SOME INSTANCES OF THE FOKKER-PLANCK EQUATION NUMERICAL ANALYSIS FOR SYSTEMS WITH GAUSSIAN NOISES

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The Fokker-Planck (FP) equation is frequently used when the response of the dynamic system subjected to additive and/or multiplicative random noises is investigated. It provides the probability density function (PDF) representing the key information for further study of the dynamic system. Various analytic and semi-analytic solution methods have been developed for various systems to obtain results requested. However numerical approaches offer a powerful alternative. In particular the Finite Element Method (FEM) seems to be very effective. A couple of single dynamic linear/non-linear systems under additive and multiplicative random excitations are discussed using FEM as a solution tool of the FP equation.

Key words : *Fokker-Planck equation, numerical solution, transition effects*

1. Introduction

There exist many methods for the response and stability analysis of dynamic systems with external excitation having the character of random noises variable in time. It can be stated that classical methods being based on spectral and correlation principles are effective in linear cases with additive Gaussian excitation only. Although their application can be considered even in more general cases, the efficiency should be always carefully premeditated. A solution procedure could shift out of the predetermined aim very easily and the result would be far from an original intention. The main reason are various hidden properties of these methods being based essentially on the superposition principle. Consequently, this modesty should be applied every time when multiplicative processes appear and mainly when non-linear systems are to be discussed.

Many difficulties can be reduced or eliminated using methods based on the theory of Markov processes. They are more general from the viewpoint of the type and structure of system which should be investigated. However they include certain conditions limiting admissible types of input processes. For instance, it is usual to presume that excitation processes are of Wiener type. In such a case the probability density function (PDF) can be described by means of the Fokker-Planck (FP) equation admitting an evolution of the PDF in time. So far as the PDF succeeds to be found, it can be treated as a natural extension of a deterministic result. It includes a complete information concerning a random character of the system response and enables to derive also its additional attributes as for example its frequency structure etc.

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2. Physical system and Fokker-Planck equation

The dynamic system behavior is commonly described by means of the differential system of the first order in the normal form. This system in general is subjected simultaneously to deterministic and random excitation processes as functions of time. Random effects are introduced separately in a form of certain linear combinations of input processes. Let us accept, that satisfactorily general formulation can be written in a form:

$$\frac{\mathrm{d}x_j(t)}{\mathrm{d}t} = f_j(\mathbf{x},t) + g_{jr}(\mathbf{x},t)\, w_r(t)\ , \quad \mathbf{x} = [x_1,\dots,x_{2n}]\ , \quad n \text{ – degrees of freedom}\ , \qquad (1)$$

$w_r(t)$ – Gaussian white noises with constant cross-density $K_{rs} = \mathbf{E}\{w_r\, w_s\}$; $r,s = 1,m$, m – number of acting noises, $\mathbf{E}\{\cdot\}$ – mathematical mean value operator in the Gaussian meaning, $f_j(\mathbf{x},t)$, $g_{jr}(\mathbf{x},t)$ – continuous deterministic functions of state variables $\mathbf{x}$ and time t; $j = 1,2n$.

If input processes w_j, w_r can be considered to be Gaussian white noises, the respective FP equation for an unknown PDF in variables $\mathbf{x}, t$ can be assigned to Eq. (1) :

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\frac{\partial}{\partial x_j}\big[\kappa_j(\mathbf{x},t)\, p(\mathbf{x},t)\big] + \frac{1}{2}\,\frac{\partial^2}{\partial x_j \partial x_k}\big[\kappa_{jk}(\mathbf{x},t)\, p(\mathbf{x},t)\big]\ , \qquad (2)$$

$$\kappa_j(\mathbf{x},t) = f_j(\mathbf{x},t) + \frac{1}{2}\, K_{rs}\, g_{ls}(\mathbf{x},t)\, \frac{\partial g_{jr}(\mathbf{x},t)}{\partial x_l}\ , \quad \kappa_{jk}(\mathbf{x},t) = K_{rs}\, g_{jr}(\mathbf{x},t)\, g_{ks}(\mathbf{x},t)\ , \quad (3)$$

$\kappa_j(\mathbf{x},t)$ – drift coefficients; $\kappa_{jk}(\mathbf{x},t)$ – diffusion coefficients.

Eq. (2) is a linear parabolic partial differential equation. It can be found, together with detailed derivation and analysis of various aspects, in many monographs devoted primarily to stochastic differential systems, see for instance : [1–5] and many others. Moreover hundreds of problem oriented papers dealing with various aspects of this topic have been published during the last several decades. More general and complex versions of FP equation exist as well (non-Gaussian inputs, problems of optimal filtering, identification problems, etc.). However the most common is the basic form, Eq. (2).

Also more general formulations of the system (1) can be considered in order to combine external random and deterministic effects absolutely. Then they would be expressed each one by means of one function on the right hand side of the respective equation in the system (1). Nevertheless they appear only exceptionally in physical applications and monographs mentioned above don't employ them in detail. Even cases when the nonlinear input of a random process is necessary to be respected, auxiliary variables can be introduced. Then the input can be modeled as a result of a non-linear filtering of a white noise being introduced in a form of a linear combination just as in the basic case.

While drift and diffusion coefficients are time independent, the applications are pointed to the stationary solution of FP equation as a rule, because it provides the most important information concerning the long term behavior of the system (1). If the system response is stationary, its PDF becomes time independent and therefore the left hand side of Eq. (2) vanishes.

In spite of that many problems require necessarily to look for a non-stationary solution of Eq. (2), even if both drift and diffusion coefficients are time independent. The reason can be a physical nature of the problem, necessity to assess a transition effect or a simple fact that the stationary solution doesn't exist, i.e. [6, 7]. The problem is getting more complicated in such a case, because a matter of a prospective post-critical convergence should be assessed, etc. Nevertheless the FP equation basic form is linear and therefore some analogies of various methods well known from the deterministic domain are applicable. However specific properties of the operator (2) must be respected. They can influence or prevent applicability of those in particular cases.

The question is what method of solution should be selected for a particular mechanical system. There have been published thousands of papers dealing with this topic. Despite the fact that a few cases of closed form analytical solution exist, the papers are mostly focussed on various approximative solution types.

The first group can be called semi-analytical processes. They are based on amendments, modifications and other treatment of primary analytical results. Various properties of the Boltzman entropy of probability can be used, some variational principles or decomposition into a series of stochastic moments or cumulants are also applicable. Asymptotic methods have been summarised in monograph [8]. Many general methods and algorithms are described in monographs above, see [1–5]. A large number of special papers, e.g. [9], are available as well.

The second group is based primarily on numerical procedures. The comprehensive state of the art concerning applications of numerical methods for analysis of the FP equation has been published by a team of twenty authors [10]. Before as well as after this date further papers have been published being oriented to the Finite Element Method (FEM) style of the FP equation handling. The first attempts at the FEM application in numerical treatment of the FP equation date back to the early seventies. As the first systematic studies oriented to FEM introduction to this task can be considered publications by Bergman, Spencer and co-authors. Let us cite for instance [11–15], etc. A possibility of a numerical solution of the FP equation by means of the FEM has reminded the Czech community the recent paper [16]. Although contributions of these studies are undisputed, the author basis still remains quite limited. Whereas earlier referenced papers are focussed to a single degree of freedom systems, the authors' objective is to reach into the near future to common solvability of the PDF for dynamic systems with multiple degrees of freedom.

Many authors have been dealing by various aspects of special FEM variants related with the Galerkin method applied to FP equation. It is not self-adjoint and therefore variational methods based on orthogonalization principles should be employed. A stationary solution has been discussed for instance in [17, 18], for a multi-scale version, see among others [19], etc. The FEM efficiency when solving FP equation seems to be enormous. The FEM tool makes it possible to abandon a supposition of Gaussian inputs in Eq. (1) without any principle difficulties. When FP equation succeeds to be derived for instance for Poisson chains, the method is working quite reliably, see e.g. [20], with extensive links to additional papers, e.g. [21–23]. Even if one or several state variables gain values within a finite interval only, the solution setup doesn't cause any difficulties.

On the other hand some shortcomings of FEM cannot be overlooked. To introduce the deterministic initial condition for PDF in a form of the Dirac function is hardly possible. Nevertheless this circumstance is not too heavy. Much worse it reveals the fact that an increase of degrees of freedom in the system (1) results in an exponential growth of independent variables. Altogether analytical methods also suffer from that. This circumstance is manifested by two factors. The first consists in a need to evaluate integrals on individual finite elements in a hyper-space with a large number of independent variables (n degrees of freedom leads to a space dimension of $2n$). Another difficulty follows from an exponential increase of the size of an ordinary differential system which is generated as a result of Eq. (2) right hand side discretisation. The solution procedure of a stationary problem (with zero left hand side of Eq. (2)) can become problematic in particular on an infinite multidimensional domain when the discrete character of FP operator eigen values partly or entirely vanishes. Another inconvenience consists in the fact that every FP equation being assigned to a particular physical system (1) every time needs a new finite element to be developed as the drift and diffusion coefficients include all information regarding the structure of the system (1).

In spite of that it seems that merits prevail in many important cases and FEM could provide a valuable and effective tool for FP equation analysis. Let us present a few examples of single degree of freedom (SDOF) systems excited by additive and multiplicative Gaussian noises. Some numerical results obtained by FEM facilitate being compared with closed form or approximative solutions, which can be obtained using the Boltzman principle, see e.g. [4]. It should be remembered that the system (1) after a transformation into the Ito form can be subdued to the direct numerical solution as a stochastic differential system. Its stochastic nature however must be carefully respected, see e.g. [24]. These steps enable verifying the FEM results comparing them with analytical or semi-analytical results and with those obtained by means of direct numerical simulation.

First of all some properties of finite elements and methods of numerical integration used in an actual case should be pointed out. With respect to non-symmetry of the FP operator and its other properties the Galerkin method in the form of the Petrov version has been applied. In order to avoid any secondary non-homogeneity the integration domain has been split in all cases into the rectangular elements of identical size without any network condensation in areas of 'dramatic' PDF changes. FP equation remains linear and in individual cases which will be discussed in two state variables $\mathbf{x} = (x_1, x_2)$ only. Therefore the problem of the element multi-dimensionality drops out in the meanwhile and a conventional integration process can be applied.

Taking into account that FP equation is of the second order in $\mathbf{x}$ coordinates, elements with linear approximation between nodes is satisfactory in order to fulfill conditions of 'smoothness' of approximation function and therefore to get a guarantee of a convergence if a stable solution exists. Let us introduce in the domain of one element an approximation function, see Fig. 1, consisting of shape functions:

$$\begin{aligned} & p^e(x_1^e, x_2^e) = \sum_{ij=1}^{2} P_{ij}^e \cdot p_{ij}^e(x_1^e, x_2^e) \, , \qquad p_{ij}^e(x_1^e, x_2^e) = p_{ij}^e \, , \\ & p_{11}^e = \frac{(h_1 + 2\,x_1^e)\,(h_2 + 2\,x_2^e)}{4\,h_1\,h_2} \, , \qquad p_{12}^e = \frac{(h_1 - 2\,x_1^e)\,(h_2 + 2\,x_2^e)}{4\,h_1\,h_2} \, , \\ & p_{21}^e = \frac{(h_1 + 2\,x_1^e)\,(h_2 - 2\,x_2^e)}{4\,h_1\,h_2} \, , \qquad p_{22}^e = \frac{(h_1 - 2\,x_1^e)\,(h_2 - 2\,x_2^e)}{4\,h_1\,h_2} \, . \end{aligned} \tag{4}$$

x_1^e, x_2^e – coordinates within one finite element, p_{ij}^e – shape functions, P_{ij}^e – PDF values in element nodes, h_1, h_2 – element dimensions.

Let us suppose for a simple demonstration that only one additive noise w_a, ($n = 2$, $m = 1$) acts in the system. Into functions g_{jr} following constants should be introduced: $g_{11} = g_1 = 0$, $g_{21} = g_2 = 1$. Approximation (4) should be substituted in Eq. (2). Making further alterations in Galerkin-Petrov meaning, matrices $\mathbf{M}^e$, $\mathbf{S}^e$ (2×2) being valid for one element can be obtained. Elements of respective matrices read:

$$M_{ij}^e = \sum_{kl=1}^{2} \int_\Omega p_{ij}^e\, p_{kl}^e \,\mathrm{d}x_1\,\mathrm{d}x_2 \ , \qquad \Omega - \text{integration domain of one element} \ ,$$

$$S_{ij}^e = \sum_{kl=1}^{2} \int_\Omega \Bigg[p_{ij}^e\, p_{kl}^e \left(\frac{\partial f_1(x_1^e, x_2^e, t)}{\partial x_1} + \frac{\partial f_2(x_1^e, x_2^e, t)}{\partial x_2} \right) + \\ + p_{ij}^e \left(f_1(x_1^e, x_2^e, t) \frac{\partial p_{kl}^e}{\partial x_1} + f_2(x_1^e, x_2^e, t) \frac{\partial p_{kl}^e}{\partial x_2} \right) + K_{aa} \frac{\partial p_{ij}^e}{\partial x_2} \frac{\partial p_{kl}^e}{\partial x_2} \Bigg] \mathrm{d}x_1\,\mathrm{d}x_2 \qquad (5)$$

Matrices (5) after transformation into the global coordinates should be uploaded into global matrices $\mathbf{M}, \mathbf{S}$. Hence the system of ordinary differential equations arises:

$$\mathbf{M}\,\dot{\mathbf{P}} = \mathbf{S}\,\mathbf{P} \qquad (6)$$

where $\mathbf{P}$ is a vector of PDF values in nodes of the network. With respect to structure of the system (1), elements of the matrix $\mathbf{S}$ in general are time dependent.

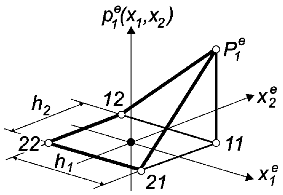


Fig.1: PDF approximation outline in a domain of one finite element

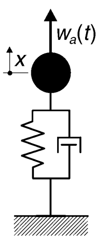


Fig.2: Outline of an SDOF linear system

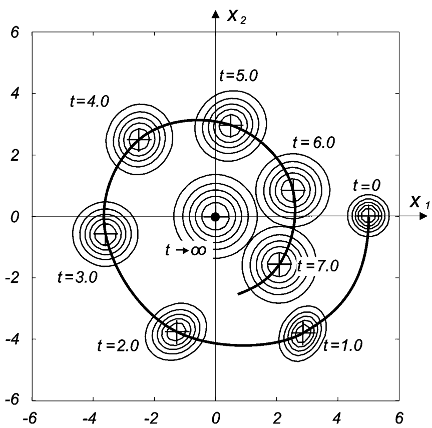


Fig.3: Contour diagram of the PDF evolution of an SDOF sytem response since the excitation beginning until the stationary state

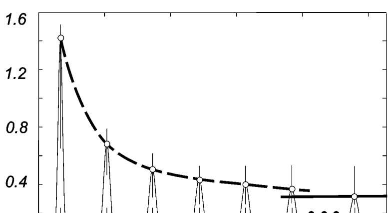


Fig.4: PDF vertical section x_1 (deflection) in selected moments starting from the initial condition until the stationary state

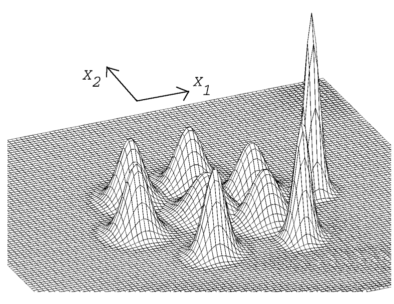


Fig.5: Axonometric display of the PDF evolution of an SDOF system response since the excitation beginning ($t=0$) until the stationary state ($t=90\,s$)

Attempts for any higher approximation using Lagrangian polynomials or smooth derivatives in nodes (l'Hermite) didn't approve. CPU time increased visibly and numerical stability didn't get better. As the most effective method of a differential system solution approved the process of predictor-corrector type based on the Adams algorithm. As the main tool applied to develop the respective element and to carry out numerical integration of system (6) the COMSOL MULTIPHYSICS$^{\mathrm{TM}}$ code has been used.

3. Linear single degree of freedom system

Let us assume that an SDOF system, see Fig. 2, is excited by an additive noise and a multiplicative noise in the damping coefficient. The respective differential equation reads:

$$\ddot{x} + 2\,\omega_b(1+w_b)\,\dot{x} + \omega_0^2\,x = w_a \quad \Rightarrow \quad \begin{aligned} \dot{x_1} &= x_2 \ , \\ \dot{x_2} &= -\omega_0^2\,x_1 - 2\,\omega_b\,x_2 - 2\,\omega_b\,x_2\,w_b + w_a \ . \end{aligned} \tag{7}$$

Processes $w_b = w_b(t)$, $w_a = w_a(t)$ are supposed to be centered Gaussian white noises. Their densities are denoted: K_{bb}, K_{ab}, K_{aa}. Drift and diffuse coefficients follows from Eqs. (3):

$$\begin{gathered} \kappa_1 = x_2 \ , \qquad \kappa_2 = -[\omega_0^2\,x_1 + 2\,\omega_b\,x_2\,(1-\omega_b\,K_{bb}) + \omega_b\,K_{ab}] \ , \\ \kappa_{22} = 4\,K_{bb}\,\omega_b^2\,x_2^2 - 4\,K_{ab}\,\omega_b\,x_2 + K_{aa} \ . \end{gathered} \tag{8}$$

The FP equation can be obtained substituting formula (8) into its general form Eq. (2). After some adaptation one obtains:

$$\begin{aligned} \frac{\partial p}{\partial t} = -\frac{\partial\,(x_2\,p)}{\partial x_1} + \frac{\partial}{\partial x_2}\left[(\omega_0^2\,x_1 + 2\,\omega_b\,x_2\,(1-\omega_b\,K_{bb}) + \omega_b\,K_{ab})\,p\right] + \\ + \frac{1}{2}\,\frac{\partial^2}{\partial x_2^2}\left[(4\,K_{bb}\,\omega_b^2\,x_2^2 - 4\,K_{ab}\,\omega_b\,x_2 + K_{aa})\,p\right] \ . \end{aligned} \tag{9}$$

It can be shown, see e.g. [3, 4], that a stationary solution of Eq. (9) (i.e. when infinite time after excitation beginning elapsed) exists having the Boltzman form which reads:

$$p(x_1, x_2) = N\,\exp\left[-\frac{\omega_b\,(\omega_0^2\,x_1^2 + x_2^2)}{K_{aa}}\right] \tag{10}$$

where N represents a normalization factor. Its value is given by a condition that the integral of the function (10) over an infinite domain should equal to one.

System parameters for the purpose of numerical solution of Eq. (9) have been chosen as follows: $\omega_0^2 = 1.0$, $\omega_b = 0.05$, $K_{aa} = 0.2$, $K_{ab} = K_{bb} = 0.0$. The excitation starts at point $t = 0$. An initial system position has been put into the point:

$$x_{1,0} = 5.0 \ , \qquad x_{2,0} = 0.0 \ . \tag{11}$$

An initial condition for PDF can be selected in a form:

$$p(x_1, x_2, 0) = N\,\exp\left[-\frac{\omega_0^2\,(x_1 - x_{1,0})^2}{\sigma^2}\right]\,\exp\left[-\frac{(x_2 - x_{2,0})^2}{\sigma^2}\right] \ , \tag{12}$$

where $N = 1/(2\,\pi\,\sigma^2)$, $\sigma^2 = 1/9$. The initial condition (12) approaches for a small value σ^2 to the Dirac function as it was primarily requested. It admits that the system response doesn't begin at point $(x_{1,0}, x_{2,0})$ for a certainty as it would be required by the Dirac function. The starting position (10) holds in compliance with the PDF (12) as almost sure only.

The numerical solution flow didn't bring any severe difficulties which would imply needs of some adaptations of the computing process. Changes of the PDF form starting from the initial condition (12) until the stationary state represent rather quantitative alterations. It doesn't generate any basic changes in the PDF form within this time interval. The whole process of numerical integration can be understood as an analogy with the dynamic relaxation process where a certain initial estimate of the static solution is inserted. The role of the initial estimate takes over here the respective PDF corresponding to the stationary and therefore to a time independent state. Consequently, the initial and final state in the form of an exponential function in both variables x_1, x_2 doesn't initiate any endangering of numerical stability.

Selected results of the FEM solution are depicted in Figs. 3–5. It is obvious from Fig. 3, that the response PDF peak follows in the plane x_1, x_2 an exponential spiral coming up to that being obtained by force of a deterministic analysis of a linear SDOF system eigen vibration for initial conditions (11). The origin becomes an attractor. The system response converges either to a standstill at this point (deterministic case without external excitation), or to stationary random movement being described by a product of two Gaussian functions (10).

The drop of PDF peak with time is obvious in Fig. 4. Individual parts of the figure demonstrate a vertical section in x_1 (deflection). It proves once again a roughly exponential drop from a level given by an initial condition as far as a horizontal asymptote outlined in the right lower part of the figure. Indeed this trend is also apparent from the axonometric display of PDF evolution in Fig. 5. Comparing the FEM solution values with those coming out of the Boltzman solution (11), their almost absolute coincidence can be learned. Comparative calculation using the spectral method provided identical results as well.

Particular specification of input noise densities excluded in fact the multiplicative noise w_b. Nevertheless the target was to demonstrate an efficiency of a FEM solution procedure applied to Eq. (9) under common conditions and to keep a verification possibility with known results. For increasing density of the additive noise some effects of parametric stability loss in a stochastic meaning is apparent.

4. Non-linear system of Duffing type

The Duffing equation in basic or normal form under white noise additive and multiplicative excitations can be written as follows:

$$\ddot{x} + 2\,\omega_b\,\dot{x} - \omega_0^2\,x\,(1 + w_s - \alpha^2\,x^2) = w_a \quad \Rightarrow \quad \Rightarrow \quad \begin{aligned} \dot{x_1} &= x_2 \;, \\ \dot{x_2} &= \omega_0^2\,x_1\,(1 - \alpha^2\,x_1^2) - 2\,\omega_b\,x_2 + \omega_0^2\,x_1\,w_s + w_a \;. \end{aligned} \tag{13}$$

Eqs. (3) imply relevant drift and diffuse coefficients:

$$\begin{aligned} \kappa_1 &= x_2 \;, \qquad \kappa_2 = \omega_0^2\,x_1\,(1 - \alpha^2\,x_1^2) - 2\,\omega_b\,x_2 \;, \\ \kappa_{11} &= \kappa_{12} = \kappa_{21} = 0 \;, \qquad \kappa_{22} = K_{ss}\,\omega_0^4\,x_1^2 + 2\,K_{as}\,\omega_0^2\,x_1 + K_{aa} \;. \end{aligned} \tag{14}$$

Using Eqs. (13), (14) relevant FP equation can be evolved:

$$\frac{\partial p}{\partial t} = -\frac{\partial (x_2 \, p)}{\partial x_1} - \frac{\partial \left[(\omega_0^2 \, x_1 \, (1 - \alpha^2 \, x_1^2) - 2 \, \omega_b \, x_2) \, p \right]}{\partial x_2} + \\ + \frac{1}{2} \, \frac{\partial^2 \left[(K_{ss} \, \omega_0^4 \, x_1^2 + 2 \cdot K_{as} \, \omega_0^2 \, x_1 + K_{aa}) \, p \right]}{\partial x_2^2} \; . \tag{15}$$

In the absence of a multiplicative noise ($K_{ss} = K_{as} = 0$) the system (13) can be carried out of a simple Hamiltonian and therefore the Boltzman solution of a stationary version of Eq. (15) can be simply formulated once again:

$$p(x_1, x_2) = N \, \exp \left[\frac{\omega_b \, \omega_0^2 \, x_1^2}{K_{aa}} \left(1 - \frac{1}{2} \, \alpha^2 \, x_1^2 \right) \right] \, \exp \left(-\frac{\omega_b \, x_2^2}{K_{aa}} \right) \; . \tag{16}$$

Eq. (13) describes the Miesess truss movement under white noise excitation, see Fig. 6. The stiffness linear part is negative and consequently the system includes an unstable stationary point in the origin $(0,0)$. Two stable stationary points have position $(\pm 1/\alpha, 0)$. The repulsivity level in the origin depends on a relation of both stiffness parts and on the multiplicative noise w_s density.

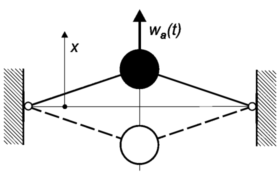


Fig.6: Outline of the SDOF Duffing system

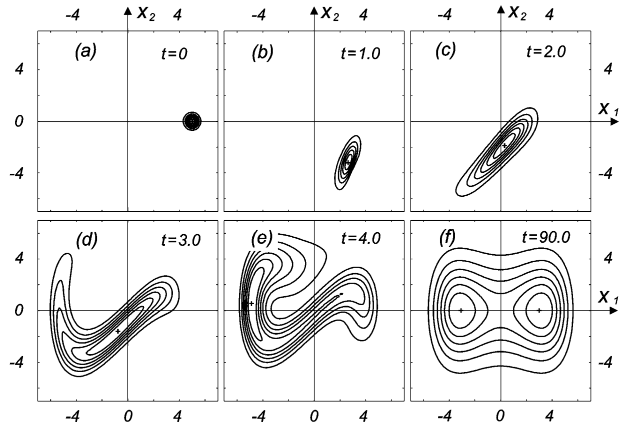


Fig.7: Contour diagrams of the PDF evolution of the Duffing system; excitation density $K_{aa} = 4.0$

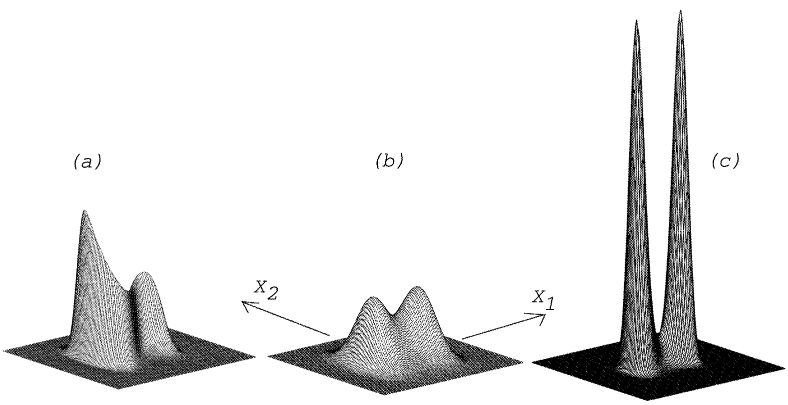


Fig.8: Axonometric display of PDF of the Duffing system response for $K_{aa} = 4.0$ in the moment $t = 4\,\mathrm{s}$ and in the stationary state – parts (a), (b); stationary PDF for $K_{aa} = 0.2$ – part (c)

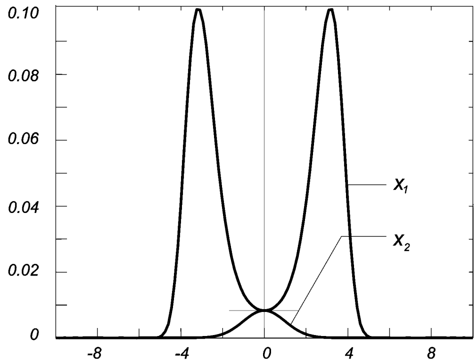


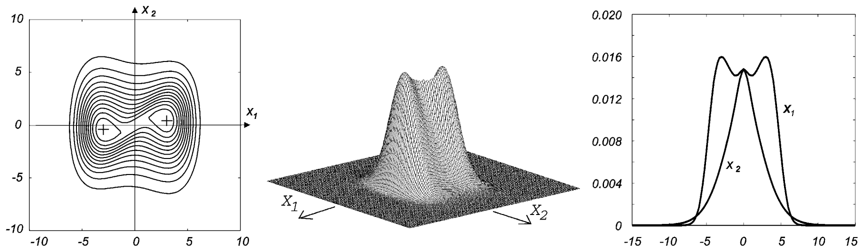
Fig.9: Vertical sections of PDF for Duffing system in the stationary state for $K_{aa} = 0.2$: x_1-deflection, x_2-velocity

The domain splitting into elements and other circumstances are similar to the preceding case : $\omega_0^2 = 1.0$, $\omega_b = 0.05$ non-linearity ratio $\alpha^2 = 0.1$. If the solution process is started at point $(x_{1,0}, x_{2,0})$ according (11) and initial condition (12), the PDF evolution in time can be followed again. Let us deal thoroughly the case of the additive excitation only.

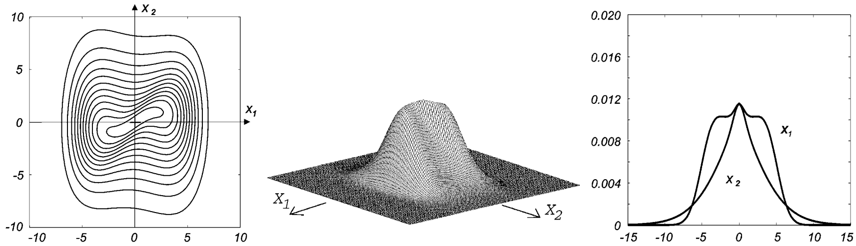
It emerged that the excitation process density K_{aa} should be inserted higher than a certain threshold value. This effect arises from dynamic relaxation properties. The resulting stationary PDF form (if it exists) cannot be basically different from the initial condition. It results from numerical experiments that $K_{aa} = 4.0$ can be considered. The integration flow for this excitation density can be observed on contour diagrams in Fig. 7. Part (a) represents the initial condition. After 1 s, 2 s and 3 s, parts (b)–(d), PDF is still unimodal and the mathematical mean value doesn't follow any exponential spiral like in the linear case. The bizarre PDF form in the moment $t = 3$ s, part (d) should be noticed. Two extremes appear after 4 s, part (e). On 90 s, part (d), PDF becomes symmetric having two equivalent maxima. Their superelevation above a saddle point is not too high. Final phase of this process for $K_{aa} = 4.0$ is visible in Fig. 8(b). It arrives approximately after 60–90 s. In the final phase, see Fig. 8(a), a periodic alternation of an absolute peak between both extremes is appreciable. This oscillating rundown successively disappears during transition into the stationary response process when PDF becomes symmetric in both axes.

Reducing the excitation density to value $K_{aa} = 0.2$ and starting from the initial condition (12), the integration process fails. However it can get through if in the meaning of the initial condition the stationary PDF for $K_{aa} = 4.0$ is used. Result of this computation is demonstrated in Fig. 8(c). Domination of both stable stationary points is well marked. The same is obvious observing PDF vertical sections along axes x_1, x_2, see Fig. 9. Response velocity PDF comes up to usual Gaussian curve, while deflection PDF is concentrated around both equilibrium points (bimodal character). Let us subjoin that an equivalence of numerical and analytical results was excellent once again.

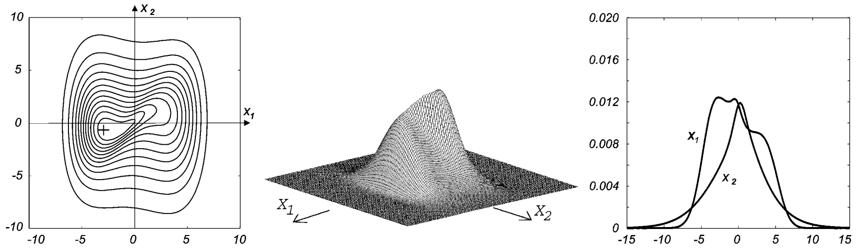
If the density of noise w_a is low, the movement prevails around one of equilibrium points in a stationary state, see Fig. 8(c). This markable feature is going to disappear with increasing excitation density and for its high values the response PDF doesn't differ much from that being valid for pure cubic stiffness characteristic. In such a case the bimodal character of PDF decays rapidly, see Fig. 8(b). The PDF finally changes almost into the unimodal form



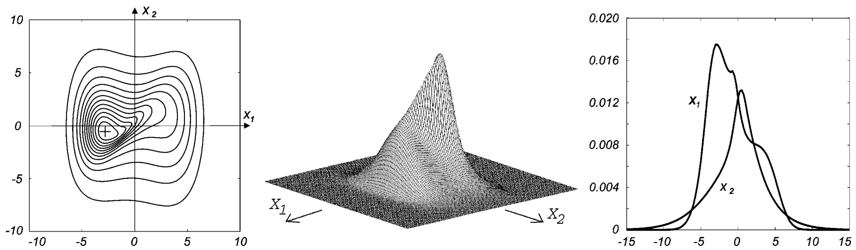
$(a)\quad \omega_0^2 = 1.0, \omega_b = 0.1, \alpha^2 = 0.1, \quad K_{aa} = 0.80, K_{ss} = 0.25, K_{as} = 0.0.$



$(b)\quad \omega_0^2 = 1.0, \omega_b = 0.1, \alpha^2 = 0.1, \quad K_{aa} = 0.80, K_{ss} = 0.50, K_{as} = 0.0.$



$(c)\quad \omega_0^2 = 1.0, \omega_b = 0.1, \alpha^2 = 0.1, \quad K_{aa} = 0.80, K_{ss} = 0.50, K_{as} = 0.25.$



$(d)\quad \omega_0^2 = 1.0, \omega_b = 0.1, \alpha^2 = 0.1, \quad K_{aa} = 0.80, K_{ss} = 0.50, K_{as} = 0.50.$

Fig.10: Duffing system in the stationary state under concurrent action of additive and multiplicative noises; excitation processes w_a, w_s are independent $(K_{as} = 0)$ – parts (a), (b); processes w_a,w_s are correlated $(K_{as} > 0)$ – parts (c), (d)

as it holds for the linear system (see the previous section). Indeed a certain difference should remain due to a dominant cubic part of the stiffness.

Under a certain coincidence of system and excitation parameters the stochastic resonance can emerge. It manifests by regular jumps between stationary points of the bistable elastic potential (interwell hopping). Then a local random movement with low variance occurs around each of stationary points. This quasi-periodic process is very stable if an exact parameter configuration is kept.

It comes to light from Eq. (16), that PDF being considered separately along x_1 (deflection) and x_2 (velocity) represents independent processes. The difference between a local extreme in the saddle point and a maximum in stable stationary points drops with the damping factor increase. This knowledge comes out not only from the well known formula (16), but also from the FEM analysis of the FP equation (15). An analytical solution (16) and FEM analysis of Eq. (15) for a solely additive excitation leads to the identical result that PDF in the stationary state is symmetric along axes x_1, x_2, see Fig. 7(f).

Let us discuss results demonstrated in Fig. 10. They deal with the stationary response for four parameter sets. Three pictures in every part (a)–(d) mean (i) PDF contour diagram, (ii) axonometric display of the same surface and (iii) vertical sections along axes x_1, x_2. The system itself and additive noise density are identical every time $(\omega_0^2, \omega_b, \alpha^2, K_{aa})$. The multiplicative noise density is variable just like the cross density K_{as}. So far the multiplicative noise density increases then the bimodal shape of PDF is going to disappear. PDF maxima at points $(\pm 1/\alpha, 0)$ are vanishing and a small peak is emerging in the origin, compare Figs. 10(a) and 10(b). This peak is growing as the density K_{ss} increases. The peak becomes successively dominant and suppresses both peaks in points $(\pm 1/\alpha, 0)$. This effect can be explained by an increasing influence of the multiplicative noise which step by step depresses the control importance of the linear part of the stiffness. Finally the system behavior remembers a case when stiffness is almost permanently positive and the most probable position of the system mass locates in the origin.

If the noises are independent, the surface of PDF in the stationary state is always symmetric with respect to the vertical axis in the origin, although it loses the symmetry along x_1, x_2 axes which are typical for the state where the multiplicative noise is absent. With rising levels the PDF surface has a tendency to twist counter clockwise. It holds especially in peak areas, see Figs. 10(a), (b). Once $K_{as} > 0$, the shape of the PDF loses any symmetry. Only one peak arises and moves into the domain nearby the point $(-1/\alpha, 0)$. The evolution of this process can be followed in the series demonstrated in Figs. 10(a)–(d).

5. Nonlinear system of Van der Pol type

The Van der Pol equation in basic or normal form being excited by additive and multiplicative random noises has a form:

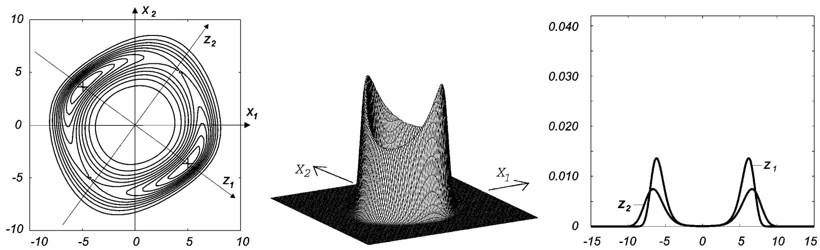
$$\ddot{x} - 2\,\omega_b\,(1 + w_b - \beta^2\,x^2)\,\dot{x} + \omega_0^2\,x = w_0 \quad \Rightarrow$$
$$\Rightarrow \quad \begin{aligned} \dot{x_1} &= x_2\ , \\ \dot{x_2} &= -\omega_0^2\,x_1 + 2\,\omega_b\,(1 - \beta^2\,x_1^2)\,x_2 + 2\,\omega_b\,x_2\,w_b + w_a\ . \end{aligned} \tag{17}$$

Using Eqs. (3) drift and diffusion coefficients can be easily derived:

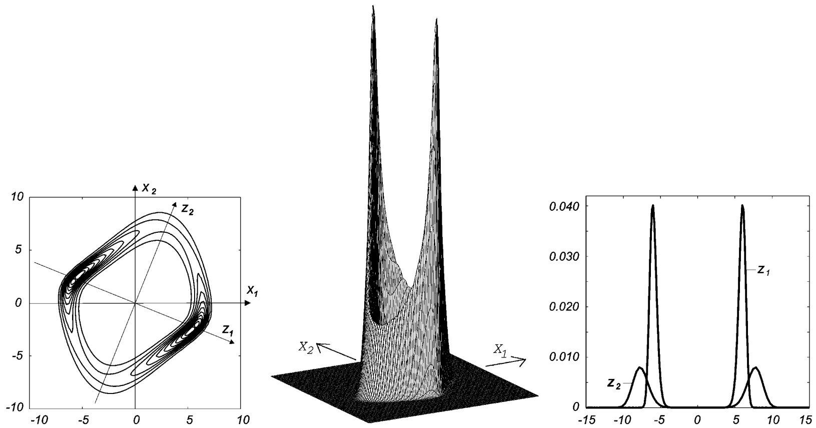
$$\begin{aligned} \kappa_1 &= x_2\ , \qquad \kappa_2 = -\omega_0^2\,x_1 + 2\,\omega_b\,x_2\,(1 - \beta^2\,x_1^2 + \omega_b\,K_{bb}) + \omega_b\,K_{ab}\ , \\ \kappa_{11} &= \kappa_{12} = \kappa_{21} = 0\ , \qquad \kappa_{22} = 4\,K_{bb}\,\omega_b^2\,x_2^2 + 4\,K_{ab}\,\omega_b\,x_2 + K_{aa}\ . \end{aligned} \tag{18}$$

FP equation corresponding to Van der Pol system (17) reads:

$$\begin{aligned} \frac{\partial p}{\partial t} = -\frac{\partial\,(x_2\,p)}{\partial x_1} &+ \frac{\partial\left[(\omega_0^2\,x_1 - 2\,\omega_b\,x_2\,(1 - \beta^2\,x_1^2 + \omega_b\,K_{bb}) - \omega_b\,K_{ab})\,p\right]}{\partial x_2} + \\ &+ \frac{1}{2}\,\frac{\partial^2\left[(4\,K_{bb}\,\omega_b^2\,x_2^2 + 4\,K_{ab}\,\omega_b\,x_2 + K_{aa})\,p\right]}{\partial x_2^2}\ . \end{aligned} \tag{19}$$



$(a)\quad \omega_0^2 = 1.0, \omega_b = 0.1, \beta^2 = 0.1, \quad K_{aa} = 0.80, K_{bb} = 0.0, K_{ab} = 0.0.$



$(b)\quad \omega_0^2 = 1.0, \omega_b = 0.3, \beta^2 = 0.1, \quad K_{aa} = 0.80, K_{bb} = 0.0, K_{ab} = 0.0.$

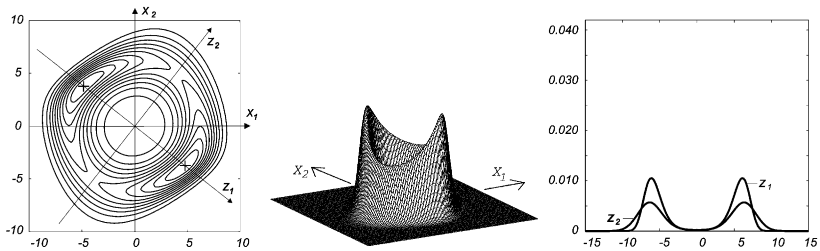
Fig.11: Van der Pol system in the stationary state under random additive excitation (multiplicative excitation prevented); lower linear damping – part (a), higher linear damping – part (b)

Contrary to the previous two cases a known simple solution of the stationary version of Eq. (19) analogous to Boltzman solution (10) or (16) doesn't exist. Nevertheless the result for $t \to \infty$ can be compared with a number of approximative analytical solutions, see e.g. [4].

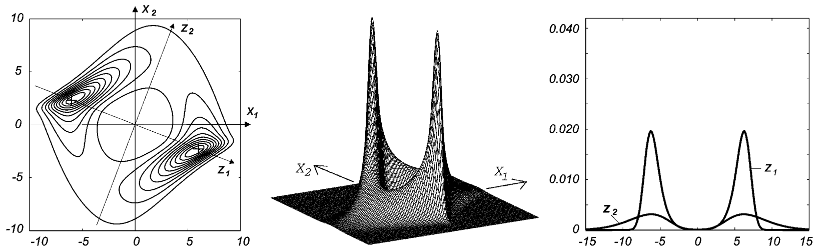
Computations have been done in two series. In the first one the initial condition of the system has been introduced in the form (12) for $x_{1,0} = 5.0$, $x_{2,0} = 0.0$, i.e. with a non-zero initial deflection. On condition that $\omega_0^2 = 1.0$, $\omega_b = 0.05$, $\beta^2 = 1.0$, the stationary state arrives in 60 to 90 seconds. Much like the analytical investigation, numerical evaluation also proved that the PDF form in the stationary state and the length of the transition process are very sensitive to an excitation density level K_{aa}.

Let us notice that from the viewpoint of detailed numerical integration any arbitrary value in the interval $K_{aa} \in (0.2, 6.0)$ didn't present a problem. To split the integration process into two or more stages as with the Duffing system was not necessary.

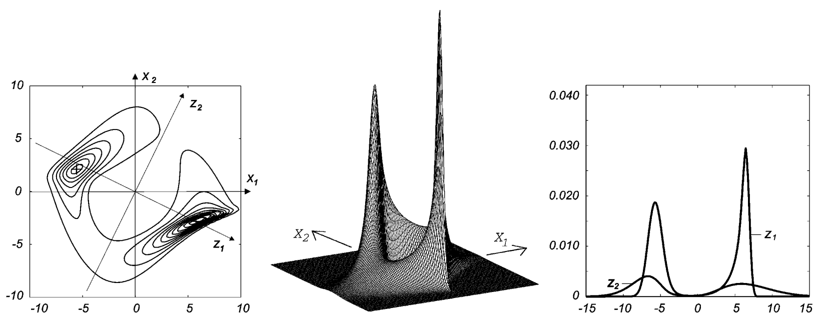
An outline of the PDF transition process in the time interval from zero to stationary state for a relatively high value $K_{aa} = 4.0$ and with absenting multiplicative excitation is similar with that concerning the Duffing system under comparable conditions. The final PDF stationary state for this excitation density level remembers in principle the basic PDF shape of the Duffing system being turned counter clockwise approximately $\pi/4$. It means in particular that two equivalent maxima and one saddle point can be recognized once again. The contour diagram is neither symmetric along axes x_1, x_2 nor z_1, z_2. It leads to an



$(a)\quad \omega_0^2 = 1.0, \omega_b = 0.1, \beta^2 = 0.1, \quad K_{aa} = 0.80, K_{bb} = 0.50, K_{ab} = 0.0.$



$(b)\quad \omega_0^2 = 1.0, \omega_b = 0.3, \beta^2 = 0.1, \quad K_{aa} = 0.80, K_{bb} = 0.50, K_{ab} = 0.0.$



$(c)\quad \omega_0^2 = 1.0, \omega_b = 0.3, \beta^2 = 0.1, \quad K_{aa} = 0.80, K_{bb} = 0.50, K_{ab} = 0.50.$

Fig.12: Van der Pol equation in the stationary state under simultaneous action of additive and multiplicative excitation; excitation processes w_a, w_b : independent ($K_{ab} = 0$) – parts (a), (b) or correlated ($K_{ab} > 0$) - part (c)

asymmetry of ascending and descending response phases in time domain from the point of view of a broad band structure of the response.

Following carefully the transition process for the given parameter specification, it reveals that once the PDF evolution leaves off a marked unimodal character extrapolating the initial condition, the process of alternately rising and dropping peaks of the bimodal PDF shape occurs. This effect is more distinctive here than in the case of the Duffing system as was pointed out in the previous section. However this effect is going to lose its alternating character with increasing time as well and PDF approaches to the stationary state which stands out by a symmetry of PDF with respect to the origin.

Essentially a different shape of PDF in the stationary state is provided for small value K_{aa}. For $K_{aa} = 0.2$, PDF approaches nearly to a rotating shape with a deep 'depression' around the origin. In such a state the system acts as a self-exciting resonator predetermined by a negative linear part of the damping. This effect originates from a stable limit cycle of the Van der Pol system. It emerges for a given deterministic part of parameter configuration and remains in force even when random low density excitation is applied.

In the second series the multiplicative noise has been concerned, moreover even its cross correlation with additive noise has been respected. The integration flow came out from the initial condition (12), i.e. from 'almost sure' home position at $x_{1,0} = 0$, $x_{2,0} = 0$ point. For some input data sets it was necessary to split the computation flow into several parts similarly as in the case of the Duffing system for low K_{aa} values.

Let us evaluate and compare five typical cases for a medium additive noise density $K_{aa} = 0.8$ while $\omega_0^2 = 1.0$, $\beta^2 = 0.1$. Respective results concerning the stationary state are depicted in Figs. 11, 12. Their layout is the same as in Fig. 10 for Duffing system. Each of five cases outlines in the contour diagram (on various clearness levels) the limit cycle being typical for the Van der Pol equation evaluated for discussed parameters in the deterministic domain. Fig. 11 deals with additive excitation only, so that it is linked up to the first series of computation. The response portrait is significantly influenced by a linear part of the damping. The PDF for higher linear damping ratio ($\omega_b = 0.3$) concentrates even more and more around the well expressed limit cycle, see Fig. 11(b). Axonometric demonstration shows a strong PDF variability on the ridge curve indicating the highest probability of the system position in two points on the z_1 axis. Fig. 11(a) ($\omega_b = 0.1$) demonstrates a visibly higher stochastic nature of the response and a dominant role of the first harmonic in the limit cycle. The drop of stochastic and increase of deterministic response component with rising damping ratio ω_b follows from the negative character of the linear part of the damping or in other words from the system instability increase in the origin. In this regime the response is given predominantly by a re-stabilization process due to non-linear part of the damping which provides positive total values as lately as for higher deflections.

Under a multiplicative noise action in a linear part of the damping the portrait of the limit cycle for higher ω_b is going to lose a unimodal character and evidently takes on higher harmonics. The PDF has a tendency to expand towards the origin and to limit itself out of a formation visible in contour diagrams in Figs. 12(a), (b). Including a positive cross-correlation of both noises the PDF shape loses a symmetry with respect to the origin, see Fig. 12(c), similarly as for the Duffing system. The most important tendencies of characteristic points of the PDF dependent on individual parameters of the system and excitation are obvious from vertical sections along z_1,z_2 axes, see Figs. 11(a), (b) and Figs. 12(a)–(c). Comparison of Figs. 12(a), (b) implies that an influence of increasing ω_b asserts itself much stronger when the multiplicative and additive noises are operating simultaneously unlike when the additive noise only is acting.

6. Conclusion

The Fokker-Planck equation represents an important tool determined for probabilistic analysis of dynamic systems subjected to additive and multiplicative random excitation by Gaussian white noises. Possibilities to solve this equation using analytical or semi-analytical methods are limited. It seems that the Finite Element Method is able to occupy an important position among other numerical methods considered for FP equation analysis. It is challenging that many aspects of FP equation remaining hidden for analytical solution procedures can be discovered by means of FEM. Demonstration examples of three SDOF systems with additive and multiplicative random excitation (linear, Duffing, Van der Pol) approved good numerical properties of FEM in widespread external excitation density. Comparison of analytical closed form and approximative results with those obtained by mean of FEM showed almost perfect equivalence. It should be remembered that FEM offers well known advantages which are unattainable using analytical methods. Let us mention a possibility of almost any arbitrary form of the definition domain, any mathematically admissible and meaningful combination of boundary conditions are acceptable, etc.

On the other hand many specific attributes of the Fokker-Planck operator should be respected. They differentiate this one from those being typical for linear or non-linear solid state mechanics. Main differences consist in several facts: non-symmetry of FP operator, mostly infinite dimensions of definition domain and subsequent need to limit them in some acceptable way, etc. In general the high multi-dimensionality of finite elements encountered is non-standard. It will require developing special integration methods being based for instance on the Monte Carlo principle. At any rate the higher approximation degrees should be avoided.

In spite of optimistic initial experiences many other questions still remained open. They refer numerical reliability and stability related to various types of multiplicative noises occurred, formulation of initial conditions, needs to split the integration flow into consecutive stages, etc. Some possible ways to solve these problems are outlined in the previous text.

Acknowledgement

The kind support of the Czech Science Foundation No. 103/06/0099, Grant Agency of the ASCR No. A2071401 and AV 0720710524 research plan are gratefully acknowledged.

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Received in editor's office: March 25, 2008
Approved for publishing: May 14, 2008