NEW MATHEMATICAL AND COMPUTATIONAL MODEL OF ELASTIC CONTINUUM INTERACTION WITH LIQUID

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This contribution is focused on the analysis of dynamic behavior of elastic continuum moving in liquid. The method of solution is demonstrated on the non-stationary movement of a bar in a real, incompressible liquid. The principle of the solution is based on the modal transformation of the bar and discrete process using the finite element method (FEM). The problem is in general non-linear. However, the method introduced in this contribution allows the solution of the problem in time steps. The solution is based upon a certain substitution which provides gradual separation of the movement of liquid and the movement of the bar. This substitution is entered analogously on the finite element and is dependent on the normal coordinates. The result is a tensor of additional effects.

Key words : *free damped vibration, real liquid, interaction of continuum with liquid, finite element*

1. Introduction

The problem of interaction of the elastic continuum with the surroundings, in which they are moving, belongs to the most difficult problems of mechanics. For instance turbine or compressor blades vibration in the surroundings or vibration of a reactor vessel in a fluid can serve as examples. From the view of mechanic tasks it basically concerns three essential problem ranges. These are the problem of eigenvalues, the problem of steady state response and the problem of analysis of transient vibration. We are recently the witnesses of so-called 'merging' of different computational systems, which were not long ago focused only on the limited range of tasks. We can mention the integration of ANSYS and FLUENT systems. This process is unavoidable and new mathematical and computational modules need to be created. This contribution is focused just on one of these.

The authors, for many years, have been engaged in the possibility of separating liquid and body from each other. They have proved that it is possible with the presumption of rigid continuum. The summarizing results of their more than six year long research are shown in [1] and [2]. The summary of computational modelling possibilities is presented in [5].

The problem of interaction of continuum and the surroundings requires a different modeling approach. The difference is apparent especially in the discrete process and, in addition, for the surroundings linked with the necessity of changing the mesh. So the calculation becomes very time consuming. Hence a number of authors are dealing with the idea of making it less time-consuming for calculations.

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One of the possibilities is presented in [3]. In the computation of fluid-continuum interactions, updated mesh methods are used consisting of mesh-moving and remeshing – as needed. When the geometries are complex and the structural displacements are large, it becomes even more important that the mesh moving techniques are designed with the objective to reduce the frequency of remeshing. To that end, mesh moving techniques are presented here where the motion of the nodes is governed by the equations of elasticity, with selective treatment of mesh deformation based on element sizes as well as deformation modes in terms of shape and volume changes. Also presented are some results from the application of these techniques to a set of two-dimensional test CASEs.

Another algorithm of interaction solution is presented in [4], where the authors are dealing with a classical exercise of rotating machine blades vibration in the surroundings.

The aeroelastic behaviour of vibrating blade assemblies is usually investigated in the frequency domain where the determination of aeroelastic stability boundaries is separated from the computation of linearized unsteady aerodynamic forces. However, nonlinear fluid-continuum interaction caused by vibration shocks or strong flow separation may significantly influence the aerodynamic damping and hence cause a shift of stability boundaries. In order to investigate such aeroelastic phenomena, the governing equations of structural and fluid motion have to be simultaneously integrated in time. In this paper a technique is presented which analyzes the aeroelastic behaviour of compressor cascade vibration in the time domain. The structural part of the governing aeroelastic equations is time- integrated according to the algorithm of Newmark, while the unsteady airloads are computed at every time step by an Euler upwind code. The link between the two time integrations is an automatic grid generation in which the used mesh is dynamically deformed as such that it conforms with the deflected blades at every time step. The computed time series of the aeroelastic simulation of an assembly of twenty compressor blades performing torsional vibrations in transonic flow are presented. For subsonic flow, the differences between time domain and frequency domain results are of negligible order. For transonic flow, however, where vibrating shocks and a temporarily choked flow in the blade channel dominate the unsteady flow, the energy transfer between fluid and continuum is no longer comparable to that of a linear system. It is demonstrated that the application of the time domain method leads to a significantly different aeroelastic behaviour of the blade assembly including a shift of the stability boundary.

2. Mathematical and computational model

The way in order to draw up the mathematical and computational model will be demonstrated on the vibration of the bar in liquid. It is possible to generalize it to the case of vibration of any elastic continuum in liquid, eventually in any surrounding. The target is to determine the relations of local matrices of additional effects by the liquid. The motion equation for the finite element of the bar – the free undamped vibration in liquid has the form:

$$m_{ij}\,\ddot{r}_j + b_{ij}\,\dot{r}_j + k_{ij}\,r_j = -\int\limits_{S_\mathrm{e}} u_i\,f\,\mathrm{d}S_\mathrm{e}\ , \tag{1}$$

$$\begin{aligned}(\mathbf{p})_i &= p\,n_i\ ,\\ (\boldsymbol{\sigma})_i &= \Pi_{ij}\,n_j\ .\end{aligned} \tag{2}$$

The above mentioned nonreversible stress tensor for uncompressible liquid has the form:

$$\Pi_{ij} = 2\,\eta\,c_{ij}\ , \qquad c_{ij} = \frac{1}{2}\left(\frac{\partial c_i}{\partial x_j} + \frac{\partial c_j}{\partial x_i}\right)\ . \tag{3}$$

The principle of the solution is based on the modal transformation of the bar. Let's extend the state vector of the bar with the help of eigenvectors on this principle. Then, with this presumption, the following is valid:

$$r_i(z,t) = v_{il}(z)\,q_l(t)\ . \tag{4}$$

Further, it is important to define the boundary conditions for liquid. On the boundaries of the area, filled with liquid, the following is valid:

$$\begin{aligned} S &: c_i = \dot{r}_i\ ,\\ \Gamma_1 &: p = 0\ ,\\ \Gamma_2 &: c_i = 0\ ,\\ \Gamma_3 &: c_i = 0\ . \end{aligned} \tag{5}$$

Establishing new defining relations for velocities and pressure in the form of convolutory integrals is a very important step for drawing up a mathematical model. With establishing velocity and pressure functions depending on the normal bar coordinates it is possible to separate continuum and liquid movements. The velocity of liquid and the pressure in an arbitrary place are defined by these relations:

$$\begin{aligned} c_i &= \int_0^t \alpha_{il}(t-\tau)\,\dot{q}_l(\tau)\,\mathrm{d}\tau\ ,\\ p &= \int_0^t \beta_k(t-\tau)\,\dot{q}_k(\tau)\,\mathrm{d}\tau\ , \end{aligned} \tag{6}$$

where α_{il}, β_k are new variables. The boundary condition for the liquid velocity on the bar surface using this presumption has the form:

$$\int_0^t \alpha_{il}(t-\tau)\,\dot{q}_l(\tau)\,\mathrm{d}\tau = v_{ik}(z)\,\dot{q}_k(t)\ . \tag{7}$$

Now we can analyze the separated liquid. It is necessary to mention in this context, that the shape of the continuum vibration doesn't only influence the velocity boundary conditions, but also by this the geometrical configuration is given in the given instant time. The modal features of the vibration continuum in the surroundings, in general, depend on the vibration amplitude for the given shape of vibration. Let's assume the liquid is real and incompressible. The initial Navier-Stokes and continuity equations for a linear task have this form:

$$\begin{aligned} \varrho\,\frac{\partial c_i}{\partial t} - \frac{\partial \Pi_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} &= 0\ ,\\ \varrho\,\frac{\partial c_k}{\partial x_k} &= 0\ . \end{aligned} \tag{8}$$

The following is valid in the finite dimension space (not FEM) using Bezier's body features:

$$\begin{gathered} \mathbf{A}\,\dot{\boldsymbol{\alpha}} - \mathbf{B}\,\boldsymbol{\alpha} + \mathbf{C}\,\boldsymbol{\beta} = \mathbf{0}\ , \\ \mathbf{D}\,\boldsymbol{\alpha} = \mathbf{0}\ . \end{gathered} \tag{9}$$

The solution is possible to suppose regarding the character of the differential equation in this form:

$$\begin{aligned} \boldsymbol{\alpha} &= \boldsymbol{\alpha}_1\,\delta + \boldsymbol{\alpha}_2(t)\ , \\ \boldsymbol{\beta} &= \boldsymbol{\beta}_1\,\dot{\delta} + \boldsymbol{\beta}_2\,\delta + \boldsymbol{\beta}_3(t)\ . \end{aligned} \tag{10}$$

We obtain after substitution into (9):

$$\begin{gathered} \mathbf{A}\,(\boldsymbol{\alpha}_1\,\dot{\delta} + \dot{\boldsymbol{\alpha}}_2) - \mathbf{B}\,(\boldsymbol{\alpha}_1\,\dot{\delta} + \boldsymbol{\alpha}_2) + \mathbf{C}\,(\boldsymbol{\beta}_1\,\dot{\delta} + \boldsymbol{\beta}_2\,\delta) = \mathbf{0}\ , \\ \mathbf{D}\,(\boldsymbol{\alpha}_1\,\delta + \boldsymbol{\alpha}_2) = \mathbf{0}\ . \end{gathered} \tag{11}$$

We obtain the next equation comparing elements of the general Dirac's function derivation in the motion equation and the general Dirac's function derivation in the continuity relation:

$$\begin{gathered} \mathbf{A}\,\boldsymbol{\alpha}_1 + \mathbf{C}\,\boldsymbol{\beta}_1 = \mathbf{0}\ , \\ \mathbf{D}\,\boldsymbol{\alpha}_1 = \mathbf{0} \end{gathered} \tag{12}$$

and the next:

$$-\mathbf{B}\,\boldsymbol{\alpha}_1 + \mathbf{C}\,\boldsymbol{\beta}_2 = \mathbf{0}\ . \tag{13}$$

Hence the pressure function $\boldsymbol{\beta}_2$ is determined:

$$\boldsymbol{\beta}_2 = \mathbf{C}^{+}\,\mathbf{B}\,\boldsymbol{\alpha}_1\ . \tag{14}$$

The velocity and pressure functions are calculated after substitution of the boundary conditions for the instant shape of vibration. Neglecting the influence of $\boldsymbol{\alpha}_2$ and $\boldsymbol{\beta}_3$ functions the following is valid for the velocities and pressures for the l-th shape of vibration:

$$\begin{aligned} c_i &= \alpha_{1il}\,\dot{q}_l\ , \\ p &= \beta_{1l}\,\ddot{q}_l + \beta_{2l}\,\dot{q}_l\ . \end{aligned} \tag{15}$$

Now it is possible to proceed with composing local matrices of additional effects by the liquid. We obtain the next equation by the substitution of relations (15) for the velocities and pressure into equation (1) using eqs. (2) and (3):

$$m_{ij}\,\ddot{r}_j + b_{ij}\,\dot{r}_j + k_{ij}\,r_j = -\int\limits_{S_\mathrm{e}} u_i \left[\beta_{1l}\,\ddot{q}_l + \beta_{2l}\,\dot{q}_l + \eta\left(\frac{\partial\alpha_{1ml}}{\partial x_n} + \frac{\partial\alpha_{1nl}}{\partial x_m}\right)\dot{q}_l\right]\mathrm{d}S_\mathrm{e}\ . \tag{16}$$

It is possible to use this assumption, when the continuum vibrates only on the chosen l-th shape of vibration. Substation of this is the elimination of the q_l from eq. (4). After this we obtain the formula with the presumption that the solution develops into the free vibration form i.e. the substitution of relation (4):

$$\begin{aligned} m_{ij}\,\ddot{r}_j + b_{ij}\,\dot{r}_j + k_{ij}\,r_j = &-\int\limits_{S_\mathrm{e}} u_i\,\beta_{1l}\,v_{jl}^{-1}\,\ddot{r}_j\,\mathrm{d}S_\mathrm{e} - \\ &-\int\limits_{S_\mathrm{e}} u_i\left[\beta_{2l} + \eta\left(\frac{\partial\alpha_{1ml}}{\partial x_n} + \frac{\partial\alpha_{1nl}}{\partial x_m}\right)\right] v_{jl}^{-1}\,\dot{r}_j\,\mathrm{d}S_\mathrm{e}\ . \end{aligned} \tag{17}$$

Hence it is obvious that local matrices of additional effects by the liquid for the l-th shape of vibration are determined by these equations:

$$m_{tij} = \int_{S_e} u_i \, \beta_{1l} \, v_{jl}^{-1} \, \mathrm{d}S_e \ ,$$
$$b_{tij} = \int_{S_e} u_i \left[\beta_{2l} + \eta \left(\frac{\partial \alpha_{1ml}}{\partial x_n} + \frac{\partial \alpha_{1nl}}{\partial x_m} \right)\right] v_{jl}^{-1} \, \mathrm{d}S_e \ . \tag{18}$$

3. Model task

The model exercise is a cantilever bar in liquid. This model was chosen with regard to the possibility of comparing with an experiment. The scheme is on fig. 1 and geometrical properties are presented in table 1. All of the dimensions are in millimeters.

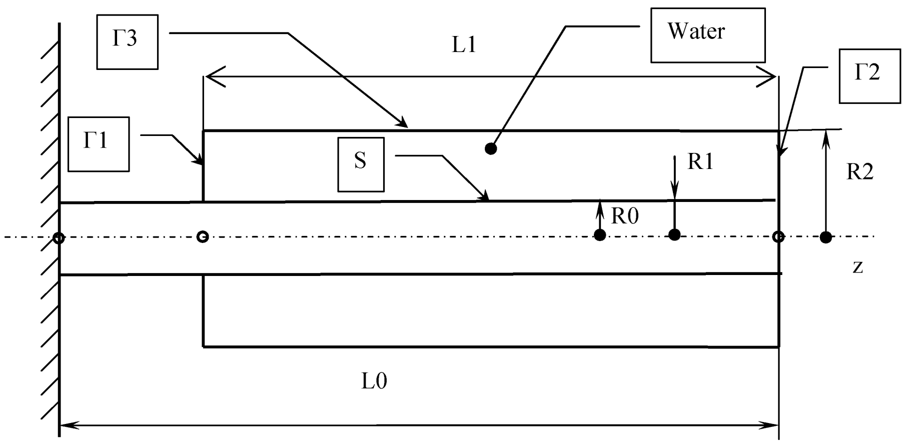

Fig.1: Scheme of the model task

L0	1100
L1	1000
R0	16.85
R1	17.85
R2	50

Tab.1: Dimensions of the model task

The results of the model task are only for matter-of-fact purposes and that is to present the possibilities of computational modelling. The whole analysis is carried out only for the first form of vibration. The velocity and pressure field is drawn for the case of liquid at a height of 1000 mm. There is a surface velocity field of the bar shown in fig. 2. The task is symmetrical hence the velocities are calculated only on one half-plane. So, the vectors of velocities correspond to this shape of vibration. The whole velocity field is shown on fig. 3 and pressure fields of pressure functions are on figs. 4 and 5. The local matrices of the additional modal matrices of masses and damping are quite interesting results. Only for information is on figs. 6 and 7 presented the distribution of modal mass and damping coefficients on the main diagonal of global matrices. The relation of the bar eigen frequencies in liquid with the different heights are drawn in fig. 8.

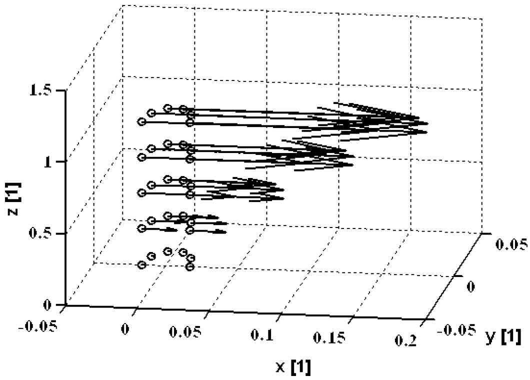

Fig.2: Velocity field on the beam surface

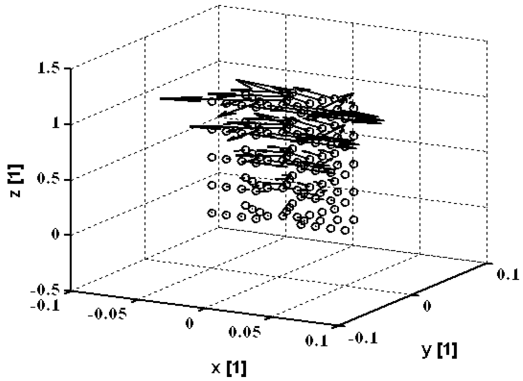

Fig.3: Velocity field

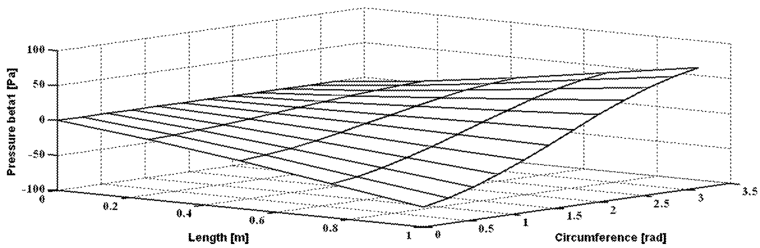

Fig.4: Pressure field of function $beta_1$ on the beam surface

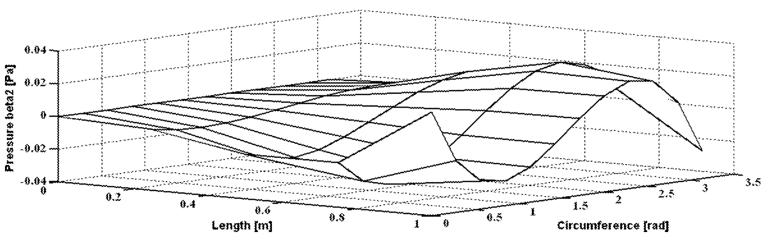

Fig.5: Pressure field of function β_2 on the beam surface

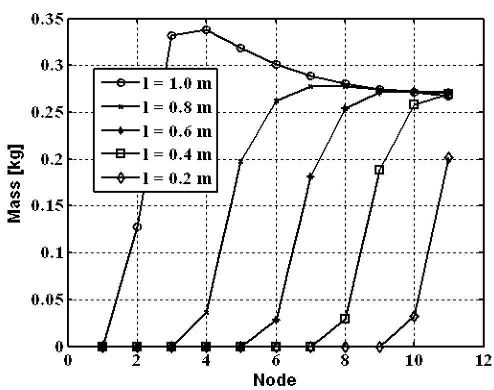

Fig.6: Mass distribution on the diagonal of the global mass matrix

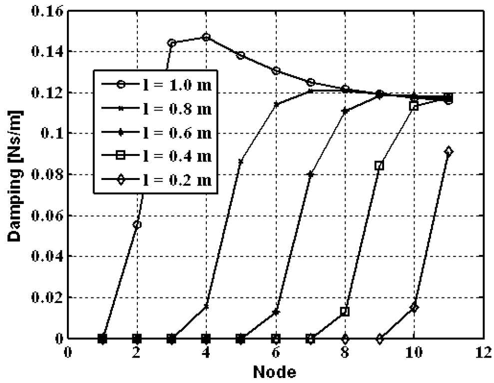

Fig.7: Damping distribution on the diagonal of the global damping matrix

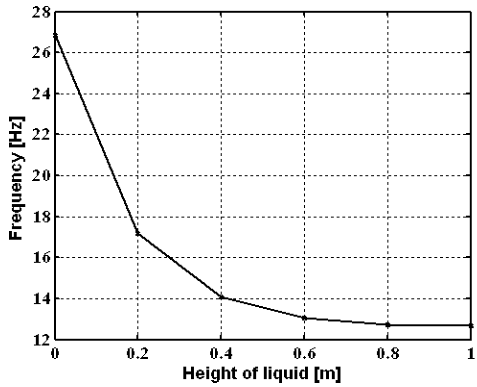

Fig.8: Frequency dependence on the high of liquid

4. Conclusion

The new mathematical and computational model for the analysis of interaction of elastic continuum and liquid with the use of finite elements was presented in this contribution. The essence of this new way allows the possibility of separating liquid and continuum motion and then applying a developed solution to the free vibration forms.

The mathematical and computational model was applied only to the model task of vibration of the cantilever bar in liquid. But it is obvious, that it is possible to generalize it on any rigid or elastic continuum in any surrounding. The effort of the authors is to extend the application of this approach to commercially supplied computational systems. Also, the possibility of extending the case to large strains and large displacements is obvious.

An experimental model is prepared for verifying the conclusions of the computational analysis of the bar with one end fixed. Presently, the experimental analysis is being performed on a bar in the centric position with different liquid (water) heights. Five containers are prepared with various diameters in which the bar is submerged.

It is essential to lastly call attention to the problem of numerical stability of the solved problems, especially when connected to the Bezier's body application. For helping numeric stability, for example, it helped to assume that when the outer surfaces of the body contacted the liquid, an ideal liquid was modeled. Furthermore is preparing combining Finite Difference and Finite Volume Methods.

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Appendix

Base functions

$$u_1(x) = 1 - \frac{3\,x^2}{L_{\mathrm{E}}^2} + \frac{2\,x^3}{L_{\mathrm{E}}^3}\,, \qquad u_2(x) = x - \frac{2\,x^2}{L_{\mathrm{E}}} + \frac{x^3}{L_{\mathrm{E}}^2}\,,$$

$$u_3(x) = \frac{3\,x^2}{L_{\mathrm{E}}^2} - \frac{2\,x^3}{L_{\mathrm{E}}^3}\,, \qquad u_4(x) = -\frac{x^2}{L_{\mathrm{E}}} + \frac{x^3}{L_{\mathrm{E}}^2}\,.$$

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