

Let us summarize the results. In case of the profile without holes we compute the Airy stress function Φ as solution of the boundary value problem (9) with boundary condition $\Phi = 0$ on the boundary Γ_0. In case of profile with holes, i.e. multi-connected domain, the Airy stress function Φ is given as the solution of Problem (P) consisting of the Poisson equation on Ω, zero boundary condition on the outer boundary Γ_0, and on each inner boundary Γ_i, $i = 1, \ldots, k$ value of Φ equals to an unknown constant c_i completed with the integral boundary condition (14), see (16).

The Airy stress function Φ can be computed from the equivalent variational formulation Problem (V). It consists of looking for a minimum of the variational functional $\mathcal{J}$ given by (18) over the set of admissible potentials $\mathcal{S}$. The variational formulation, in addition, yields existence and uniqueness of the potential Φ.

Then (23) yields the torsion constant J and (22) describes dependence of the twist rate α on the torque M. The only non-zero components τ_{xy}, τ_{xz} of the stress tensor can be computed from Φ by (7).

The displacement vector (u, v, w) is given by (2) with the deflection function f. It can be computed from (12) as a potential of a given vector field. In the case of multi-connected profile Ω, existence of the potential f is ensured by the integral boundary condition (14). Condition (15) yields uniqueness of the deflection f.

Let us complete the theory by dimensions of the quantities. The displacements u, v, w and deflection function f are in meters [m], strain tensor e is dimensionless, twist rate α is in [m^{-1}], Airy stress function Φ is in [m^2], moment of the cross-section J in [m^4], the twist section modulus W in [m^3], the sheer modulus μ, the stress tensor τ, the stress vector T in [$\mathrm{N\,m}^{-2}$] and the torque M in [N m].

3. Examples

In general, finding the exact solution to the Problem (P) is not simple. Thus we shall deal with the ring profile which was already solved in [2]. We shall compare results for this complete ring (annulus) with those of a 'broken' ring which is in fact a profile without hole. We derive also the solution for the incomplete ring of the angle β (annulus sector). We shall use the polar coordinates (ρ, φ) given by $y = \rho\,\cos\varphi$, $z = \rho\,\sin\varphi$. In the end we shall deal also with elliptic ring profiles.

3.1. Ring profile

Let us consider a ring profile Ω with the outer radius R and the inner radius r, $0 < r < R$, i.e. $\Omega = \{[y,z] \in \mathbf{R}^2 \mid r^2 < y^2 + z^2 < R^2\}$. In this case the solution can be taken from the case of full circle. Let us denote $\lambda = r/R$, i.e. $r = R\,\lambda$. Let us take the function

$$\Phi(y,z) = \frac{1}{2}\,(R^2 - y^2 - z^2) = \frac{1}{2}\,(R^2 - \rho^2)\ , \tag{27}$$

which is the solution to problem on the full circle profile $\Omega_0 = \{[y,z] \in \mathbf{R}^2 \mid y^2 + z^2 < R^2\}$.

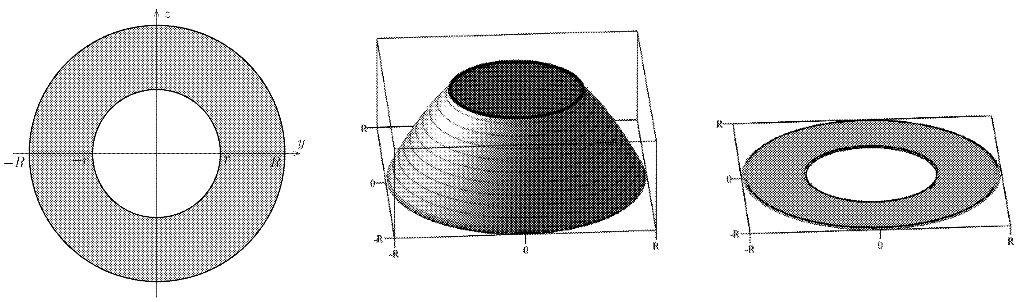

Fig.4: Ring profile, the corresponding stress function Φ and the zero deflection function f

Simple computation verifies that the function Φ satisfies the problem (16) with constant $c_1 = \frac{1}{2}\,(R^2 - r^2)$ and $|\Omega_1| = \pi\,r^2$. Indeed, the stress function Φ, in the polar coordinates $\Phi^*(\rho,\varphi) = \frac{1}{2}\,(R^2 - \rho^2)$ on Γ_1 has normal derivative $\frac{\partial}{\partial n}\Phi^*(r,\varphi) = -\frac{\partial}{\partial \rho}\Phi^*(r,\varphi) = r$ and thus

$$\int\limits_{\Gamma_1} \frac{\partial \Phi}{\partial n}\,\mathrm{d}s = |\Gamma_1| \cdot r = 2\pi\,r \cdot r = 2\,|\Omega_1|\ .$$

Let us calculate the other quantities. Using (23) simple computation yields

$$J = 2\left(\iint\limits_{\Omega} \Phi(y,z)\,\mathrm{d}y\,\mathrm{d}z + c_1\,|\Omega_1|\right) = \frac{\pi}{2}\,(R^4 - r^4) = \frac{\pi\,R^4}{2}\,(1 - \lambda^4)\ .$$

Since the maximum of the gradient $\nabla\Phi$ is on the outer boundary, using (26) and (25), the maximum stress $|T|_{\max}$ and the twist section modulus W equals

$$|T|_{\max} = \alpha\,\mu\,R\ , \qquad W = \frac{\pi}{R}\,(R^4 - r^4) = \pi\,R^3\,(1-\lambda^4)\,.$$

Finally (16) yields $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 0$, thus the deflection f is constant. Assuming zero mean value of f there is $f(y,z) = 0$, as can be expected.

3.2. Incomplete ring profile

In order to compare the results for the closed profile with results for an analogous open profile, we calculate solution to the problem for the β-angle segment of the previous ring, see Fig. 5. It is a simply connected profile without hole.

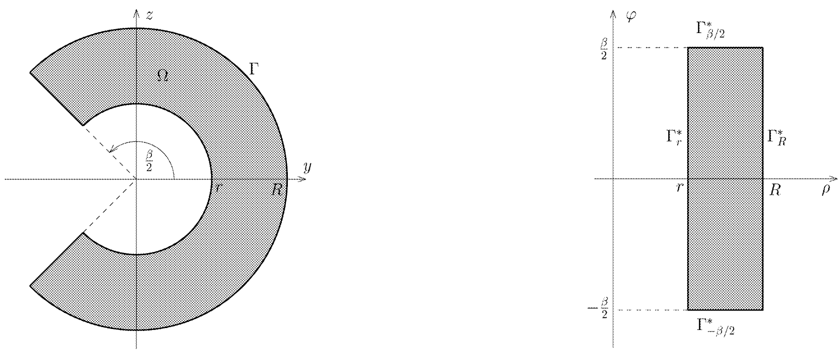

Fig.5: The angle segment ring profile in Cartesian and the polar coordinates

We transform the problem (16) into the polar coordinates $(\rho\,\varphi)$. The transformed function Φ in the polar coordinates will be denoted by Φ^*, i.e.

$$\Phi^*(\rho(y,z),\varphi(y,z)) = \Phi(y,z)\ .$$

The profile Ω in the polar coordinates is $\Omega^* = (r,R)\times(-\beta/2,\beta/2)$. Since the Laplace operator $\Delta = \partial_y^2 + \partial_z^2$ in the polar coordinates reads $\partial_\rho^2 + \frac{1}{\rho}\,\partial_\rho + \frac{1}{\rho^2}\,\partial_\varphi^2$, we obtain the equation

$$\frac{\partial^2\Phi^*}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial\Phi^*}{\partial\rho} + \frac{1}{\rho^2}\,\frac{\partial^2\Phi^*}{\partial\varphi^2} = -2 \quad \text{on } \Omega^* \tag{28}$$

completed with boundary conditions $\Phi^* = 0$ on the arches Γ_r^* and Γ_R^*, i.e.

$$\Phi^*(r,\varphi) = \Phi^*(R,\varphi) = 0 \quad \text{for } \varphi\in(-\tfrac{\beta}{2},\tfrac{\beta}{2}) \tag{29}$$

and also $\Phi^* = 0$ on both radiuses $\Gamma^*_{\pm\beta/2}$, i.e.

$$\Phi^*(\rho,-\tfrac{\beta}{2}) = \Phi^*(\rho,\tfrac{\beta}{2}) = 0 \quad \text{for } \rho\in(r,R)\ . \tag{30}$$

Approximate solution

The solution $\Phi(y,z) = \frac{1}{2}\,(R^2 - y^2 - z^2)$ of the ring profile, see (27), in polar coordinates $\Phi^*(\rho,\varphi) = \frac{1}{2}(R^2-\rho^2)$ satisfies $\Phi^* = 0$ on Γ_R^*. The simplest possibility is to correct it by a multiple of the elementary solution of the Laplace equation, in polar coordinate $\ln(\rho)$, such that $\Phi^* = 0$ also on Γ_r^*. Simple calculation yields (with the ratio $\lambda = r/R < 1$, $\ln\lambda < 0$)

$$\Phi_0^*(\rho,\varphi) = \frac{1}{2}\left[R^2 - \rho^2 + \frac{R^2 - r^2}{-\ln(r/R)}\ln\frac{\rho}{R}\right] = \frac{R^2}{2}\left[1 - \frac{\rho^2}{R^2} + \frac{1-\lambda^2}{-\ln\lambda}\ln\frac{\rho}{R}\right] . \tag{31}$$

The function Φ_0^* satisfies the equation and zero boundary condition on both Γ_r^* and Γ_R^*, the boundary conditions are not satisfied on $\Gamma_{\pm\beta/2}^*$.

Using (23) we can compute the corresponding approximate moment J_0

$$J_0 = 2\iint_\Omega \Phi_0(y,z)\,\mathrm{d}y\,\mathrm{d}z = 2\int_r^R \left(\int_{-\beta/2}^{\beta/2} \Phi_0^*(\rho,\varphi)\,\rho\,\mathrm{d}\varphi\right)\mathrm{d}\rho ,$$

which yields

$$J_0 = \frac{\beta}{4}\left[R^4 - r^4 + \frac{(R^2-r^2)^2}{\ln(r/R)}\right] = \frac{\beta R^4}{4}\left[1 - \lambda^4 - \frac{(1-\lambda^2)^2}{-\ln\lambda}\right] . \tag{32}$$

Maximum stress is in the middle point $(r,0)$ of the inner arch Γ_r. In the polar coordinates $|\nabla\Phi_0(r,0)| = |\frac{\partial\Phi_0^*}{\partial\rho}(r,0)|$. According to (26) we have

$$|T|_{\max} = \alpha\,\mu\,\frac{\partial\Phi_0^*}{\partial\rho}(r,0) = \alpha\,\mu\,R\left[\frac{1-\lambda^2}{-2\,\lambda\,\ln\lambda} - \lambda\right] .$$

Finally let us compute the approximate deflection f_0. The system (12) in the polar coordinates reads

$$\frac{\partial f^*}{\partial\rho} = \frac{1}{\rho}\frac{\partial\Phi^*}{\partial\varphi} , \qquad \frac{\partial f^*}{\partial\varphi} = -\rho\,\frac{\partial\Phi^*}{\partial\rho} - \rho^2 . \tag{33}$$

Inserting for Φ_0^* we obtain

$$\frac{\partial f_0^*}{\partial\rho} = 0 , \qquad \frac{\partial f_0^*}{\partial\varphi} = -R^2\,\frac{1-\lambda^2}{-2\,\ln\lambda}$$

and integration with condition (15) yields

$$f_0^*(\rho,\varphi) = -R^2\,\frac{1-\lambda^2}{-2\,\ln\lambda}\,\varphi . \tag{34}$$

Exact solution

To satisfy the boundary conditions (29) on Γ_R^* and Γ_r^*, in the previous approach we added two additional terms to the term $-\frac{1}{2}\rho^2$ and obtained an approximate solution Φ_0^*. To obtain the exact solution we shall use the Fourier series method with a different strategy. We start with the particular solution Φ_p^*

$$\Phi_p^*(\rho,\varphi) = \frac{\rho^2}{2}\left[-1 + \frac{\cos(2\varphi)}{\cos\beta}\right] , \tag{35}$$

which satisfies the equation (28), boundary conditions (30) on $\Gamma^*_{\pm\beta/2}$ for any angle $\beta \in (0, 2\pi)$ except for two cases $\beta = \pi/2$ and $\beta = 3\pi/2$, when $\cos\beta = 0$. We shall deal with these two singular cases later. Then we decompose the solution $\Phi^* = \Phi^*_p - \Phi^*_h$, where the subtracted homogeneous part Φ^*_h should satisfy the Laplace equation in the polar coordinates

$$\frac{\partial^2 \Phi^*_h}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi^*_h}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi^*_h}{\partial \varphi^2} = 0, \tag{36}$$

zero boundary conditions on $\Gamma^*_{\pm\beta/2}$ and the condition $\Phi^*_h = \Phi^*_p$ on Γ^*_R and Γ^*_r. It will be looked for in the form of a series of functions with separated variables

$$\Phi^*_h(\rho, \varphi) = \sum_k c_k\, F_k(\rho)\, G_k(\varphi)\ ,$$

where each product $F_k\, G_k$ satisfies the Laplace equation (36) and boundary conditions on $\Gamma^*_{\pm\beta/2}$. The constants c_k will be chosen such that the combination $\sum_k c_k\, F_k\, G_k$ satisfies the boundary condition on Γ^*_R and Γ^*_r. Inserting $F(\rho)\, G(\varphi)$ into the Laplace equation (36) and multiplying it with $\rho^2/(F(\rho)\, G(\varphi))$ we obtain

$$\frac{\rho^2\, F''(\rho) + \rho\, F'(\rho)}{F(\rho)} + \frac{G''(\varphi)}{G(\varphi)} = 0\,.$$

Since the first term is independent of φ and the second is independent of ρ, both terms are constant (denoted by $\pm\kappa$) and we obtain two ordinary differential equations

$$\rho^2\, F''(\rho) + \rho\, F'(\rho) - \kappa\, F(\rho) = 0\ , \qquad G''(\varphi) + \kappa\, G(\varphi) = 0\ .$$

General solution to the second equation is $G(\varphi) = a_1 \cos(p\varphi) + a_2 \sin(p\varphi)$. Boundary conditions $\Phi^*_h = 0$ on $\Gamma^*_{\pm\beta/2}$ yield $G(-\beta/2) = G(\beta/2) = 0$ which is satisfied for $a_2 = 0$ and $\cos(\pm p\,\beta/2) = 0$. The last equality gives $p\,\beta/2$ being equal to odd multiples of $\pi/2$. The constant a_1 can be chosen $a_1 = 1$. In this way we obtain a sequence of solutions

$$G_k(\varphi) = \cos(p_k\, \varphi)\ , \qquad p_k = \frac{(2k+1)\,\pi}{\beta}\ , \qquad \kappa_k = p_k^2\ , \qquad k = 0, 1, 2, 3, \ldots \tag{37}$$

Let us turn our attention to the first equation. It is the second order Euler's equation $x^2\, y'' + a_1\, x\, y' + a_0\, y = 0$. Its solution can be found in the form $y(x) = x^\nu$, where ν is a root of the polynomial $P(\nu) = \nu(\nu - 1) + a_1\, \nu + a_0$. In our case the polynomial is $P(\nu) = \nu^2 - \kappa_k$ and its roots are $\nu_{1,2} = \pm\sqrt{\kappa_k} = \pm p_k$. Thus the general solution is

$$F_k(\rho) = b_1\, \rho^{p_k} + b_2\, \rho^{-p_k}\ .$$

According to the boundary conditions on Γ^*_R and Γ^*_r let us we choose the constants b_1, b_2 such that $F_k(R) = \frac{1}{2}\, R^2$ and $F_k(r) = \frac{1}{2}\, r^2$. Simple computation with $r = R\,\lambda$ yields

$$F_k(\rho) = \frac{R^2}{2}\left[\frac{1 - \lambda^{p_k+2}}{1 - \lambda^{2p_k}} \left(\frac{\rho}{R}\right)^{p_k} + \frac{\lambda^{p_k+2} - \lambda^{2p_k}}{1 - \lambda^{2p_k}} \left(\frac{R}{\rho}\right)^{p_k}\right]. \tag{38}$$

Then all the boundary conditions $\Phi^*_h = \Phi^*_p$ on Γ^*_R and Γ^*_r will be satisfied if

$$\sum_{k=1}^{\infty} c_k \cos(p_k\, \varphi) = f(\varphi) \equiv -1 + \frac{\cos(2\varphi)}{\cos\beta}\,, \qquad \varphi \in \left(-\tfrac{\beta}{2}, \tfrac{\beta}{2}\right)\ .$$

We have to expand the function $f(\varphi) = -1 + \cos(2\varphi)/\cos\beta$ into the cosine Fourier series with basis $\{G_k\}_{k=0}^{\infty}$, where $G_k(\varphi) = \cos(p_k\,\varphi)$. Since the basis is orthogonal on the interval $(-\beta/2, \beta/2)$, the coefficients c_k can be computed by the formula, see e.g. [7], Section 16.3

$$c_k = \frac{1}{\|G_k\|^2_{L^2(I)}} \int\limits_I f(\varphi)\, G_k(\varphi)\, \mathrm{d}\varphi = \frac{2}{\beta} \int\limits_{-\beta/2}^{\beta/2} \left[-1 + \frac{\cos(2\varphi)}{\cos\beta}\right] \cos(p_k\,\varphi)\, \mathrm{d}\varphi \ .$$

Both f and G_k are even functions, thus we use $\int_{-\beta/2}^{\beta/2} f(\varphi)\, G_k(\varphi)\, \mathrm{d}\varphi = 2\int_0^{\beta/2} f(\varphi)\, G_k(\varphi)\, \mathrm{d}\varphi$. Using the formula $\cos x\, \cos y = \frac{1}{2}\,[\cos(x+y) + \cos(x-y)]$ the second term can be rewritten to $\cos(2\varphi)\, \cos(p_k\,\varphi) = \frac{1}{2}\, \cos((p_k+2)\,\varphi) + \cos((p_k-2)\,\varphi)$ and integration yields

$$c_k = -\frac{4}{\beta}\left[\frac{\sin(p_k\,\varphi)}{p_k}\right]_0^{\beta/2} + \frac{2}{\beta\,\cos\beta}\left[\frac{\sin((p_k+2)\,\varphi)}{p_k+2} + \frac{\sin((p_k-2)\,\varphi)}{p_k-2}\right]_0^{\beta/2} \ .$$

For $p_k = (2k+1)\,\pi/\beta$ we have $\sin(p_k\,\beta/2) = \sin(2k+1)\,\pi/2 = (-1)^k$ and $\sin((p_k \pm 2)\,\beta/2) =$ $= \sin((2k+1)\,\pi/2 \pm \beta) = (-1)^k\, \cos\beta$. Thus by simple calculation we obtain

$$\begin{aligned} c_k = -\frac{4\,(-1)^k}{\pi\,(2\,k+1)} + \frac{2\,(-1)^k}{(2\,k+1)\,\pi + 2\,\beta} + \frac{2\,(-1)^k}{(2\,k+1)\,\pi - 2\,\beta} &= \\ = \frac{16\,(-1)^k\,\beta^2}{(2\,k+1)\,\pi\,[(2\,k+1)^2\,\pi^2 - 4\,\beta^2]} & \ . \end{aligned} \tag{39}$$

The exact solution to the problem is

$$\Phi^*(\rho,\varphi) = \frac{\rho^2}{2}\left[-1 + \frac{\cos(2\varphi)}{\cos\beta}\right] - \sum_{k=0}^{\infty} c_k\, F_k(\rho)\, \cos(p_k\,\varphi) \ , \tag{40}$$

where c_k are given by (39), p_k by (37) and $F_k(\rho)$ by (38). For β in $(0, 2\pi)$ except for $\pi/2$, $3\pi/2$ the Weierstrass convergence criterion yields uniform converge of the series. Indeed, for $\rho \in (r, R)$ in (38) $\rho/R \le 1$ and $\lambda\, R/\rho = r/\rho \le 1$, the functions $F_k(\rho)$ are bounded by R^2, $G_k(\varphi) \le 1$ and c_k decays at rate k^{-3}.

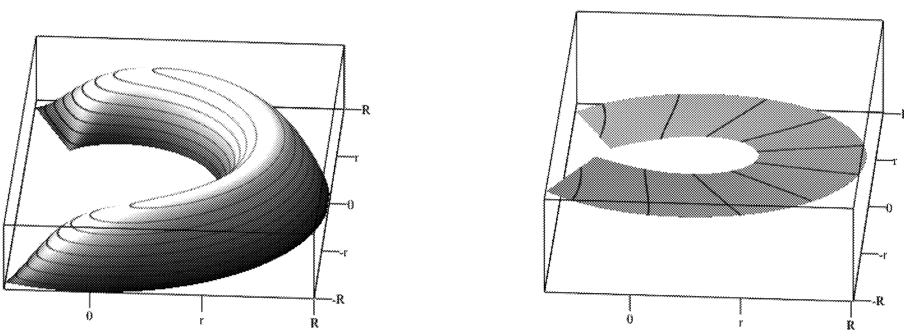

Fig.6: Stress function Φ and deflection function for $5\pi/4$-angle segment ring profile (annulus sector)

Let us compute the moment J. In the polar coordinates (ρ, φ) it is given by

$$J = 2 \iint_{\Omega} \Phi(y,z)\,\mathrm{d}y\,\mathrm{d}z = 2 \iint_{\Omega^*} \Phi^*(\rho,\varphi)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi$$

and simple computation yields

$$J = \frac{1}{4} R^4 (1-\lambda^4)(-\beta + \tan\beta) - 2 \sum_{k=0}^{\infty} c_k \int_r^R F_k(\rho)\,\rho\,\mathrm{d}\rho \int_{-\beta/2}^{\beta/2} G_k(\varphi)\,\mathrm{d}\varphi\ ,$$

where

$$\int_r^R F_k(\rho)\,\rho\,\mathrm{d}\rho = \frac{R^4}{2}\left[\frac{(1-\lambda^{p_k+2})^2}{(1-\lambda^{2p_k})(p_k+2)} + \frac{(\lambda^2-\lambda^{p_k})^2}{(1-\lambda^{2p_k})(p_k-2)}\right]$$

and

$$\int_{-\beta/2}^{\beta/2} G_k(\varphi)\,\mathrm{d}\varphi = \frac{2\beta}{(2k+1)\pi}(-1)^k\ ,$$

which yields

$$J = \frac{R^4(1-\lambda^4)}{4}(\tan\beta - \beta) - \frac{2^5 R^4 \beta^3}{\pi^2} \sum_{k=0}^{\infty} \frac{\frac{(1-\lambda^{p_k+2})^2}{(1-\lambda^{2p_k})(p_k+2)} + \frac{(\lambda^2-\lambda^{p_k})^2}{(1-\lambda^{2p_k})(p_k-2)}}{(2k+1)^2\,[(2k+1)^2\pi^2 - 4\beta^2]}\ . \tag{41}$$

Similarly, for $\beta \in (0, 2\pi) \setminus \{\pi/2, 3\pi/2\}$ one can verify that the series converges.

Due to the symmetry the maximal stress is attained at the point $[r, 0]$, i.e.

$$|T|_{\max} = \alpha\,\mu\,R\,\lambda \left[\left(-1 + \frac{1}{\cos\beta}\right) - \frac{1}{2} \sum_{k=0}^{\infty} c_k\,p_k\,\frac{2\lambda^{p_k-2} - \lambda^{2p_k} - 1}{1-\lambda^{2p_k}}\right]. \tag{42}$$

Finally let us compute the deflection f. The system (12) in the polar coordinates has the form (33). Inserting for Φ^* we obtain

$$\begin{aligned}
\frac{\partial f^*}{\partial \rho} &= -\rho\,\frac{\sin(2\varphi)}{\cos\beta} + \sum_{k=0}^{\infty} c_k\,\frac{F_k(\rho)}{\rho}\,p_k\,\sin(p_k\varphi)\ ,\\
\frac{\partial f^*}{\partial \varphi} &= -\rho^2\,\frac{\cos(2\varphi)}{\cos\beta} + \sum_{k=0}^{\infty} c_k\,F_k'(\rho)\,\rho\,\cos(p_k\varphi)\ .
\end{aligned}$$

Integrating each term separately we obtain

$$\begin{aligned}
f^*(\rho,\varphi) = &-\frac{\rho^2\,\sin(2\varphi)}{2\,\cos\beta} + \\
&+ \sum_{k=0}^{\infty} \frac{R^2 c_k}{2}\left[\frac{1-\lambda^{p_k+2}}{1-\lambda^{2p_k}}\left(\frac{\rho}{R}\right)^{p_k} - \frac{\lambda^{p_k+2}-\lambda^{2p_k}}{1-\lambda^{2p_k}}\left(\frac{R}{\rho}\right)^{p_k}\right]\sin(p_k\,\varphi).
\end{aligned} \tag{43}$$

Singular cases $\beta = \pi/2$ and $\beta = 3\pi/2$

Let us look at the singular cases. In the first case $\beta = \pi/2$ and $\cos\beta = \cos(\pi/2) = 0$ and the term in (40) is undefined due to zero in the denominator. At the same time also the coefficient c_0 has also zero $\pi - 2\beta = 0$ in the denominator of the third term of (39). Let us process the sum of these two singular terms. Nontrivial computation proves that limit $\beta \to \pi/2$ of sum of these two terms is finite. In general, formula for the limit is rather complicated, let us introduce the simpler case $\rho = R$

$$\lim_{\beta\to\frac{\pi}{2}} \left[\frac{\rho^2\,\cos(2\varphi)}{2\,\cos\beta} - c_0(\beta)\,F_0(\rho)\,G_0(\phi)\right]_{\rho=R} = R^2\,\frac{3\,\cos(2\varphi) - \pi + 4\,\varphi\,\sin(2\,\varphi)}{2\pi}\ .$$

The other terms are finite and form a converging series.

The same problem appears in the second case, when $\beta = 3\pi/2$. Again $\cos\beta = \cos(3\pi/2) = 0$ and the coefficient c_1 has $3\pi - 2\beta = 0$ in the denominator. A nontrivial computation proves that limit $\beta \to 3\pi/2$ of sum of these two terms is finite. In case $\rho = R$ the limit is

$$\lim_{\beta\to\frac{3\pi}{2}} \left[\frac{\rho^2\,\cos(2\varphi)}{2\,\cos\beta} - c_1(\beta)\,F_1(\rho)\,G_1(\phi)\right]_{\rho=R} = -R^2\,\frac{3\,\cos(2\varphi) + 6\,\pi + 4\,\varphi\,\sin(2\,\varphi)}{6\pi}\ .$$

Similar problem appears in (41) for J, (42) for $|T|_{\max}$ and (43) for deflection. In all these situations pair of the singular terms must be replaced by the corresponding limit.

3.3. Comparison of the approximative and the exact solution

Let us compare the approximative and the exact solution of angular segment of the ring profile. Approximate Airy stress function Φ_0 is given by (31), the exact Φ is given by (40). For $\lambda = 1/2$ and $\beta = 2\pi$ the functions are plotted on Fig. 7.

More instructive is to compare the moments J for various angle β. Constants $K = K(\beta, \lambda)$ of $J = K \cdot R^4$ for constant ratio $\lambda = 1/2$ are in the table 1 and for various ratios $\lambda = r/R$ with constant angle $\beta = \pi$ in the table 2.

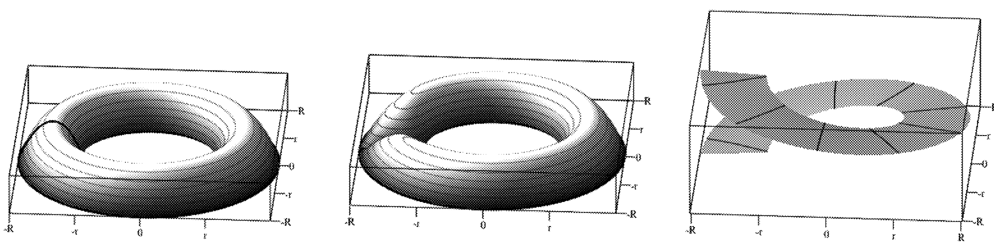

Fig.7: The approximate and the exact function Φ and the exact deflection function for 2π-angle ring profile

Angle β	$\frac{\pi}{8} \approx 22.5°$	$\frac{\pi}{4} \approx 45°$	$\frac{\pi}{2} \approx 90°$	$\pi \approx 180°$	$\frac{3\pi}{2} \approx 270°$	$2\pi \approx 360°$
Approximate	0.01237	0.02474	0.04947	0.09895	0.1484	0.19790
Exact	0.00270	0.01197	0.03600	0.08545	0.1349	0.18440

Tab.1: Constants of the moment J for various angle β and ratio $\lambda = 1/2$

Ratio $\lambda = r/R$	0.3	0.5	0.7	0.8	0.9	0.95
Approximate	0.2388	0.0989	0.02408	0.00755	0.000995	0.0001276
Exact	0.1843	0.0855	0.02237	0.00721	0.000974	0.0001263

Tab.2: Constants of the moment J for various ratio $\lambda = r/R$ and angle $\beta = \pi \approx 180°$

3.4. Comparison of solutions for the ring and the broken ring

Let us briefly compare the complete ring profile, which is a double connected domain, and a broken ring profile, which is a simply connected domain. The corresponding stress functions Φ are on Fig. 4b and Fig. 7b, the deflections f are on Fig. 4c and Fig. 7c.

Finally we can compare the moment J for different ratios λ with full and broken ring, constants $K = K(\beta, \lambda)$ of $J = K \cdot R^4$ for different ratio r/R are in the table 3.

Ratio $\lambda = r/R$	0.3	0.5	0.7	0.8	0.9	0.95
Full ring	1.5581	1.4726	1.936	0.92740	0.540197	0.291373
Broken ring (exact)	0.4231	0.1844	0.0464	0.01476	0.001969	0.000254

Tab.3: Constants for the moment J of the complete (unbroken) and broken ring for various ratio $\lambda = r/R$

3.5. Elliptic ring

Similar study can be made for elliptic rings, i.e. for the profile

$$\Omega = \left\{ [y,z] \in \mathbf{R}^2 \;\middle|\; \lambda^2 < \frac{y^2}{a^2} + \frac{z^2}{b^2} < 1 \right\} ,$$

where $a, b > 0$ are the half-axes of the ellipse and $\lambda > 0$ is ratio of the inner and the outer ellipse radius. Its outer boundary Γ_0 is the ellipse with half-axes a and b and the inner boundary Γ_1 is the ellipse with half-axes $a\,\lambda$ and $b\,\lambda$ encircling the hole Ω_1. Again, we can use the stress function from the full elliptic profile

$$\Phi(y,z) = \frac{a^2\,b^2}{a^2+b^2} \left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2} \right) .$$

We shall use the elliptic polar coordinates $y = a\,\rho\cos\varphi$ and $z = b\,\rho\sin\varphi$ with the Jacobian $a\,b\,\rho$. The profile Ω is converted to $\Omega^* = (\lambda, 1) \times (-\pi, \pi)$. Let us compute the moment. With $c_1 = (1-\lambda^2)\,a^2\,b^2/(a^2+b^2)$ and $|\Omega_1| = \lambda^2\,\pi\,a\,b$ we obtain

$$J = 2 \iint_{\Omega^*} \Phi^*(\rho,\varphi)\,a\,b\,\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi + 2\,c_1\,|\Omega_1| = \pi\,\frac{a^3\,b^3}{a^2+b^2}\,(1-\lambda^4) .$$

For $a > b$ the maximal stress $|T|_{\max} = 2\,\alpha\,\mu\,a^2\,b/(a^2+b^2)$ is in points $[0, \pm b]$ and the deflection function is $f(y,z) = -y\,z\,(a^2-b^2)/(a^2+b^2)$.

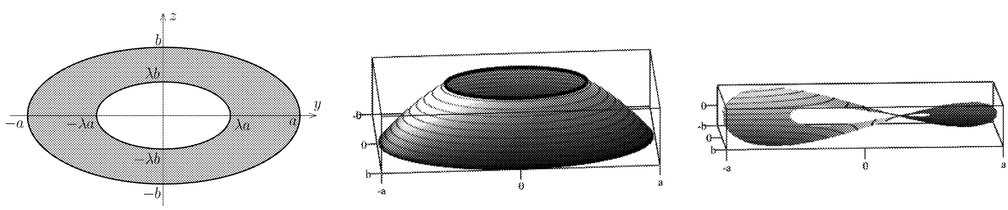

Fig.8: Elliptic (a:b=2:1) ring profile and its stress Φ and deflection f function

4. Conclusion

The first part of the paper dealt with the mathematical model of torsion of a bar with profile with holes. It led to a boundary value problem for the Airy stress function Φ. Case of the profile with holes, i.e. multiply connected domain, has brought several difficulties: the constants c_i on the boundary of holes were undefined which led to loss of uniqueness of the solution Φ. The problem has been solved by potentiality conditions for the deflection function f which led to the integral condition on boundary holes and ensured uniqueness of solutions to the Problem (P). It also has given physical meaning to these additive integral conditions coming from the variational formulation.

The second part was devoted to examples. To compare solutions of the complete ring and the broken ring profiles we derived exact solution to angular segments of ring (annulus sector) profile in form of a Fourier series. Two angles $\beta = \pi/2$ and $\beta = 3\pi/2$ led to problems: two members of the series were undefined – zero denominator. Fortunately their sum has finite limit for β tending to these critical values. Thus we have obtained solution for any angle $\beta \in (0, 2\pi\rangle$. The approximate solutions which do not satisfy zero boundary condition on radiuses $\Gamma_{\pm\beta/2}$ of the ring segment are easy to compute. But, according to numerical experiments in Tables 1,2, they yielded higher values, particularly for small angles. Table 3 also showed that solution to complete (unbroken) and broken ring profiles differ substantially, as could be expected. In the end the elliptic ring profile was computed.

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