OPTIMAL PPF CONTROLLER FOR MULTIMODAL VIBRATION SUPPRESSION

Štefan Fenik, Ladislav Starek*

Positive Position Feedback (PPF) is one of the most attractive vibration control method due to its stability and ease of implementation. On the other hand, low robustness makes the PPF design more complicated in multimodal control case. It is known that a little change in optimal parameters setup, especially the change in controller frequency, can strongly degrade the control effort. Thus knowing a good approximation of optimal PPF parameters can be very helpful in practical implementations and simplified analytical relations between optimal parameters and modal properties of the structure are inevitable for efficient control design. In this paper derivation of such relations is introduced, based on simplified transfer function of controlled structure. Furthermore influence of the parallel PPF controllers in multimodal vibration suppression is analyzed and formulae for optimal parameters updating are suggested. Optimal multimodal PPF control design is demonstrated on experimental example of vibration suppression of beam structure.

Key words : *vibration suppression, PPF control, root-locus method, optimization, piezoactuator*

1. Introduction

Structural vibration control has experienced rapid development in the last 30 years. Although there have been many new algorithms for vibration suppression developed, the most useful and attractive technique is still Positive Position Feedback (PPF), which has achieved much success because of its stability and ease of implementation. PPF controller, basically a special form of second order compensator, was first introduced by Caughey and Goh [1], who also published a study com- paring collocated velocity feedback to PPF [2]. In [3] authors showed that PPF is capable of controlling several vibration modes simultaneously and has global stability conditions, which are easy to fulfil even in the presence of actuator dynamics. On the other hand, as any narrow-band active control system, PPF reaches its best results if tuned properly to the characteristics of the structure to be controlled. The PPF transfer function contains three parameters to be determined in designing the control law for each mode. The three parameters are gain, filter damping ratio and frequency. To determine the values of these parameters a tuning principle is often used. Most authors suggest a value for the filter frequency slightly greater than the structural frequency to be damped. While in [4] authors specified a factor of 1.3, in [5] author chose 1.45. The range for the filter damping ratio found in the literature reaches from 0.01 to 0.5. In many experiments described in the literature the authors would find a compromise value for the damping ratio, and leave it constant through all their experiments. Some authors ([2], [5])

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used a pole placement technique to compute the three parameters that ensures minimal gains at a maximum closed-loop damping ratio. However, an exact model of the poles of the structure to be damped is required. Other authors only used information about the frequency of the pole to be damped and determine the parameters with a trial and error technique experimentally. Goh and Lee realized that effective vibration control with PPF depends on the accuracy of the modal parameters used in the control design [6]. They extended the original feedback technique with an adaptive estimation procedure to identify the structural parameters and presented design parameters for the PPF filter depending on the structural damping ratios and frequencies to achieve the maximum amount of damping. However, for the multi-mode case and persistent excitation, it is shown that simultaneous parameter estimation and control may possibly lead to erroneous results. Another technique to adaptively tune the PPF filter parameters in real time was suggested in [7], where genetic algorithm (GA) was utilized to adapt the filter frequency of a single second order filter in a simulation. The filter damping ratio and the gain were kept constant. McEver [8] states that the optimal parameters can be derived from the ratio of the structural zero frequencies to the structural poles. He proposes a very simple algorithm for obtaining the two PPF filter parameters damping and frequency depending on gain, which is related to stability margin. In this way, McEver was able to automate the tuning process of one second-order PPF filter, while keeping the gain constant [8].

In this paper, there are presented new relations for calculation of all three parameters for optimal single PPF controller. These relations take into account not only the pole and zero of the desired mode, but also the natural damping and dynamical stiffness, which determines the gain as an absolute parameter. Derivation of the relations is based on numerical root-locus analysis and proper simplification of frequency response function (FRF) in the narrow band within the controlled mode of vibration. In the next step multimodal PPF control is analyzed and simple correction formulae for optimal parameters updating are derived. Experimental example of multimodal vibration suppression is also presented.

2. Mathematical modelling

The best suited mathematical model for analysing the vibration suppression problem in practical technical applications is well-known Finite Element Method (FEM) model of the form:

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{D}\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} = \mathbf{B}\,\mathbf{u}\ , \tag{1}$$

where $\mathbf{q}$ is the vector of generalized displacements, $\mathbf{M}$ is the symmetric positive definite mass matrix, $\mathbf{D}$ is the symmetric positive semi-definite viscous damping matrix, $\mathbf{K}$ is the symmetric positive (semi-)definite stiffness matrix, $\mathbf{u}$ is the vector of control variables, $\mathbf{B}$ is the input matrix, which transforms control variables to generalized actuator forces.

The measured output needed for control purposes can be represented, in the case of position feedback, by linear combination of the generalized displacements:

$$\mathbf{y} = \mathbf{C}\,\mathbf{q}\ , \tag{2}$$

where $\mathbf{C}$ is corresponding output matrix for position measurement.

Providing a modal analysis of the system (1) without damping (i.e. $\mathbf{D} = \mathbf{0}$) we obtain the values of undamped natural frequencies and set of appropriate mode shape vectors, which can be easily normalized to fulfil the following statements:

$$\mathbf{V}^\mathrm{T}\,\mathbf{M}\,\mathbf{V} = \mathbf{I}\ , \qquad \mathbf{V}^\mathrm{T}\,\mathbf{K}\,\mathbf{V} = \mathbf{\Omega}\ , \qquad \mathbf{\Omega} = \mathrm{diag}(\omega_{n1}^2, \omega_{n2}^2, \dots, \omega_{ni}^2, \dots, \omega_{nn}^2)\ , \tag{3a,b}$$

where $\mathbf{V}$ is real and non-singular modal matrix, which columns are normalized mode shape vectors of undamped system (1), $\mathbf{I}$ is an identity matrix, ω_{ni} is the i-th undamped natural frequency, $\mathbf{\Omega}$ is diagonal matrix containing the squares of ω_{ni}. All matrices in (3) are square of dimension n, where n is number of DOFs (i.e. length of the vector $\mathbf{q}$).

In the case of modal viscous damping, which is very close to the case of damping behaviour of all lightweight structures with no passive dampers or dashpots, modal transformation of the damping matrix gives:

$$\mathbf{V}^\mathrm{T}\,\mathbf{D}\,\mathbf{V} = 2\,\mathbf{\Delta}\ , \qquad \mathbf{\Delta} = \mathrm{diag}(\delta_1, \delta_2, \dots, \delta_i, \dots, \delta_n)\ , \tag{4}$$

where δ_i is modal damping constant, i.e. product of damping ratio and undamped natural frequency ($\delta_i = \zeta_i\,\omega_{ni}$), $\mathbf{\Delta}$ is diagonal matrix containing δ_i.

An identification of modal properties mentioned above is usually based on Experimental Modal Analysis (EMA) in given frequency range. As analytical FRFs have to match with their experimental counterparts, the estimation of undamped frequencies, modal damping constants and some components of mode shape vectors can be done by proper optimization technique, based on fact that analytical FRFs for the system (1) has the form of truncated partial fraction expansion. Bearing in mind only frequency band of given modes one can take the influence of the lower (including rigid body modes) and upper frequencies into account like this [9]:

$$H_{rs}(\omega) = -\frac{R_{0rs}}{\omega^2} + \sum_{i=1}^{m} \frac{R_{irs}}{-\omega^2 + 2\,\mathrm{j}\,\omega\,\delta_i + \omega_{ni}^2} + Z_{irs}\ , \tag{5}$$

where $H_{rs}(\mathrm{j}\,\omega)$ is the FRF for the s-th input a the r-th output, R_{irs} is so called residue for the i-th mode within the given input-output pair, δ_i and ω_{ni} are the damping constant ($\delta_i = \zeta_i\,\omega_{ni}$) and undamped natural frequency for the i-th mode, R_{0rs} is the residue representing rigid body modes (if present), Z_{irs} is the constant representing the contribution of higher modes, m is the number of vibration modes within the frequency band of interest.

When modal viscous damping (4) is present the residues of FRF (5) are all real constants and can be computed as (see Appendix A):

$$R_{irs} = \mathbf{c}_r\,\mathbf{v}_i\,\mathbf{v}_i^\mathrm{T}\,\mathbf{b}_s\ , \tag{6}$$

where $\mathbf{b}_s$ is the s-th column of the input matrix $\mathbf{B}$, $\mathbf{c}_r$ is the r-th row of the output matrix $\mathbf{C}$, $\mathbf{v}_i$ is normalized mode shape vector of the i-th mode (i.e. column of modal matrix $\mathbf{V}$).

If we analyze the control performance in the narrow frequency band **within only one given mode**, we can simplify the FRF (5) and convert it into a transfer function in the following way:

$$H_{rs}(s) \doteq \frac{R_{irs}}{s^2 + 2\,\delta_i\,s + \omega_{ni}^2} + Z_{irs} = Z_{irs}\,\frac{s^2 + 2\,\delta_i\,s + \omega_{ni}^2 + \dfrac{R_{irs}}{Z_{irs}}}{s^2 + 2\,\delta_i\,s + \omega_{ni}^2} \quad \text{for} \quad s \approx \mathrm{j}\,\omega_{ni}\ , \tag{7}$$

where $s = \mathrm{j}\,\omega$ is imaginary variable (assuming the poles are stable), Z_{irs} is real constant representing the contribution of the rest of vibration modes (in both lower and upper frequency range).

In the numerator of (7) there is (same as in the denominator) polynomial of the second order, which roots are zeros of the transfer function. Imaginary part of the zero plays important role in the control design and for lightly damped modes it can be expressed as:

$$\mathrm{Im}(s_\mathrm{Z}) \doteq \omega_{\mathrm{ar}i} = \sqrt{\omega_{ni}^2 + \frac{R_{irs}}{Z_{irs}}} \quad \text{if} \quad \omega_{ni}^2 + \frac{R_{irs}}{Z_{irs}} \geq 0 \;, \tag{8}$$

where s_Z is the zero of the transfer function (7), $\omega_{\mathrm{ar}i}$ is antiresonant circular frequency for the i-th undamped mode.

If the antiresonant frequency can be easily determined from FRF (either analytical or experimental) the constant Z_{irs} will be simply:

$$Z_{irs} \doteq \frac{R_{irs}}{\omega_{\mathrm{ar}i}^2 - \omega_{ni}^2} \;. \tag{9}$$

In a special case of collocated actuator-sensor arrangement, which is the crucial case in the active control of vibration, the FRFs have interesting property of alternating resonances and antiresonances. Actually, it is easy to show that for collocated actuator-sensor pair all residues of given FRF have the same sign:

$$\mathbf{c}_s = \alpha\,\mathbf{b}_s^\mathrm{T} \Rightarrow R_{iss} = \mathbf{c}_s\,\mathbf{v}_i\,\mathbf{v}_i^\mathrm{T}\,\mathbf{b}_s = \alpha\,(\mathbf{v}_i^\mathrm{T}\,\mathbf{b}_s)^2 \geq 0 \quad \forall i,s \quad \text{if} \quad \alpha \geq 0 \;, \tag{10}$$

where R_{iss} is the residue of the i-th mode for collocated input-output pair within the s-th control variable, α is constant ratio.

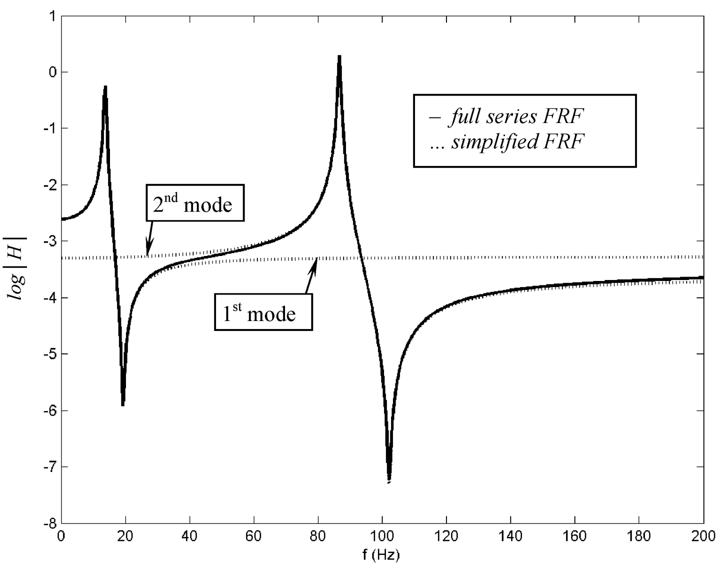

Fig.1: Typical FRF of beam structure with well-separated modes

Thus from (10) and (5) it is obvious that real part of $H_{ss}(\mathrm{j}\,\omega)$ changes its sign just once between two neighbouring resonances, which implies the existence of sharp antiresonance for low natural damping. The frequency bandwidth of accurate approximation of FRF depends, in general, on resonant frequencies and natural damping of neighbouring modes. Figure 1 shows the typical FRF of beam structure with well-separated modes. For such type of structure the simplified FRF (7) gives very good approximation of full series FRF in sufficient frequency range within given mode.

Let's now discuss the closed-loop control problem in the terms of transfer function analysis. In single-input, single-output (SISO) case (see Figure 2) the closed-loop transfer can be expressed as:

$$Y(s) = \frac{H_{\mathrm{MS}}(s)}{1 - H_{\mathrm{MS}}(s)\, H_{\mathrm{C}}(s)}\, D(s) \ , \tag{11}$$

where $Y(s)$ is the Laplace transform of the output variable y, $D(s)$ is the Laplace transform of the input disturbance d, $U(s)$ is the Laplace transform of the control variable u, $H_{\mathrm{MS}}(s)$ is the transfer function of the mechanical structure (i.e. open-loop transfer function), $H_{\mathrm{C}}(s)$ is the transfer function of the controller $(U(s)/Y(s))$.

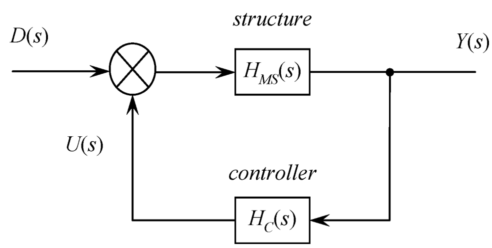

Fig.2: SISO control loop with positive feedback

If both the controller and the structure are linear systems the transfer functions in (11) are rational functions of s:

$$H_{\mathrm{MS}}(s) = \left[\frac{Y(s)}{D(s)}\right]_{U=0} = \frac{A_{\mathrm{MS}}(s)}{B_{\mathrm{MS}}(s)} \ , \qquad H_{\mathrm{C}} = \frac{U(s)}{Y(s)} = \frac{A_{\mathrm{C}}(s)}{B_{\mathrm{C}}(s)} \ , \tag{12}$$

where A_{MS} (B_{MS}) is the polynomial numerator (denominator), which roots are zeros (poles) of $H_{\mathrm{MS}}(s)$, A_{C} (B_{C}) is the polynomial numerator (denominator), which roots are zeros (poles) of $H_{\mathrm{C}}(s)$.

Thus the transfer between the disturbance d and the measured output y takes the form:

$$\frac{Y(s)}{D(s)} = \frac{A_{\mathrm{MS}}\, B_{\mathrm{C}}}{B_{\mathrm{MS}}\, B_{\mathrm{C}} - A_{\mathrm{MS}}\, A_{\mathrm{C}}} \ . \tag{13}$$

According to the fact that only poles of (13) are needed to determine overall dynamical behaviour of the controlled structure, following characteristic equation is vital for *root-locus* based control design:

$$B_{\mathrm{MS}}\, B_{\mathrm{C}} - A_{\mathrm{MS}}\, A_{\mathrm{C}} = 0 \ . \tag{14}$$

If transfer function of the controller has simple form (similar to transfer function (7)), analytical relations between optimal control parameters and modal properties (i.e. parameters of simplified FRF) can be derived by solving equation (14) with respect to given *root-locus* objectives. For instance, in vibration suppression problem, the objective is to add the maximum damping, which is represented by the real part of appropriate root of (14), to given mode of vibration.

3. PPF control

In the area of structural vibration suppression, the technique with perhaps the greatest immunity from the destabilizing effects of spillover is collocated direct velocity feedback, which, in the absence of actuator dynamics, is unconditionally stable. In the presence of actuator dynamics, however, instability may result if a priori precaution is not taken. It has been shown that the stability boundary of modes near the natural frequency of the actuators critically dependent on the inherent natural damping in these modes a quantity not well known in most cases. In addition, the technique requires rate measurement a quantity that becomes vanishingly small at low frequencies.

The technique implemented in this contribution, *Positive Position Feedback* (PPF), was originally suggested by Caughey and Goh [2], as an alternative to collocated direct velocity feedback. Like velocity feedback, the method is not sensitive to spillover but in addition, it is not destabilized by finite actuator dynamics. PPF requires only generalized displacement measurements which makes it amenable to a strain-based sensing approach. While PPF is not unconditionally stable, the stability condition is non-dynamic and minimally restrictive [2].

Single PPF controller, working as vibration compensator, is essentially an auxiliary system similar to the mechanical vibration absorber. The compensator is driven by position of the structure, thus, the output variable y takes form (2), while the control variable u is proportional to the coordinate (position) of compensator and fed back to the structure with positive gain (hence the name of this method). When parameters of the single compensator are properly tuned desired amount of damping is added in narrow frequency range (usually within only one mode of interest). For multimodal control an appropriate number of parallel compensators are needed. In such case the mathematical model of the PPF controller is in the matrix form:

$$\mathbf{q}_\mathrm{c} + 2\,\boldsymbol{\Delta}_\mathrm{c}\,\mathbf{q}_\mathrm{c} + \boldsymbol{\Omega}_\mathrm{c}\,\mathbf{q}_\mathrm{c} = \mathbf{B}_\mathrm{c}\,\mathbf{C}\,\mathbf{q}\ , \tag{15}$$

where $\mathbf{q}_\mathrm{c}$ is the vector of controller modal coordinates, $\boldsymbol{\Delta}_\mathrm{c}$ is diagonal matrix containing damping constants (products of damping ratios and natural frequencies) of the controller, $\boldsymbol{\Omega}_\mathrm{c}$ is diagonal matrix containing the squares of controller natural frequencies, $\mathbf{C}$ is the output matrix for displacements, $\mathbf{q}$ is displacement vector of the structure, $\mathbf{B}_\mathrm{c}$ is the controller input matrix.

Now the control law is:

$$\mathbf{u} = \mathbf{F}\,\boldsymbol{\Omega}_\mathrm{c}\,\mathbf{q}_\mathrm{c}\ , \tag{16}$$

where $\mathbf{F}$ is the constant gain matrix, $\mathbf{u}$ is the vector of control variables.

State space model of mechanical structure with multimodal PPF controller can be derived from the equation of motion (1) and equations (15) and (16). After extension of the displacement vector $\mathbf{q}$ by the controller coordinates $\mathbf{q}_\mathrm{c}$ we get:

$$\hat{\mathbf{M}}\,\ddot{\hat{\mathbf{q}}} + \hat{\mathbf{D}}\,\dot{\hat{\mathbf{q}}} + \hat{\mathbf{K}}\,\hat{\mathbf{q}} = \mathbf{0} \quad \text{for} \quad \hat{\mathbf{q}} = \begin{bmatrix} \mathbf{q} \\ \mathbf{q}_\mathrm{c} \end{bmatrix}\ , \tag{17}$$

where:

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} , \qquad \hat{\mathbf{D}} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & 2\,\boldsymbol{\Delta}_\mathrm{c} \end{bmatrix} , \qquad \hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & -\mathbf{B}\,\mathbf{F}\,\boldsymbol{\Omega}_\mathrm{c} \\ -\mathbf{B}_\mathrm{c}\,\mathbf{C} & \boldsymbol{\Omega}_\mathrm{c} \end{bmatrix} . \tag{18}$$

Now if the state vector is given as:

$$\mathbf{x} = \begin{bmatrix} \hat{\mathbf{q}} \\ \dot{\hat{\mathbf{q}}} \end{bmatrix} \tag{19}$$

the state-space representation is:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} , \qquad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\hat{\mathbf{M}}^{-1}\,\hat{\mathbf{K}} & -\hat{\mathbf{M}}^{-1}\,\hat{\mathbf{D}} \end{bmatrix} . \tag{20}$$

Analysing the eigenvalues of state matrix $\mathbf{A}$ we can find the optimal values of the controller parameters with respect to the control objective. Using only single PPF to damp only one mode we have three parameters: matrix $\mathbf{F}$ reduced to gain K_p, and matrices $\boldsymbol{\Delta}_\mathrm{c}$ and $\boldsymbol{\Omega}_\mathrm{c}$ reduced to damping constant δ_c $(= \zeta_\mathrm{c}\,\omega_\mathrm{c})$ and controller natural frequency ω_c.

Let's now analyze, for example, the eigenvalues of the free-free beam (see section 6 for details) with the lowest deformation mode controlled by single PPF controller with collocated actuator-sensor pair. As we have three independent parameters, it will be useful to keep one of them constant and search the traces of appropriate closed-loop eigenvalues in Gauss plane as the result of the changes in the remaining two parameters. At first, we start with constant gain, set to $K_\mathrm{p} = -1$ ($K_\mathrm{p}\,R_{irs} > 0$ for PPF). Tuning the controller damping δ_c and the natural frequency ω_c in valid range we can find the eigenvalues related to the controlled mode of the beam and the eigenvalues of PPF controller moving along the *root-loci* curves plotted in Figure 3.

Bold lines belong to approximately optimal values, for wich the maximum of closed-loop damping of the controlled mode is achieved (at intersection). We can see that for constant

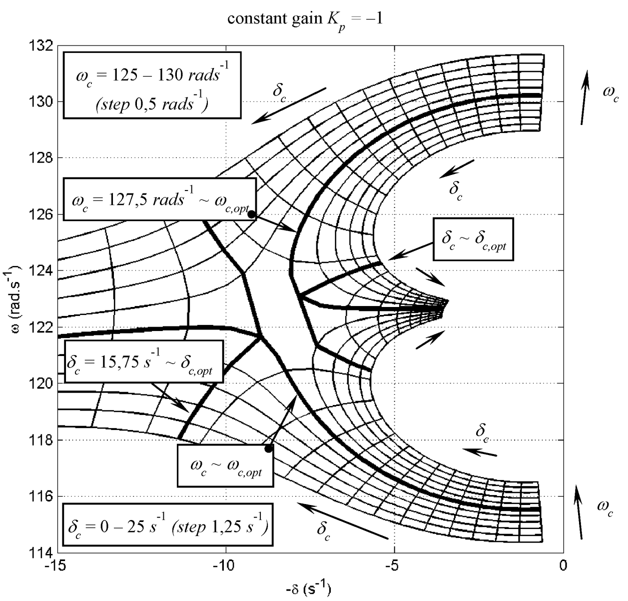

Fig.3: Root-loci curves for free-free beam with single PPF – constant gain (subscript 'opt' for approximately optimal values)

gain $K_\mathrm{p} = -1$ the closed-loop damping has distinct maximum for δ_c between $14.5\,\mathrm{s}^{-1}$ and $15.75\,\mathrm{s}^{-1}$. For δ_c greater than $15.75\,\mathrm{s}^{-1}$ closed-loop damping of the controller is growing, but closed-loop damping of the controlled mode is decreasing very quickly. In the next, let us try to plot *root-loci* while the damping is kept constant near the optimal value from Figure 3. We obtain the curves presented in Figure 4. Here again bold lines belong to approximately optimal values. Now we can see that for gain $|K_\mathrm{p}| < 1$ the closed-loop damping of the controlled mode is decreasing for any value of ω_c. On the other hand for gain $|K_\mathrm{p}| > 1$ the closed-loop damping of the controlled mode can grow only at the cost of decreasing the closed-loop damping of the controller. Furthermore, the average damping can not exceed the value of approximately $0.5\,\delta_\mathrm{c}$. Gain $|K_\mathrm{p}| = 1$ is the minimum gain for which the closed-loop damping of both modes is on the same level and equals to $0.5\,\delta_\mathrm{c}$.

Now results of numerical *root-locus* analysis can be summarized as follows:

1. When the gain is constant, eigenvalue related to the mode of the structure lies always in the interior of the circle, which has centre located in the pole of uncontrolled structure and which radius depends on the value of the gain. For positive feedback ($K_\mathrm{p}\,R_{irs} > 0$) damping of this mode is increased.
2. When the damping of the structure is increasing, regardless of which parameter is being adjusted, controlled eigenvalue is approaching toward the eigenvalue of the controller. **In the optimal case of vibration suppression, when the maximum damping of both modes is desired, the appropriate eigenvalues coincide**. This very important property allows us to derive the formulae for computing the optimal parameters in the next section.

4. Optimal PPF control for single-mode vibration suppression

From the numerical analysis presented in previous section we can see that little change in optimal parameters setup, especially the change in frequency ω_c, can strongly degrade the

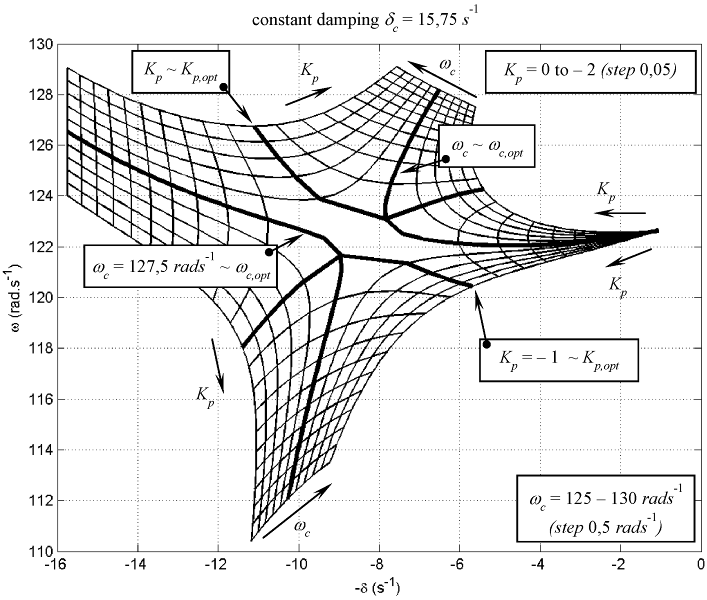

Fig.4: Root-loci curves for free-free beam with single PPF – constant damping (subscript 'opt' for approximately optimal values

control effort. Thus knowing the good approximation of optimal parameter values can be very helpful in practical implementations. Hereby, according to the computational intensity of numerical *root-locus* analysis, simplified analytical relations between optimal parameters and modal properties of the structure are inevitable for efficient control design.

Going out from simplified FRF (7) and taking the transfer function of single PPF in the form:

$$H_{\rm PPF}(s) = \frac{K_{\rm p}\,\omega_{\rm c}^2}{s^2 + 2\,\delta_{\rm c}\,s + \omega_{\rm c}^2} = \frac{A_{\rm C}}{B_{\rm C}} \tag{21}$$

we have polynomial characteristic equation (14) of the fourth order:

$$(s^2 + 2\,\delta_i\,s + \omega_{ni}^2)\,(s^2 + 2\,\delta_{\rm c}\,s + \omega_{\rm c}^2) - K_{\rm p}\,Z_i\,\omega_i^2\,(s^2 + 2\,\delta_i\,s + \omega_{{\rm ar}i}^2) = 0\ , \tag{22}$$

where Z_i is real constant representing the contribution of the rest of vibration modes from (9) (index rs dropped for simplicity), δ_i is damping constant of the i-th mode, ω_{ni} is the i-th natural frequency and $\omega_{{\rm ar}i}$ is antiresonant circular frequency for the i-th undamped mode.

Assuming there are two pairs of stable complex conjugate poles we can write eq. (22) in the form:

$$(s^2 + 2\,\delta_{i{\rm r}}\,s + \omega_{ni{\rm r}}^2)\,(s^2 + 2\,\delta_{\rm cr}\,s + \omega_{\rm cr}^2) = 0 \tag{23}$$

where $\delta_{i{\rm r}}$ and $\omega_{ni{\rm r}}$ are closed-loop damping constant and frequency of the i-th mode, $\delta_{\rm cr}$ and $\omega_{\rm cr}$ are closed-loop damping constant and frequency of the controller (real positive values).

Extracting and comparing the corresponding coefficients of polynomials (22) and (23) we get four equations:

$$\begin{aligned} \delta_{i{\rm r}} + \delta_{\rm cr} &= \delta_i + \delta_{\rm c}\ ,\\ \omega_{\rm cr}^2 + \omega_{ni{\rm r}}^2 + 4\,\delta_{i{\rm r}}\,\delta_{\rm cr} &= \omega_{\rm c}^2\,(1 - K_{\rm p}\,Z_i) + \omega_{ni}^2 + 4\,\delta_i\,\delta_{\rm c}\ ,\\ \delta_{i{\rm r}}\,\omega_{\rm cr}^2 + \delta_{\rm cr}\,\omega_{ni{\rm r}}^2 &= \delta_i\,\omega_{\rm c}^2\,(1 - K_{\rm p}\,Z_i) + \delta_{\rm c}\,\omega_{ni}^2\ ,\\ \omega_{\rm cr}^2\,\omega_{ni{\rm r}}^2 &= \left[1 - K_{\rm p}\,Z_i\,\frac{\omega_{{\rm ar}i}^2}{\omega_{ni}^2}\right]\omega_{ni}^2\,\omega_{\rm c}^2\ . \end{aligned} \tag{24a–d}$$

As we have mentioned in previous section, maximum damping is achieved when controlled pole of the structure coincides with pole of the controller (see Appendix B for proof). Hence two conditions can be stated to reduce the set of equations (24):

$$\delta_{i{\rm r}} = \delta_{\rm cr}\ , \qquad \omega_{\rm cr}^2 = \omega_{ni{\rm r}}^2\ . \tag{25a,b}$$

Using these conditions we get two expressions from (24a,d):

$$\delta_{i{\rm r}} = \delta_{\rm cr} = \frac{1}{2}\,(\delta_i + \delta_{\rm c})\ , \qquad \omega_{\rm cr}^2 = \omega_{ni{\rm r}}^2 = \omega_{\rm c}\,\omega_{ni}\,\sqrt{1 - K_{\rm p}\,Z_i\,\frac{\omega_{{\rm ar}i}^2}{\omega_{ni}^2}}\ . \tag{26a,b}$$

Now substituting from (26a,b) to (24b,c) we obtain two equations relating only control parameters and modal properties of the structure:

$$\omega_{\rm c}^2\,(1 - K_{\rm p}\,Z_i) + \omega_{ni}^2 - 2\,\omega_{\rm c}\,\omega_{ni}\,\sqrt{1 - K_{\rm p}\,Z_i\,\frac{\omega_{{\rm ar}i}^2}{\omega_{ni}^2}} - (\delta_{\rm c} - \delta_i)^2 = 0\ , \tag{27a,b}$$

$$\delta_i\left[\omega_{\rm c}^2\,(1 - K_{\rm p}\,Z_i) - \omega_{\rm c}\,\omega_{ni}\,\sqrt{1 - K_{\rm p}\,Z_i\,\frac{\omega_{{\rm ar}i}^2}{\omega_{ni}^2}}\right] + \delta_{\rm c}\left[\omega_{ni}^2 - \omega_{\rm c}\,\omega_{ni}\,\sqrt{1 - K_{\rm p}\,Z_i\,\frac{\omega_{{\rm ar}i}^2}{\omega_{ni}^2}}\right] = 0\ .$$

Finally, equations (27) allow us to express two arbitrary PPF parameters as functions of the third. Choosing the gain K_p as independent, formula for optimal controller frequency follows:

$$\omega_\mathrm{c,opt} = \frac{\omega_{ni}^2 + \delta_i\,(\delta_\mathrm{c} - \delta_i)}{\omega_{ni}\sqrt{1 - K_\mathrm{p}\,Z_i\,\dfrac{\omega_{\mathrm{ar}i}^2}{\omega_{ni}^2}}} \doteq \frac{\omega_{ni}}{\sqrt{1 - K_\mathrm{p}\,Z_i\,\dfrac{\omega_{\mathrm{ar}i}^2}{\omega_{ni}^2}}}\ . \tag{28}$$

Relative error caused by neglecting the damping δ_i in (28) equals to the product of damping ratio of the controlled mode and damping ratio of the controller (this product is usually less then one percent). Substituting (28) to (27a) and rearranging terms we have the optimal value of the controller damping constant as:

$$\delta_\mathrm{c,opt} = \delta_i + \sqrt{\frac{K_\mathrm{p}\,Z_i\,(\omega_{\mathrm{ar}i}^2 - \omega_{ni}^2)}{1 - K_\mathrm{p}\,Z_i\,\dfrac{\omega_{\mathrm{ar}i}^2}{\omega_{ni}^2}}} = \delta_i + \sqrt{\frac{K_\mathrm{p}\,R_i}{1 - K_\mathrm{p}\,Z_i\,\dfrac{\omega_{\mathrm{ar}i}^2}{\omega_{ni}^2}}} \tag{29}$$

where R_i is the residue of the i-th mode from (9) (index rs dropped for simplicity).

Amount of the damping added to controlled mode in optimal case (i.e. maximum) follows directly from (26a). For lightly damped structures it is approximately the half of δ_c. In optimal control case the undamped closed-loop natural frequency is same as the open-loop one, as follows from (26b) after substituting from (28). We can also see that optimal PPF control (with respect to conditions (25)) make sense only if the product $K_\mathrm{p}\,R_i$ is positive and the magnitude of the gain is less than critical value K_cr:

$$|K_\mathrm{p}| < K_\mathrm{cr} = \frac{\omega_{ni}^2}{|Z_i|\,\omega_{\mathrm{ar}i}^2}\ . \tag{30}$$

From this point of view, it is better to express the gain K_p as function of the damping δ_c, which can be set apriori (as double of the desired increase of structural damping). Here inversion of (29) gives:

$$K_\mathrm{p,opt} = \frac{1}{Z_i + R_i\left[\dfrac{1}{\omega_{ni}^2} + \dfrac{1}{(\delta_\mathrm{c} - \delta_i)^2}\right]}\ . \tag{31}$$

Note that for any real value of δ_c substituted into (31) condition (30) holds, so it is no more needed. However, according to the physical limitations, the gain K_p has adequate limit, depending on actuator's working range and level of measured deformation. Thus there is always some limit of maximum possible damping increase we can desire. Note that formula (31) is only approximation for low open-loop damping δ_i. For exact formula, which is more complicated, see Appendix B.

5. Optimal parameters for multimodal PPF control

Important attribute of the PPF control, which made it popular, is roll-off character of its transfer function. From (21) it is clear that frequency band within the control action is narrow, with cut-off frequency being slightly greater than natural frequency of the controller (see Figure 5). On the other side, PPF controller affects all lower modes to some extent, even when they are well-separated. Therefore it is required to design the PPF controllers in descending order with respect to their natural frequencies and taking the influence of the higher (currently designed) controllers into account. If we use the same input-output pair for all the PPF controllers, then designing process is quite straightforward. Substituting simplified transfer function of the structure from (7) to (11) we have the estimation of closed-loop transfer function in neighbourhood of the i-th mode in the form:

$$H_{\mathrm{CL}}(s) = \frac{Z_i}{1 - H_{\mathrm{AGR}}(s)\, Z_i} \, \frac{s^2 + 2\,\delta_i\, s + \omega_{ni}^2 + \dfrac{R_i}{Z_i}}{s^2 + 2\,\delta_i\, s + \omega_{ni}^2 - \dfrac{H_{\mathrm{AGR}}(s)\, R_i}{1 - H_{\mathrm{AGR}}(s)\, Z_i}} \;, \qquad s \approx \mathrm{j}\,\omega_{ni} \tag{32}$$

while:

$$H_{\mathrm{AGR}}(s) = \sum_k H_{\mathrm{PPF},k}(s) = \sum_k \frac{K_{\mathrm{p}k}\, \omega_{\mathrm{c}k}^2}{s^2 + 2\,\delta_{\mathrm{c}k}\, s + \omega_{\mathrm{c}k}^2} \;, \tag{33}$$

where $H_{\mathrm{CL}}(s)$ is closed-loop transfer function, $H_{\mathrm{AGR}}(s)$ is aggregated transfer function representing the set of parallel PPF controllers aiming the higher modes, $K_{\mathrm{p}k}$, $\delta_{\mathrm{c}k}$ and $\omega_{\mathrm{c}k}$ are design parameters for the k-th PPF controller.

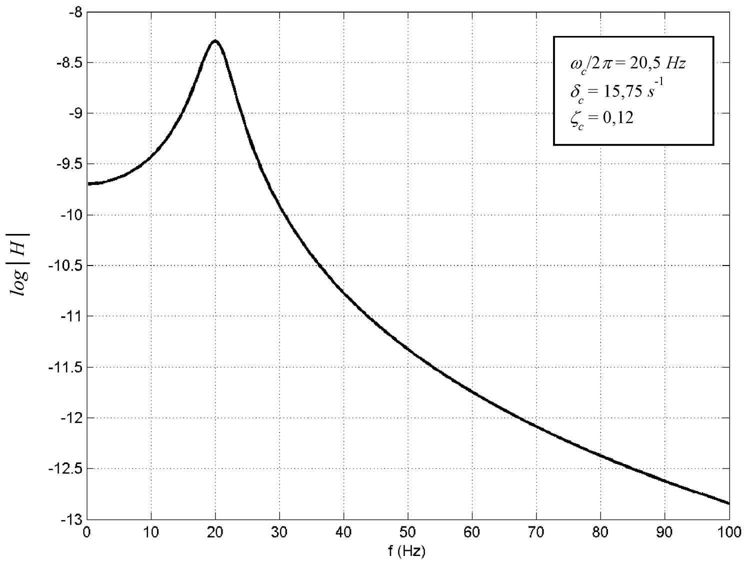

Fig.5: Typical amplitude plot of PPF controller

The aggregated transfer function $H_{\mathrm{AGR}}(s)$ is nearly constant in any narrow frequency range below the lowest value of $\omega_{\mathrm{c}k}$:

$$H_{\mathrm{AGR}}(\mathrm{j}\,\omega) \doteq \sum_k \frac{K_{\mathrm{p}k}\, \omega_{\mathrm{c}k}^2}{\omega_{\mathrm{c}k}^2 - \omega_{ni}^2} = G_i = \text{const.} \quad \text{for} \quad \omega \approx \omega_{ni} < \min(\omega_{\mathrm{c}k}) \;, \tag{34}$$

where G_i is the gain representing the influence of the higher PPF controllers in the frequency range of the i-th mode.

Now the closed-loop transfer function can be transformed to the simplified form by substituting G_i for $H_{\mathrm{AGR}}(s)$:

$$H_{\mathrm{CL}}(s) = Z_{\mathrm{up}i}\,\frac{s^2 + 2\,\delta_i\,s + \omega_{ni}^2 + \dfrac{R_i}{Z_i}}{s^2 + 2\,\delta_i\,s + \omega_{\mathrm{up}i}^2} = \frac{R_{\mathrm{up}i}}{s^2 + 2\,\delta_i\,s + \omega_{\mathrm{up}i}^2} + Z_{\mathrm{up}i} \tag{35}$$

if following statements hold:

$$Z_{\mathrm{up}i} = \frac{Z_i}{1 - G_i\,Z_i}\,, \qquad R_{\mathrm{up}i} = \frac{R_i}{(1 - G_i\,Z_i)^2}\,, \qquad \omega_{\mathrm{up}i}^2 = \omega_{ni}^2 - \frac{G_i\,R_i}{1 - G_i\,Z_i}\,, \tag{36a–c}$$

where R_i, Z_i and ω_{ni} are parameters of open-loop transfer function (7), $R_{\mathrm{up}i}$ is updated value of the residue for the i-th mode, $Z_{\mathrm{up}i}$ is updated value of the constant Z_i, $\omega_{\mathrm{up}i}$ is the i-th natural frequency affected by higher PPF controllers, G_i is the gain from (34) representing all higher PPF controllers in frequency range of the i-th mode.

So when we design any lower PPF controller (aiming the i-th mode), first we have to compute the gain G_i, which is then used for updating of the modal parameters and finally optimal parameters of the controller are obtained from (28) and (29). Note that there is no change in antiresonant frequency when only one input-output pair is present. In the case of collocated control all residues R_i have the same sign, thus updated value of natural frequency (36c) is less than open-loop one, assuming there is at least one higher PPF controller ($G_i\,R_i > 0$). This interesting property implicates that more damping can be added with the same value of gain K_{p} in comparison with single mode control. Figure 6 illustrates the effect of updating the optimal parameters on example of vibration suppression of cantilever beam within two lowest modes (see Table 4 for PPF parameters). First the single PPF is applied on the first mode ($\omega_{n1} \approx 86\,\mathrm{rad\,s^{-1}}$) and damping ratio about 26 % is achieved with gain $K_{\mathrm{p}} = -4.2$. When optimal PPF for the second mode is applied with the same gain, PPF for the first mode is affected and damping decrease to 21 %. But after updating of PPF parameters for the first mode damping ratio reaches almost 34 %.

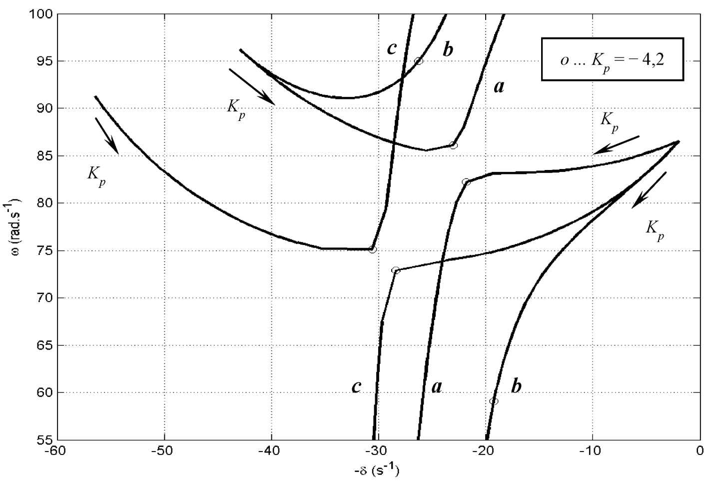

Fig.6: Root-loci of canteliver beam with collocated PPF control: a) single PPF – only 1st mode; b) multi PPF – parameters for 1st mode unchanged; c) multi PPF – parameters for 1st mode updated

6. Experiment – vibration suppression of beam

In this section we demonstrate the designing of multimodal PPF control on experimental example of vibration suppression, where prismatic aluminium beam with cross-section area of 40 mm × 4 mm is used as controlled structure in two configurations: free-free (hanging on soft rubber bands) and one-end clamped (i.e. cantilever beam). One 100mm long piezopatch, product QP40N of Midé Technology Corporation, is attached on the surface of the beam and serves as self-sensing actuator [4].

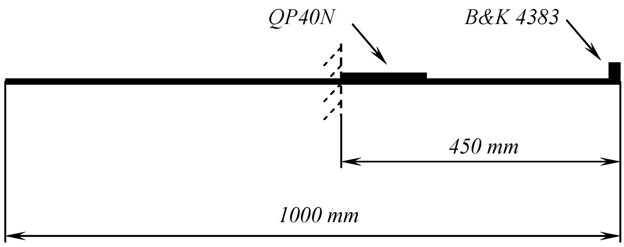

Fig.7: Experimental aluminium beam with self-sensing piezoactuator and acceleration sensor

Optimal actuator placement was found (see [10] for details) that allows us to control the four lowest modes in both configurations, as it can be seen in Figure 7. There is also one accelerometer measuring the transverse vibrations attached at the free end of the beam. Inertia and elastic properties of both piezoelement and accelerometer are included in FEM model. Damping matrix is computed using measured damping (Table 1), assuming modal viscous damping behaviour of the beam.

	free-free beam		cantilever beam	
mode	$\omega_{ni}/2\pi$ (Hz)	ζ_i (%)	$\omega_{ni}/2\pi$ (Hz)	ζ_i (%)
1	19.57	2.5	13.69	3.3
2	54.24	3.9	87.7	2.4
3	106.54	2.6	243.1	2.2
4	176.5	1.3	492	—

Tab.1: Experimental natural frequencies and damping ratios of the beam

Piezoelement QP40N acts as bending actuator with bending moment of 1.45×10^{-3} Nm/V and sensor of slope angle difference with sensitivity of 7630 V/rad [10]. Using analytical mode shape vectors and full series FRF we can compute appropriate residues R_i from (6), antiresonant frequencies and constants Z_i for controlled modes (see Table 2).

	free-free beam			cantilever beam		
mode	R_i (s^{-2})	Z_i (–)	$\omega_{\mathrm{ar}i}/2\pi$ (Hz)	R_i (s^{-2})	Z_i (–)	$\omega_{\mathrm{ar}i}/2\pi$ (Hz)
1	−200.0	−0.0614	21.53	−271.25	−0.0376	19.3
2	−560.3	−0.0564	56.13	−3061	−0.0265	102.15
3	−1828	−0.0572	110.1	—	—	—
4	−13316	−0.0432	198.1	—	—	—

Tab.2: Parameters of simplified FRF of the beam

mode	G_i	δ_c (s^{-1})		ω_c (rad s^{-1})		$\zeta_{i\max}$ (%)	
		single	multi	single	multi	single	multi
1	−6.8	23.7	44.4	134.1	137.8	10.1	18.5
2	−5.09	40.35	57.3	363.5	370.2	6.4	8.9
3	−3.15	68.6	84.9	716.9	724.6	5.2	6.4
4	—	185.6	185.6	1187	1187	8.6	8.6

Tab.3: Optimal parameters and achievable damping for PPF control of free-free beam ($K_p = -2.15$)

mode	G_i	δ_c (s^{-1})		ω_c (rad s^{-1})		$\zeta_{i\max}$ (%)	
		single	multi	single	multi	single	multi
1	−4.3	42.6	56.5	105.4	107.6	25.7	33.7
2	—	125.4	125.4	593.0	593.0	11.7	11.7

Tab.4: Optimal parameters and achievable damping for PPF control of cantilever beam ($K_p = -4.2$)

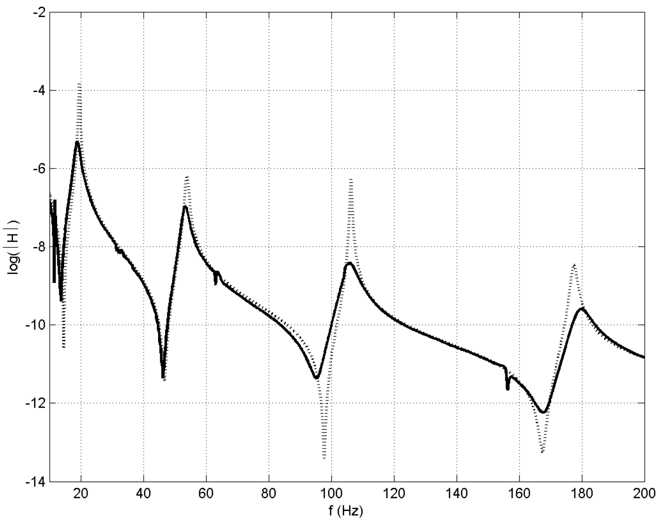

Fig.8: Amplitude plot for free-free beam with multi-modal PPF

In the next the optimal parameters for single- and multi-modal PPF can be computed using formulae presented in previous sections. The gain K_p was set to be constant according to physical limits. Values of the parameters together with theoretical closed-loop damping ratios are shown in Tables 3 and 4.

Experimental FRFs measured for transverse displacement of the free end of the beam are shown in Fig. 8 (free-free beam) and Fig. 9 (cantilever beam).

It can be seen that increase of structural damping as result of multimodal PPF control is approximately 5–10 times the natural damping for each mode, in agreement with predicted values in Tables 3 and 4.

7. Conclusions

Relations between optimal PPF control parameters and modal properties of actively damped structure were derived, which allows the designing of PPF controller to be straight-

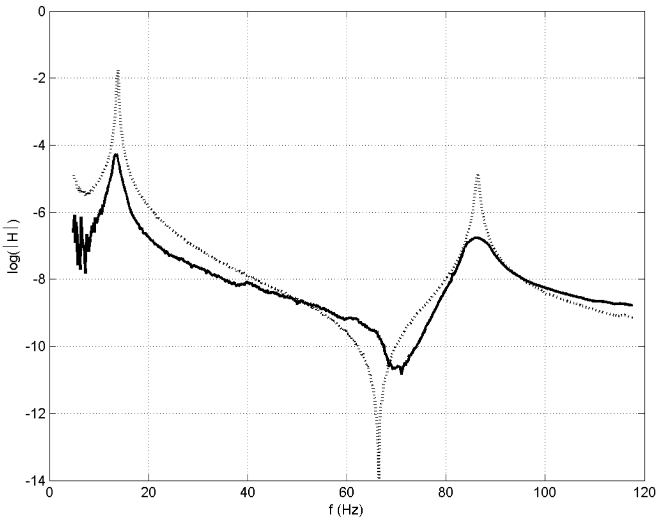

Fig.9: Amplitude plot for cantilever beam with multi-modal PPF

forward and efficient. Modal properties needed for calculation of the optimal parameters can be easily obtained experimentally (EMA) or analytically, using updated FEM model of the structure, thus an easy implementation into an adaptive PPF control is also possible. Using derived relations desired increase of structural damping can be achieved with minimum control gain, or maximum damping can be achieved with given gain (with respect to limitations of control aparatus). It has been also shown, that predicted control performance is fully consistent with numerical analysis, assuming the structure is lightly damped with well-separated modes. In multimodal case the interconnection between parallel PPF controllers has been analysed. In general, influence of given PPF controller in lower frequency range is evident regardless of modal properties of the structure and without proper updating the controll effort is degraded (comparing to single PPF). However, after parameter updating suggested in this paper simultaneous PPF controllers for multimodal control can achieve better results than in single-mode control. Finally, the experimental example of multimodal vibration suppression of the beam (4 controlled modes for free-free beam or 2 controlled modes for cantilever beam respectively) has demonstrated very good agreement with theoretical results.

Appendix A: Computation of transfer functions for vibrating mechanical systems

Generalized displacements of the structure $\mathbf{q}$ and measured outputs $\mathbf{y}$ can be expressed as linear combination of so called modal coordinates using non-singular modal matrix $\mathbf{V}$:

$$\mathbf{q} = \mathbf{V}\,\boldsymbol{\xi}\ , \qquad \mathbf{y} = \mathbf{C}\,\mathbf{V}\,\boldsymbol{\xi}\ , \tag{A-1a,b}$$

where $\mathbf{C}$ is the output matrix.

Substituting (A-1a) into eq. of motion (1), multiplying with the transpose of $\mathbf{V}$ from left and using eqs. (3) and (4) we get :

$$\mathbf{I}\,\ddot{\boldsymbol{\xi}} + 2\,\boldsymbol{\Delta}\,\dot{\boldsymbol{\xi}} + \boldsymbol{\Omega}\,\boldsymbol{\xi} = \mathbf{V}^{\mathrm{T}}\,\mathbf{B}\,\mathbf{u}\ , \tag{A-2}$$

where $\boldsymbol{\Delta}$, $\boldsymbol{\Omega}$ are diagonal matrices of damping constants and squares of natural frequencies.

Taking Laplace transform of equation (A-2) we get relation between modal coordinates and input variables in the form:

$$\mathbf{X}(s) = (s^2\,\mathbf{I} + s\,2\,\boldsymbol{\Delta} + \boldsymbol{\Omega})^{-1}\,\mathbf{V}^{\mathrm{T}}\,\mathbf{B}\,\mathbf{U}(s)\ , \tag{A-3}$$

where $\mathbf{X}(s)$ is Laplace transform of vector $\boldsymbol{\xi}(t)$, and $\mathbf{U}(s)$ is Laplace transform of the input $\mathbf{u}(t)$.

Taking Laplace transform of (A-1b) and substituting from (A-3) we get the open-loop relation between input $\mathbf{u}(t)$ and output $\mathbf{y}(t)$:

$$\mathbf{Y}(s) = \mathbf{H}(s)\,\mathbf{U}(s)\ , \qquad \mathbf{H}(s) = \mathbf{C}\,\mathbf{V}\,(s^2\,\mathbf{I} + s\,2\,\boldsymbol{\Delta} + \boldsymbol{\Omega})^{-1}\,\mathbf{V}^{\mathrm{T}}\,\mathbf{B}\ , \tag{A-4}$$

where $\mathbf{H}(s)$ is matrix of transfer functions for given input and output variables. Transfer $H_{rs}(s)$ between s-th input $u_s(t)$ and r-th output $y_r(t)$ is an element of the transfer matrix in the r-th row and s-th column and can be computed as follows:

$$H_{rs}(s) = \frac{Y_r(s)}{U_s(s)} = \mathbf{c}_r\,\mathbf{V}\,(s^2\,\mathbf{I}s\,2\,\boldsymbol{\Delta} + \boldsymbol{\Omega})^{-1}\,\mathbf{V}^{\mathrm{T}}\,\mathbf{b}_s\ , \tag{A-5}$$

where $\mathbf{c}_r$ is the r-th row of the output matrix $\mathbf{C}$, $\mathbf{b}_s$ is the s-th column of the input matrix $\mathbf{B}$.

Since expression in the brackets in (A-5) is diagonal matrix, its inversion is simply:

$$(s^2\,\mathbf{I}s\,2\,\boldsymbol{\Delta} + \boldsymbol{\Omega})^{-1} = \mathrm{diag}\left(\frac{1}{s^2 + 2\,\delta_i\,s + \omega_{ni}^2}\right)\ , \qquad i = 1, 2, \ldots, n\ , \tag{A-6}$$

where δ_i is damping constant of the i-th mode, ω_{ni} is the i-th natural frequency, n is number of generalized displacements.

Now the transfer function $H_{rs}(s)$ can be expressed as the sum of partial fractions:

$$H_{rs}(s) = \sum_{i=1}^{n} \frac{R_{irs}}{s^2 + 2\,\delta_i\,s + \omega_{ni}^2}\ , \tag{A-7}$$

where R_{irs} is the residue for the i-th mode within the given input-output pair (r, s).

Finally comparing (A-7) and (A-5) and using (A-6) we have the formula for computation of the residues R_{irs}:

$$R_{irs} = \mathbf{c}_r\,\mathbf{v}_i\,\mathbf{v}_i^{\mathrm{T}}\,\mathbf{b}_s\ . \tag{A-8}$$

Appendix B: Conditions for achieving the maximum closed-loop damping using PPF control

Analytical relations between open-loop and closed-loop modal parameters of the PPF controller (index c) and controlled mode of the structure (index i) are given by equations (24):

$$\begin{aligned}
\delta_{i\mathrm{r}} + \delta_{\mathrm{cr}} &= \delta_i + \delta_{\mathrm{c}}\ ,\\
\omega_{\mathrm{cr}}^2 + \omega_{ni\mathrm{r}}^2 + 4\,\delta_{i\mathrm{r}}\,\delta_{\mathrm{cr}} &= \omega_{\mathrm{c}}^2\,(1 - K_{\mathrm{p}}\,Z_i) + \omega_{ni}^2 + 4\,\delta_i\,\delta_{\mathrm{c}}\ ,\\
\delta_{i\mathrm{r}}\,\omega_{\mathrm{cr}}^2 + \delta_{\mathrm{cr}}\,\omega_{ni\mathrm{r}}^2 &= \delta_i\,\omega_{\mathrm{c}}^2\,(1 - K_{\mathrm{p}}\,Z_i) + \delta_{\mathrm{c}}\,\omega_{ni}^2\ ,\\
\omega_{\mathrm{cr}}^2\,\omega_{ni\mathrm{r}}^2 &= \left[1 - K_{\mathrm{p}}\,Z_i\,\frac{\omega_{\mathrm{ar}i}^2}{\omega_{ni}^2}\right]\omega_{ni}^2\,\omega_{\mathrm{c}}^2\ .
\end{aligned} \tag{B-1a–d}$$

where δ_{ir} and ω_{nir} are closed-loop damping constant and frequency of the i-th mode, δ_{cr} and ω_{cr} are the closed-loop damping constant and frequency of the controller, Z_i is real constant representing the contribution of the rest of vibration modes given by eq. (9), δ_i is open-loop damping constant of the i-th mode, ω_{ni} is the i-th open-loop natural frequency, $\omega_{\mathrm{ar}i}$ is corresponding antiresonant frequency, δ_c is open-loop damping constant of the PPF controller, ω_c is the controller open-loop natural frequency, K_p is the controller gain.

When control objective is to add the maximum of damping to the both modes (controller and structure), which are contributing to overall closed-loop performance equally, from the first equation of (B-1) directly follows:

$$\delta_{ir} = \max \quad \wedge \quad \delta_{cr} = \max \quad \Rightarrow \quad \delta_{ir} = \delta_{cr} = \frac{1}{2}(\delta_i + \delta_c) = \max \; . \tag{B-2}$$

Equation (B-2) is in fact the claim (25a). Now the main question is: what is the maximum value of δ_c we can set for given values of ω_c or K_p? Or, what is the minimum control gain for achieving desired damping increase? To answer these questions, let us analyze the system of equations (B-1).

Substituting (B-2) into (B-1) we get three equations relating open-loop and closed-loop modal quantities:

$$\begin{aligned} \omega_{cr}^2 + \omega_{nir}^2 + (\delta_c - \delta_i)^2 &= \omega_c^2 (1 - K_p Z_i) + \omega_{ni}^2 \; , \\ \omega_{cr}^2 + \omega_{nir}^2 &= \frac{2\,\delta_i}{\delta_c + \delta_i}\,\omega_c^2 (1 - K_p Z_i) + \frac{2\,\delta_c}{\delta_c + \delta_i}\,\omega_{ni}^2 \; , \\ \omega_{cr}^2\,\omega_{nir}^2 &= \omega_{ni}^2\,\omega_c^2 \left(1 - K_p Z_i \frac{\omega_{\mathrm{ar}i}^2}{\omega_{ni}^2}\right) \; . \end{aligned} \tag{B-3a–c}$$

From (B-3c) it is clear, that value of the gain K_p is not arbitrary, but has to fullfil following inequality:

$$1 - K_p Z_i \frac{\omega_{\mathrm{ar}i}^2}{\omega_{ni}^2} > 0 \; . \tag{B-4}$$

Now subtracting c) from b) in (B-3) we have relation between δ_c and ω_c:

$$(\delta_c - \delta_i)^2 = \frac{\delta_c - \delta_i}{\delta_c + \delta_i}\left[\omega_c^2 (1 - K_p Z_i) - \omega_{ni}^2\right] \quad \text{or} \quad \delta_c^2 - \delta_i^2 = \omega_c (1 - K_p Z_i) - \omega_{ni}^2 \; . \tag{B-5}$$

From (B-5) the square of the open-loop frequency ω_c is:

$$\omega_c^2 = \frac{1}{1 - K_p Z_i}\,(\omega_{ni}^2 + \delta_c^2 - \delta_i^2) > 0 \; . \tag{B-6}$$

Substituting (B-6) into (B-3b,c) we get two equations relating the closed-loop frequencies and open-loop parameters δ_c and K_p:

$$\begin{aligned} \omega_{cr}^2 + \omega_{nir}^2 &= 2\left[\omega_{ni}^2 + \delta_i (\delta_c - \delta_i)\right] = 2\,\omega_s^2 \; , \\ \omega_{cr}^2\,\omega_{nir}^2 &= \frac{1 - K_p Z_i \dfrac{\omega_{\mathrm{ar}i}^2}{\omega_{ni}^2}}{1 - K_p Z_i}\,\omega_{ni}^2 (\omega_{ni}^2 + \delta_c^2 - \delta_i^2) = C(K_p)\,\omega_{ni}^2 (\omega_{ni}^2 + \delta_c^2 - \delta_i^2) \; , \end{aligned} \tag{B-7a,b}$$

where frequency ω_s is function of ω_c only and $C(K_p)$ is function of K_p only.

From (B-7b) it is clear that, for real non-zero values of closed-loop frequencies, function $C(K_\mathrm{p})$ is positive:

$$C(K_\mathrm{p}) > 0 \ . \tag{B-8}$$

Now using (B-7) we can express $\omega_{ni\mathrm{r}}$ and ω_cr as solutions of quadratic equation:

$$(\Omega^2 - \omega_{ni\mathrm{r}}^2)\,(\Omega^2 - \omega_\mathrm{cr}^2) = \Omega^4 - 2\,\omega_\mathrm{s}^2\,\Omega^2 + C(K_\mathrm{p})\,\omega_{ni}^2\,(\omega_{ni}^2 + \delta_\mathrm{c}^2 - \delta_i^2) = 0 \ , \tag{B-9}$$

which roots are:

$$\Omega_{1,2}^2 = \begin{pmatrix} \omega_\mathrm{cr}^2 \\ \omega_{ni\mathrm{r}}^2 \end{pmatrix} = \omega_\mathrm{s}^2 \pm \sqrt{\omega_\mathrm{s}^4 - C(K_\mathrm{p})\,\omega_{ni}^2\,(\omega_{ni}^2 + \delta_\mathrm{c}^2 - \delta_i^2)} = \omega_\mathrm{s}^2 \pm \sqrt{D(\delta_\mathrm{c}, K_\mathrm{p})} \ , \tag{B-10}$$

where D is discriminant of equation (B-9). Note that D is function of two independent parameters, controller damping δ_c and gain K_p.

To get real non-zero values of $\omega_{ni\mathrm{r}}$ and ω_cr the discriminant D must satisfy next conditions:

$$\omega_\mathrm{s}^4 > D(\delta_\mathrm{c}, K_\mathrm{p}) \ , \qquad D(\delta_\mathrm{c}, K_\mathrm{p}) \geq 0 \ . \tag{B-11a,b}$$

From (B-8) it is clear, that (B-11a) holds automatically. For further analysis of (B-11b) we suppose that damping of the controller is greater than open-loop damping of the controlled mode (we want to add damping to the structure):

$$\delta_\mathrm{c} > \delta_i \ . \tag{B-12}$$

Let ω_i be the damped open-loop frequency of the controlled mode:

$$\omega_i^2 = \omega_{ni}^2 - \delta_i^2 > 0 \ . \tag{B-13}$$

Using (B-13) we can write the discriminant of (B-9) in the form:

$$D(\delta_\mathrm{c}, K_\mathrm{p}) = (\omega_i^2 + \delta_\mathrm{c}\,\delta_i)^2 - C(K_\mathrm{p})\,(\omega_i^2 + \delta_i^2)\,(\omega_i^2 + \delta_\mathrm{c}^2) \geq 0 \ . \tag{B-14}$$

Now extracting $C(K_\mathrm{p})$ from (B-14) we have:

$$C(K_\mathrm{p}) \leq \frac{(\omega_i^2 + \delta_\mathrm{c}\,\delta_i)^2}{(\omega_i^2 + \delta_i^2)\,(\omega_i^2 + \delta_\mathrm{c}^2)} \ . \tag{B-15}$$

Since for $\delta_\mathrm{c} > \delta_i$ the ratio on the right side of (B-15) is less than 1, we can complete condition (B-8) as follows:

$$1 > C(K_\mathrm{p}) = \frac{1 - K_\mathrm{p}\,Z_i\,\dfrac{\omega_{\mathrm{ar}i}^2}{\omega_{ni}^2}}{1 - K_\mathrm{p}\,Z_i} > 0 \ . \tag{B-16}$$

Solving (B-16) for K_p and assuming that from (B-6) is $(1 - K_\mathrm{p}\,Z_i) > 0$ we get these gain limits:

$$\begin{aligned} &\omega_{\mathrm{ar}i}^2 > \omega_{ni}^2 \Rightarrow 1 > \frac{\omega_{ni}^2}{\omega_{\mathrm{ar}i}^2} > K_\mathrm{p}\,Z_i > 0 \ , \\ &\omega_{\mathrm{ar}i}^2 < \omega_{ni}^2 \Rightarrow K_\mathrm{p}\,Z_i < 0 \ . \end{aligned} \tag{B-17a,b}$$

Note that for collocated control holds (B-17a) so we must use positive feedback to add damping to the structure. Now let us try to solve (B-14) for δ_c. Making some re-arrangement we can write:

$$(1-C)\,(\omega_i^2+\delta_c\,\delta_i)^2 \geq C\,\omega_i^2\,(\delta_c-\delta_i)^2 > 0\;, \tag{B-18}$$

and finally, assuming that both (B-12) and (B-16) hold, we have:

$$\omega_i^2+\delta_c\,\delta_i \geq \sqrt{\frac{C}{1-C}}\,\omega_i\,(\delta_c-\delta_i) > 0 \tag{B-19}$$

or after re-gruping terms within δ_c it is:

$$\omega_i\left(\omega_i+\sqrt{\frac{C}{1-C}}\,\delta_i\right) \geq \delta_c\left(\sqrt{\frac{C}{1-C}}\,\omega_i-\delta_i\right)\;. \tag{B-20}$$

There are two solutions for δ_c depending on sign of the right side of (B-20). Since this right side changes its sign when value of $C(K_p)$ reaches square of open-loop damping ratio ζ_i:

$$\begin{aligned} C > \zeta_i^2 = \frac{\delta_i^2}{\omega_i^2+\delta_i^2} = \frac{\delta_i^2}{\omega_{ni}^2} \quad &\Rightarrow \quad \sqrt{\frac{C}{1-C}}\,\omega_i-\delta_i > 0\;, \\ 0 < C \leq \zeta_i^2 \quad &\Rightarrow \quad \sqrt{\frac{C}{1-C}}\,\omega_i-\delta_i \leq 0 \end{aligned} \tag{B-21a,b}$$

we can write:

$$\begin{aligned} C(K_p) > \zeta_i^2 \quad &\Rightarrow \quad \delta_c \leq \frac{\omega_i\left(\omega_i+\sqrt{\frac{C}{1-C}}\,\delta_i\right)}{\sqrt{\frac{C}{1-C}}\,\omega_i-\delta_i} = \delta_{c,\max} \in \mathrm{Re}\;, \\ C(K_p) \leq \zeta_i^2 \quad &\Rightarrow \quad \delta_c > 0 \quad \Rightarrow \quad \delta_{c,\max} \to \infty\;. \end{aligned} \tag{B-22a,b}$$

For gain K_p such that $C(K_p) > \zeta_i^2$ the damping constant δ_c has its maximum defined by (B-22a), which we can set as controller parameter in agreement with (B-2).

Using (B16) we can re-define gain limits as:

$$C(K_p) > \zeta_i^2 \quad \wedge \quad \omega_{ari}^2 > \omega_{ni}^2 \quad \Rightarrow \quad \frac{\omega_{ni}^2}{\omega_{ari}^2} \geq \frac{\omega_{ni}^2-\delta_i^2}{\omega_{ari}^2-\delta_i^2} > K_p\,Z_i > 0\;, \tag{B-23}$$

where ω_{ari} is corresponding antiresonant frequency.

For gain K_p such that $C(K_p) \leq \zeta_i^2$ the damping constant δ_c has no real bound and theoretically can be set as infinity. In technical applications it is of course impossible. Note that for decaying $C(K_p)$ the gain K_p is raising, so there is no reason to set K_p greater than upper bound in (B-23). In such case (i.e. $C(K_p) > \zeta_i^2$) maximum controller damping can be computed from (B-22a) using (B-13):

$$\delta_{c,\max} = \delta_i + \frac{\omega_{ni}^2}{\sqrt{\frac{C(K_p)}{1-C(K_p)}}\,\omega_i-\delta_i}\;. \tag{B-24}$$

Alternatively, if desired damping is given, the minimal gain needed for achieving this damping can be computed from (B-16):

$$(K_{\rm p}\, Z_i)_{\rm min} = \frac{1 - C_{\rm max}}{\dfrac{\omega_{{\rm ar}i}^2}{\omega_{ni}^2} - C_{\rm max}} \ , \tag{B-25}$$

where $C_{\rm max}$ is on the right side of (B-15):

$$C_{\rm max} = \frac{(\omega_i^2 + \delta_{\rm c}\, \delta_i)^2}{(\omega_i^2 + \delta_i^2)\,(\omega_i^2 + \delta_{\rm c}^2)} \ . \tag{B-26}$$

Substituting (B-26) into (B-25) and using (B-13) we can express minimum gain as:

$$(K_{\rm p}\, Z_i)_{\rm min} = \frac{1}{1 + (\omega_{{\rm ar}i}^2 - \omega_{ni}^2)\, \dfrac{\omega_{ni}^2 + \delta_{\rm c}^2 - \delta_i^2}{(\omega_{ni}^2 - \delta_i^2)\,(\delta_{\rm c} - \delta_i)^2}} \ , \tag{B-27}$$

and finally using (9) we can write:

$$(K_{\rm p})_{\rm min} = \frac{1}{Z_i + R_i \left[\dfrac{\delta_{\rm c}^2}{(\omega_{ni}^2 - \delta_i^2)\,(\delta_{\rm c} - \delta_i)^2} + \dfrac{1}{(\delta_{\rm c} - \delta_i)^2}\right]} \ . \tag{B-28}$$

Comparing (B-28) and (31) we see, that for $\delta_i \to 0$ we get same result.

Setting $K_{\rm p,min}$ for given $\delta_{\rm c}$ or $\delta_{\rm c,max}$ for given $K_{\rm p}$ and substituting into (B-14) we get:

$$D(\delta_{\rm c,max}, K_{\rm p}) = D(\delta_{\rm c}, K_{\rm p,min}) = 0 \ . \tag{B-29}$$

Equation (B-29) is, in fact, proof of the claim (25b), because after substituting (B-29) into (B-10) the closed-loop frequencies are the same:

$$K_p = K_{\rm p,min} \quad \vee \quad \delta_{\rm c} = \delta_{\rm c,max} \quad \Rightarrow \quad \omega_{ni{\rm r}}^2 = \omega_{\rm cr}^2 = \omega_{\rm s}^2 = \omega_{ni}^2 + \delta_i\,(\delta_{\rm c} - \delta_{\rm i}) \ . \tag{B-30}$$

Now using (B-30) we can easily compute the open-loop frequency of the controller from (B-3c):

$$\omega_{\rm c,opt} = \omega_{\rm c}(K_{\rm p}, \delta_{\rm c,max}) = \omega_{\rm c}(K_{\rm p,min}, \delta_{\rm c}) = \frac{\omega_{ni}^2 + \delta_i\,(\delta_{\rm c} - \delta_i)}{\omega_{ni}\sqrt{1 - K_{\rm p}\, Z_i\, \dfrac{\omega_{{\rm ar}i}^2}{\omega_{ni}^2}}} \ , \tag{B-31}$$

where $\omega_{\rm c,opt}$ is controller frequency for achieving the maximum damping (or minimum gain) with respect to condition (B-2).

As can be expected, formula (B-31) is same as formula (28).

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