NUMERICAL SIMULATION OF AEROELASTIC PROBLEMS WITH CONSIDERATION OF NONLINEAR EFFECTS

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In this paper the numerical approximation of a two dimensional aeroelastic problem is addressed, where nonlinear effects are considered. For the flow model we use the Navier-Stokes equations, spatially discretized by the FE method and stabilized with a modification of the Galerkin Least Squares (GLS) method. The motion of the computational domain is treated with the aid of the Arbitrary Lagrangian Eulerian (ALE) method. The structure model is considered as a solid body with two degrees of freedom (bending and torsion). The motion is described with the aid of a system of nonlinear differential equations and coupled with the flow model by the strongly coupled algorithm.

Keywords: *aeroelasticity, finite element method, Arbitrary Lagrangian Eulerian method*

1. Introduction

The numerical approximation of fluid-structure interaction (FSI) is becoming to be important in many technical and scientific applications, cf. [2], [1]. In order to properly approximate the mutual interaction between fluid and structure, various strategies are used. During last years, significant advances have been made in the development of computational methods for fluid-structure interaction problems. The arbitrary Lagrangian-Eulerian (ALE) formulations are usually employed. The application of the ALE method is straightforward, cf. [9], [8], but there is still a number of important computational issues which need to be properly addressed, cf. [4]. The application of the ALE method depends on the discretization method used.

In this paper the main attention is paid to the numerical approximation of fluid-structure interaction problems particulary for the near- and post-critical regimes. The conservative ALE formulation of Navier-Stokes system is considered (see, e.g. [8]) and numerically approximated by the finite element method. In the finite element context the ALE non-conservative formulation of the incompressible Navier-Stokes system is usually used, cf. [6]. For the finite volume schemes the ALE conservative formulation are used, cf. [4], but it can be applied also in the finite element context, see [8].

We focus here on the use of ALE conservative formulation of the Navier-Stokes system in the finite element context. In order to explain the differences between conservative and non-conservative formulation a simplified elliptic problem is considered, formulated weakly on moving grids with the aid of ALE conservative and non-conservative formulations, and discretized by the finite element method. The numerical solution of a benchmark problem (cf. [8]) is found and the results of different formulations are compared.

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Further, the ALE conservative formulation of the Navier-Stokes equations is formulated weakly, discretized by the finite element method, stabilized and applied to an aeroelastic problem. The numerical solution of the post-critical case, where the loss of stability occurs, is presented. This paper is an extended version of the paper [12].

2. Mathematical model

The mathematical formulation of the problem consists of the flow model, the structure model and the interface conditions. We assume that the fluid motion is described by the incompressible Navier-Stokes system written in the *ALE conservative* form, cf. [8],

$$\begin{aligned} \frac{1}{\mathcal{J}}\frac{\mathrm{D}^{\mathcal{A}}}{\mathrm{D}t}\Big(\mathcal{J}\,v_i\Big) + \operatorname{div}\left[(\mathbf{v}-\mathbf{w}_{\mathrm{D}})\,v_i\right] - \nu\,\triangle v_i + \frac{\partial p}{\partial x_i} &= 0\ , \quad i = 1,2\ , \\ \operatorname{div}\mathbf{v} &= 0\ , \quad \text{in } \Omega_t \subset \mathbb{R}^2\ , \end{aligned} \tag{1}$$

where Ω_t is the computational domain at time t, $\mathbf{v} = (v_1, v_2)$ is the fluid velocity vector, p is the kinematic pressure (i.e., the pressure divided by the constant fluid density ρ), ν is the kinematic viscosity of the fluid (i.e. the viscosity divided by the density ρ).

Here, the ALE method is used. We assume that $\mathcal{A} = \mathcal{A}(\xi,t) = \mathcal{A}_t(\xi)$ is an ALE mapping defined for all $t \in (0,T)$ and $\xi \in \Omega_0$, which satisfies the following assumptions:

(A1) The mapping $t \in (0,T)$, $\xi \in \Omega_0 \mapsto x = \mathcal{A}(\xi,t) \in \Omega_t$ has continuous first order derivatives with respect to t, ξ_1, ξ_2 and continuous second order derivatives $\partial^2\varphi/(\partial\xi_i\,\partial t)$ and $\partial^2\varphi/(\partial t\,\partial\xi_i)$ for $i = 1,2$.

(A2) The mapping $\xi \mapsto x = \mathcal{A}(\xi,t)$ is continuously differentiable mapping of Ω_0 onto Ω_t with the Jacobian $\hat{\mathcal{J}} = \hat{\mathcal{J}}(\xi,t)$, which is continuous, bounded and

$$\mathcal{J}(x,t) = \hat{\mathcal{J}}(\xi,t) = \det\frac{\mathrm{D}\mathcal{A}}{\mathrm{D}\xi}(\xi,t) > 0\ , \quad \text{where } x = \mathcal{A}_t(\xi)\ .$$

(A3) The *domain velocity* $\mathbf{w}_{\mathrm{D}} : \mathcal{M} \to \mathbb{R}$ satisfies

$$\mathbf{w}_{\mathrm{D}}(\mathcal{A}(\xi,t),t) = \frac{\partial\mathcal{A}}{\partial t}(\xi,t) \qquad \forall\xi \in \Omega_0.$$

In what follows by symbol $\widehat{\mathbf{w}}_{\mathrm{D}}$ we shall denote the domain velocity defined on the reference configuration Ω_0, i.e.

$$\widehat{\mathbf{w}}_{\mathrm{D}}(\xi,t) = \mathbf{w}_{\mathrm{D}}(\mathcal{A}(\xi,t),t).$$

Similary for any function $f(x,t)$ defined on the open set

$$\mathcal{M} = \Big\{(x,t); x \in \Omega_t, t \in (0,T)\Big\}\ . \tag{2}$$

we shall define the function $\hat{f}(\xi,t)$ defined for any $\xi \in \Omega_0$ and $t \in (0,T)$ by

$$\hat{f}(\xi,t) = f(\mathcal{A}(\xi,t),t)\ .$$

Furthermore the symbol $\mathrm{D}^{\mathcal{A}}/\mathrm{D}t$ denotes the ALE derivative, i.e. the time derivative with respect to the reference configuration:

$$\frac{\mathrm{D}^{\mathcal{A}}f}{\mathrm{D}t}(x,t) = \frac{\partial\hat{f}}{\partial t}(\xi,t)\ , \quad \text{where } x = \mathcal{A}_t(\xi)\ .$$

Under the assumptions (A1)-(A3) the ALE derivative satisfies (cf. [6, 8])

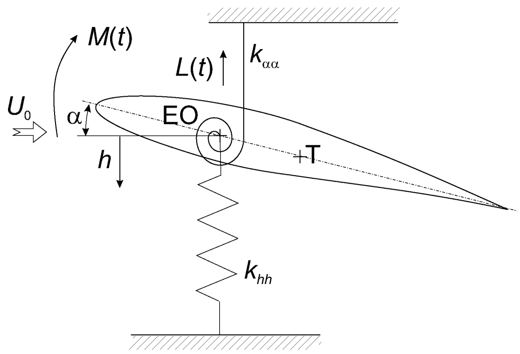
$$\frac{\mathrm{D}^{\mathcal{A}} f}{\mathrm{D}t}(x,t) = \frac{\partial f}{\partial t}(x,t) + \mathbf{w}_{\mathrm{D}}(x,t) \cdot \nabla f(x,t) \ .$$

The *non-conservative ALE* formulation follows from (1) with the aid of the relation

$$\frac{\mathrm{D}^{\mathcal{A}} \mathcal{J}}{\mathrm{D}t} = \mathcal{J}(\nabla \cdot \mathbf{w}_{\mathrm{D}}) \ , \qquad \frac{\partial \hat{\mathcal{J}}}{\partial t}(\xi,t) = \hat{\mathcal{J}}(\xi,t) \ \mathrm{div}\, \mathbf{w}_{\mathrm{D}}(\mathcal{A}(\xi,t),t) \ , \tag{3}$$

which is equivalent to the Euler's expansion formula, see, e.g. [5]. The equivalent (non-conservative) formulation of equations (1) reads

$$\frac{\mathrm{D}^{\mathcal{A}} v_i}{\mathrm{D}t} + (\mathbf{v} - \mathbf{w}_{\mathrm{D}}) \cdot \nabla v_i - \nu \, \triangle v_i + \frac{\partial p}{\partial x_i} = 0 \ . \tag{4}$$



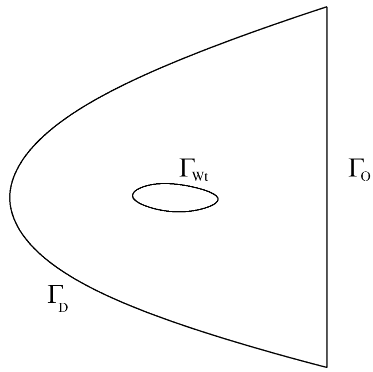


Fig.1: The flexibly supported airfoil shown in the deformed position (left); sketch of the computational domain Ω_t and the parts of the boundary $\partial\Omega_t$ (right)

The system (1) is equipped with boundary conditions prescribed on mutually disjoint parts of the boundary $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{O}} \cup \Gamma_{\mathrm{Wt}}$:

$$\begin{aligned} &\text{a) } \mathbf{v} = \mathbf{v}_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}} \ , \\ &\text{b) } \mathbf{v} = \mathbf{w}_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{Wt}} \ , \\ &\text{c) } -\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} + (p - p_{\mathrm{ref}})\, \mathbf{n} = 0 \quad \text{on } \Gamma_{\mathrm{O}} \ , \end{aligned} \tag{5}$$

where p_{ref} denotes a reference pressure. Further, the system (1) is equipped with an initial condition $\mathbf{v}(x,0) = \mathbf{v}_0(x)$, $x \in \Omega_0$.

The flow model is coupled with the structure model representing the flexibly supported airfoil (see Figure 1). The airfoil can be vertically displaced by h (downwards positive) and rotated by angle α (clockwise positive). The equations of motion then read (see [6])

$$\begin{aligned} m\ddot{h} + S_\alpha \, \ddot{\alpha} \cos\alpha - S_\alpha \, \dot{\alpha}^2 \sin\alpha + d_{\mathrm{hh}} \, \dot{h} + k_{\mathrm{hh}} \, h &= -L(t) \ , \\ S_\alpha \, \ddot{h} \cos\alpha + I_\alpha \, \ddot{\alpha} + d_{\alpha\alpha} \, \dot{\alpha} + k_{\alpha\alpha} \, \alpha &= M(t) \ . \end{aligned} \tag{6}$$

where m is the mass of the airfoil, S_α is the static moment around the elastic axis EO, and I_α is the inertia moment around the elastic axis EO. The parameters $d_{\alpha\alpha}$ and $d_{\rm hh}$ denote the structural damping coefficients, $k_{\rm hh}$ and $k_{\alpha\alpha}$ denote the stiffness coefficients. On the right-hand side the aerodynamical lift force $L(t)$ and aerodynamical torsional moment $M(t)$ are involved.

The aerodynamical lift force and torsional moment satisfy

$$L = -l \int\limits_{\Gamma_{\rm Wt}} \sum_{j=1}^{2} \tau_{2j}\, n_j \,\mathrm{d}S \ , \qquad M = l \int\limits_{\Gamma_{\rm Wt}} \sum_{i,j=1}^{2} \tau_{ij}\, n_j\, r_i^{\rm ort} \,\mathrm{d}S \ , \tag{7}$$

where

$$\tau_{ij} = \rho \left[-p\,\delta_{ij} + \nu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right] \ , \qquad r_1^{\rm ort} = -(x_2 - x_{\rm EO2}) \ , \quad r_2^{\rm ort} = x_1 - x_{\rm EO1} \ , \tag{8}$$

(see Fig. 2) and l denotes the depth of the airfoil section.

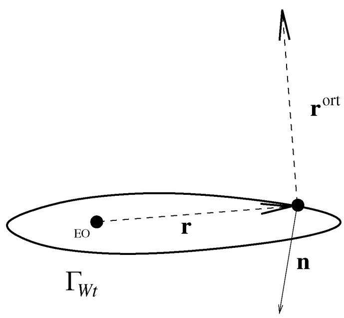


Fig.2: The outward unit normal $\mathbf{n}$ *and vectors* $\mathbf{r}$, $\mathbf{r}^{\rm ort}$

3. Weak formulation and time discretization

In order to introduce the weak formulation of the Navier-Stokes equations problem, we multiply equations (1) by a test function $\mathbf{z}$, integrate over the domain Ω_t and apply Green's theorem. Because the computational domain varies in time, it is natural to use also time-dependent test functions $\mathbf{z}$.

We consider a test function $\mathbf{z} = \mathbf{z}(x,t)$ in the form $\mathbf{z} = \hat{\mathbf{z}} \circ \mathcal{A}_t^{-1}$, which means that $\mathbf{z}(x,t) = \hat{\mathbf{z}}(\xi)$, where $x = \mathcal{A}_t(\xi)$ for all $\xi \in \Omega_0$, and $\hat{\mathbf{z}} \in \mathbf{H}^1(\Omega_0)$ ($\mathbf{H}^1(\Omega_0) = \left[H^1(\Omega_0)\right]^2$ and $H^1(\Omega_0)$ is the Sobolev space of functions which are square integrable over Ω_0 together with their first-order derivatives). The space of all such test functions $\mathbf{z} = \mathbf{z}(x,t)$ will be denoted by $\widetilde{\mathcal{X}}$ and we define $\mathcal{X}$ by

$$\mathcal{X} = \left\{ \mathbf{z} \in \widetilde{\mathcal{X}} \, : \, \mathbf{z}(x,t) = 0, x \in \Gamma_{\rm D} \cup \Gamma_{\rm Wt}, t \in (0,T) \right\} \ .$$

Further, for any time instant t we define the spaces $\mathcal{W}_t$, $\mathcal{Q}_t$ by

$$\mathcal{W}_t = \mathbf{H}^1(\Omega_t) \ , \qquad \mathcal{Q}_t = L^2(\Omega_t) \ .$$

The weak formulation is introduced with the aid of the following identity: Let the mapping $\mathcal{A}_t$ and $\mathbf{w}_\mathrm{D}$ satisfy assumptions (A1)-(A3) and let $\mathbf{v} \in C^1(\mathcal{M})$ and $\mathbf{z} \in \mathcal{X}$. Then

$$\int\limits_{\Omega_t} \frac{1}{\mathcal{J}} \frac{\mathrm{D}^\mathcal{A}}{\mathrm{D}t} \big(\mathcal{J}\, v_i\big)\, z_i \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Omega_t} v_i(x,t)\, z_i(x,t)\,\mathrm{d}x \;. \tag{9}$$

Proof. We start with the use of the substitution theorem, use the fact that function $\hat{\mathbf{z}} = \hat{z}(\xi)$ does not depend on time and exchange the order of integration and differentiation:

$$\begin{aligned}
\int\limits_{\Omega_t} \frac{1}{\mathcal{J}} \frac{\mathrm{D}^\mathcal{A}}{\mathrm{D}t} \big(\mathcal{J}\, v_i\big)\, z_i \,\mathrm{d}x &= \int\limits_{\Omega_0} \frac{\partial}{\partial t} \Big(\hat{\mathcal{J}}\, \hat{v}_i\Big)\, \hat{z}_i(\xi)\,\mathrm{d}\xi = \\
&= \int\limits_{\Omega_0} \frac{\partial}{\partial t} \Big(\hat{\mathcal{J}}(\xi,t)\, \hat{v}_i(\xi,t)\, \hat{z}_i(\xi)\Big)\,\mathrm{d}\xi = \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Omega_0} \hat{\mathcal{J}}(\xi,t)\, \hat{v}_i(\xi,t)\, \hat{z}_i(\xi)\,\mathrm{d}\xi = \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Omega_t} v_i(x,t)\, z_i(x,t)\,\mathrm{d}x \;,
\end{aligned}$$

where the last equation holds with the use of substitution theorem again. □

Now, the weak formulation of the problem (1) reads: Find $U = (\mathbf{v}, p) \in \mathcal{W}_t \times \mathcal{Q}_t$ such that it satisfies the identity

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \Big[\big(\mathbf{v}, \mathbf{z}\big)_{\Omega_t} \Big] + \nu \left(\nabla \mathbf{v}, \nabla \mathbf{z}\right)_{\Omega_t} + \left((\overline{\mathbf{w}} \cdot \nabla)\, \mathbf{v}, \mathbf{z}\right)_{\Omega_t} - \left(p, \nabla \cdot \mathbf{z}\right)_{\Omega_t} - \\
- \big((\nabla \cdot \mathbf{w}_\mathrm{D})\, \mathbf{v}, \mathbf{z}\big)_{\Omega_t} + \big(\nabla \cdot \mathbf{v}, q\big)_{\Omega_t} = - \int\limits_{\Gamma_\mathrm{O}} p_\mathrm{ref}\, \mathbf{n} \cdot \mathbf{z}\,\mathrm{d}S \;,
\end{aligned} \tag{10}$$

for all $V = (\mathbf{z}, q) \in \mathcal{X} \times \mathcal{Q}_t$, where $\overline{\mathbf{w}} = \mathbf{v} - \mathbf{w}_\mathrm{D}$ and $\mathbf{v}(\cdot, t)$ satisfies

$$\begin{aligned}
\mathbf{v}(x,t) &= \mathbf{v}_\mathrm{D}(x) \;, \quad & x \in \Gamma_\mathrm{D} \;, \\
\mathbf{v}(x,t) &= \mathbf{w}_\mathrm{D}(x,t) \;, \quad & x \in \Gamma_\mathrm{Wt} \;.
\end{aligned}$$

Remark 3.1 (Weak formulation of ALE non-conservative form). *The weak formulation of equation ALE (1) reads Find $U = (\mathbf{v}, p) \in \mathcal{W}_t \times \mathcal{Q}_t$ such that for all $V = (\mathbf{z}, q) \in \widetilde{X}_t \times \mathcal{Q}_t$ holds*

$$\begin{aligned}
\left(\frac{\mathrm{D}^\mathcal{A} \mathbf{v}}{\mathrm{D}t}, \mathbf{z}\right)_{\Omega_t} + \nu \left(\nabla \mathbf{v}, \nabla \mathbf{z}\right)_{\Omega_t} + \left((\overline{\mathbf{w}} \cdot \nabla)\, \mathbf{v}, \mathbf{z}\right)_{\Omega_t} - \left(p, \nabla \cdot \mathbf{z}\right)_{\Omega_t} + \\
+ \big(\nabla \cdot \mathbf{v}, q\big)_{\Omega_t} = - \int\limits_{\Gamma_\mathrm{O}} p_\mathrm{ref}\, \mathbf{n} \cdot \mathbf{z}\,\mathrm{d}S \;,
\end{aligned} \tag{11}$$

where $\overline{\mathbf{w}} = \mathbf{v} - \mathbf{w}_\mathrm{D}$ and $\mathbf{v}(\cdot, t)$ satisfies

$$\begin{aligned}
\mathbf{v}(x,t) &= \mathbf{v}_\mathrm{D}(x) \;, \quad & x \in \Gamma_\mathrm{D} \;, \\
\mathbf{v}(x,t) &= \mathbf{w}_\mathrm{D}(x,t) \;, \quad & x \in \Gamma_\mathrm{Wt} \;.
\end{aligned}$$

3.1. Time discretization

In order to discretize problem (10) in time, we consider a time step $\Delta t > 0$, denote $t_k = k\,\Delta t$ and at every time instant t_k employ the approximations $\mathbf{v}^k \approx \mathbf{v}(\cdot, t_k)$ and $p^k \approx p(\cdot, t_k)$. Moreover we approximate the domain velocity $\mathbf{w}_{\mathrm{D}}$ at time level t_k by $\mathbf{w}^k_{\mathrm{D}}$. Similary we shall use a simplified notation for the test function $\mathbf{z}$. By $\mathbf{z}^k$ we shall denote the function $\mathbf{z}^k(x) = \mathbf{z}(x, t_k)$ defined for any $x \in \Omega_k = \Omega_{t_k}$. In what follows we shall relate the value of the test function $\mathbf{z}^k$ to the value of the test function on the time level t_{n+1} by the identity

$$\mathbf{z}^k(x) = \mathbf{z}\left(\mathcal{A}_{t_{n+1}}\left(\mathcal{A}^{-1}_{t_k}(x)\right), t_{n+1}\right) = \mathbf{z}^{n+1}\left(\mathcal{A}_{t_{n+1}}\left(\mathcal{A}^{-1}_{t_k}(x)\right)\right)\ , \quad x \in \Omega_k\ . \tag{12}$$

The time derivative in the weak formulation (10) is approximated at time $t = t_{n+1}$ by the second order backward difference formula, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\Big(\mathbf{v}, \mathbf{z}\Big)_{\Omega_t}\right] \approx \frac{3\Big(\mathbf{v}^{n+1}, \mathbf{z}^{n+1}\Big)_{\Omega_{n+1}} - 4\Big(\mathbf{v}^n, \mathbf{z}^n\Big)_{\Omega_n} + \Big(\mathbf{v}^{n-1}, \mathbf{z}^{n-1}\Big)_{\Omega_{n-1}}}{2\,\Delta t}.$$

Further, we shall consider the function spaces at the time instant t_k

$$\mathcal{W}^k = \mathbf{H}^1(\Omega_k)\ , \qquad \mathcal{Q}^k = L^2(\Omega_k)\ .$$

Let us define the forms at time instant $t = t_{n+1}$

$$\begin{aligned} a(U^*, U, V) = {} & \frac{3}{2\,\Delta t}\left(\mathbf{v}^{n+1}, \mathbf{z}^{n+1}\right)_{\Omega_{n+1}} + \left(\left(\overline{\mathbf{w}}^{n+1}\cdot\nabla\right)\mathbf{v}^{n+1}, \mathbf{z}^{n+1}\right)_{\Omega_{n+1}} + \\ & + \nu\left(\nabla\mathbf{v}^{n+1}, \nabla\mathbf{z}^{n+1}\right)_{\Omega_{n+1}} + \left(\nabla\cdot\mathbf{v}^{n+1}, q\right)_{\Omega_{n+1}} - \\ & - \left(p^{n+1}, \nabla\cdot\mathbf{z}^{n+1}\right)_{\Omega_{n+1}} - \left(\left(\nabla\cdot\mathbf{w}^{n+1}_D\right)\mathbf{v}^{n+1}, \mathbf{z}^{n+1}\right)_{\Omega_{n+1}}\ , \\ L(V) = {} & -\int\limits_{\Gamma_{\mathrm{O}}} p_{\mathrm{ref}}\,\mathbf{n}\cdot\mathbf{z}^{n+1}\,\mathrm{d}S + \frac{4}{2\,\Delta t}\left(\mathbf{v}^n, \mathbf{z}^n\right)_{\Omega_n} - \frac{1}{2\,\Delta t}\left(\mathbf{v}^{n-1}, \mathbf{z}^{n-1}\right)_{\Omega_{n-1}}\ , \end{aligned}$$

where $U = (\mathbf{v}^{n+1}, p^{n+1})$, $V = (\mathbf{z}^{n+1}, q)$, $U^* = (\mathbf{v}^{*,n+1}, p^{n+1})$, $\overline{\mathbf{w}}^{n+1} = \mathbf{v}^{*,n+1} - \mathbf{w}^{n+1}_{\mathrm{D}}$, and the functions $\mathbf{z}^n$, $\mathbf{z}^{n-1}$ are defined by the relation (12).

Problem 3.1 (Weak ALE conservative-formulation of the time discretized problem). *For $t = t_{n+1}$ find $U = (\mathbf{v}^{n+1}, p^{n+1}) \in \mathcal{W}^{n+1} \times \mathcal{Q}^{n+1}$ such that for all test functions $V = (\mathbf{z}, q)$ where $q \in \mathcal{Q}^{n+1}$ and $\mathbf{z} \in \mathcal{X}$ holds*

$$a(U, U, V) = L(V)\ . \tag{13}$$

ALE non-conservative formulation

The finite element method is usually applied to the Navier-Stokes equations written in the non-conservative form. Let us start from equations (4), where the ALE time derivative is approximated with the aid of second order backward difference formula (BDF2) by

$$\frac{\mathrm{D}^{\mathcal{A}}\mathbf{v}}{\mathrm{D}t}(x, t_{n+1}) \approx \frac{3\mathbf{v}^{n+1}(x) - 4\,\tilde{\mathbf{v}}^n(x) + \tilde{\mathbf{v}}^{n-1}(x)}{2\,\Delta t} \quad \text{for any } x \in \Omega_{t_{n+1}}\ ,$$

where $\tilde{\mathbf{v}}^k(x) = \mathbf{v}^k\left(\mathcal{A}_{t_k}\left(\mathcal{A}^{-1}_{t_{n+1}}(x)\right)\right)$.

Similary as for problem (13) we define the forms a_{non} and L_{non} on the time level t_{n+1}:

$$a_{\mathrm{non}}(U^*,U,V) = \left(\frac{3\,\mathbf{v}^{n+1}}{2\,\Delta t},\mathbf{z}^{n+1}\right)_{\Omega_{n+1}} + \big((\overline{\mathbf{w}}^{n+1}\cdot\nabla)\,\mathbf{v}^{n+1},\mathbf{z}^{n+1}\big)_{\Omega_{n+1}} + \\ + \nu\,(\nabla\mathbf{v}^{n+1},\nabla\mathbf{z}^{n+1})_{\Omega_{n+1}} + \big(\nabla\cdot\mathbf{v}^{n+1},q\big)_{\Omega_{n+1}} - \big(p^{n+1},\nabla\cdot\mathbf{z}^{n+1}\big)_{\Omega_{n+1}}\;,$$

$$L_{\mathrm{non}}(V) = -\int_{\Gamma_{\mathrm{O}}} p_{\mathrm{ref}}\,\mathbf{n}\cdot\mathbf{z}^{n+1}\,\mathrm{d}S + \left(\frac{4\,\tilde{\mathbf{v}}^n-\tilde{\mathbf{v}}^{n-1}}{2\,\Delta t},\mathbf{z}^{n+1}\right)_{\Omega_{n+1}},$$

where $U=(\mathbf{v}^{n+1},p^{n+1})$, $V=(\mathbf{z}^{n+1},q)$, $U^*=(\mathbf{v}^{*,n+1},p^{n+1})$, $\overline{\mathbf{w}}^{n+1}=\mathbf{v}^{*,n+1}-\mathbf{w}_{\mathrm{D}}^{n+1}$.

Problem 3.2 (Weak ALE non-conservative formulation of the time discretized problem). *For $t=t_{n+1}$ find $U=(\mathbf{v}^{n+1},p^{n+1})\in\mathcal{W}^{n+1}\times\mathcal{Q}^{n+1}$ such that for all test functions $V=(\mathbf{z}^{n+1},q)$ where $q\in\mathcal{Q}^{n+1}$ and $\mathbf{z}^{n+1}\in\mathcal{X}^{n+1}$ holds*

$$a_{\mathrm{non}}(U,U,V) = L_{\mathrm{non}}(V)\;. \tag{14}$$

4. Finite element approximation and stabilization

In order to apply the Galerkin FEM, we approximate the spaces $\mathcal{W}^k$, $\mathcal{X}^k$, $\mathcal{Q}^k$ from the weak formulation by finite dimensional subspaces $\mathcal{W}^k_\triangle$, $\mathcal{X}^k_\triangle$, $\mathcal{Q}^k_\triangle$, $\triangle\in(0,\triangle_0)$, $\triangle_0>0$, $\mathcal{X}^k_\triangle=\{\mathbf{v}_\triangle\in\mathcal{W}_\triangle;\mathbf{v}_\triangle|_{\Gamma_{\mathrm{D}}\cap\Gamma_{\mathrm{Wt}}}=0\}$. In practical computations we assume that the domain Ω_{n+1} is a polygonal approximation of the region occupied by the fluid at time t_{n+1} and the spaces $\mathcal{W}^{n+1}_\triangle$, $\mathcal{X}^{n+1}_\triangle$, $\mathcal{Q}^{n+1}_\triangle$ are defined over a triangulation $\mathcal{T}^{n+1}_\triangle$ of the domain Ω_{n+1}, formed by a finite number of closed triangles $K\in\mathcal{T}^{n+1}_\triangle$. We use the standard assumptions on the system of triangulation, cf. [3]. Here $\triangle$ denotes the size of the mesh $\mathcal{T}^{n+1}_\triangle$. The spaces $\mathcal{W}^{n+1}_\triangle$, $\mathcal{X}^{n+1}_\triangle$ and $\mathcal{Q}^{n+1}_\triangle$ are formed by piecewise polynomial functions:

$$\begin{aligned} \mathcal{H}^{n+1}_\triangle &= \{v\in C(\overline{\Omega}_{n+1}); v|_K\in P_k(K) \text{ for each } K\in\mathcal{T}^{n+1}_\triangle\}\;, \\ \mathcal{W}^{n+1}_\triangle &= \left[\mathcal{H}^{n+1}_\triangle\right]^d\;, \qquad \mathcal{X}^{n+1}_\triangle=\mathcal{W}^{n+1}_\triangle\cap\mathcal{X}^{n+1}\;, \\ \mathcal{Q}^{n+1}_\triangle &= \{v\in C(\overline{\Omega}_{n+1}); v|_K\in P_k(K) \text{ for each } K\in\mathcal{T}^{n+1}_\triangle\}\;. \end{aligned} \tag{15}$$

The standard Galerkin approximation of the weak formulations (13) and (14) may suffer from two sources of instabilities. One instability is caused by the incompatibility of the pressure and velocity pairs of finite elements, cf. [11], [10]. Further, the finite element scheme is unstable due to the dominating convection.

In order to overcome both difficulties, we apply a modification of the Galerkin Least Squares method together with *div-div* stabilization, cf. [7]. We start from the definition of two parts $\mathcal{R}^a_K$ and $\mathcal{R}^f_K$ of the *local element residual* on the element $K\in{\mathcal{T}_\triangle}^{n+1}$:

$$\mathcal{R}^a_K(\overline{\mathbf{w}}^{n+1};\mathbf{v}^{n+1},p^{n+1}) = \frac{3\,\mathbf{v}^{n+1}}{2\,\Delta t} - \nu\,\triangle\mathbf{v}^{n+1} + \big(\overline{\mathbf{w}}^{n+1}\cdot\nabla\big)\,\mathbf{v}^{n+1} + \nabla p^{n+1}\;, \tag{16}$$

where the function $\overline{\mathbf{w}}^{n+1}$ stands for the transport velocity and $\mathcal{R}^f_K$ is defined by

$$\mathcal{R}^f_K(\tilde{\mathbf{v}}^n,\tilde{\mathbf{v}}^{n-1}) = \frac{1}{2\,\Delta t}\,(4\,\tilde{\mathbf{v}}^n-\tilde{\mathbf{v}}^{n-1}), \tag{17}$$

where $\tilde{\mathbf{v}}^k=\mathbf{v}^k\circ\mathcal{A}_{t_k}\circ\mathcal{A}^{-1}_{t_{n+1}}$.

The stabilizing terms are defined by

$$
\begin{aligned}
\mathcal{L}_{\mathrm{GLS}}(U^*_\triangle, U_\triangle, V_\triangle) &= \sum_{K\in\mathcal{T}^{n+1}_\triangle} \delta_K \Big(\mathcal{R}^a_K(\overline{\mathbf{w}}^{n+1}; \mathbf{v}^{n+1}, p^{n+1}), \big(\overline{\mathbf{w}}^{n+1}\cdot\nabla\big)\,\mathbf{z}^{n+1} + \nabla q^{n+1}\Big)_K ,\\
\mathcal{F}_{\mathrm{GLS}}(V_\triangle) &= \sum_{K\in\mathcal{T}^{n+1}_\triangle} \delta_K \Big(\mathcal{R}^f_K(\tilde{\mathbf{v}}^n, \tilde{\mathbf{v}}^{n-1}), \big(\overline{\mathbf{w}}^{n+1}\cdot\nabla\big)\,\mathbf{z}^{n+1} + \nabla q^{n+1}\Big)_K ,
\end{aligned}
\tag{18}
$$

where the function $\overline{\mathbf{w}}^{n+1}$ stands for the transport velocity, i.e. $\overline{\mathbf{w}}^{n+1} = \mathbf{v}^{*,n+1} - \mathbf{w}^{n+1}_{\mathrm{D}}$. Furthermore, the *div-div* stabilizing terms $\mathcal{P}_\triangle(U_\triangle, V_\triangle)$ read

$$
\mathcal{P}_\triangle(U_\triangle, V_\triangle) = \sum_{K\in\mathcal{T}^{n+1}_\triangle} \tau_K\left(\nabla\cdot\mathbf{v}^{n+1}, \nabla\cdot\mathbf{z}^{n+1}\right)_K . \tag{19}
$$

Here, $\tau_K \geq 0$ and $\delta_K \geq 0$ are suitably chosen parameters. The stabilized discrete problem reads:

Problem 4.1 (GLS stabilized problem). *Find $U_\triangle = (\mathbf{v}^{n+1}, p^{n+1}) \in \mathcal{W}^{n+1}_\triangle \times \mathcal{Q}^{n+1}_\triangle$ such that $\mathbf{v}^{n+1}$ satisfies approximately the Dirichlet boundary conditions (5a,b) and the condition*

$$
a(U_\triangle, U_\triangle, V_\triangle) + \mathcal{L}_{\mathrm{GLS}}(U_\triangle, U_\triangle, V_\triangle) + \mathcal{P}_\triangle(U_\triangle, V_\triangle) = f(V_\triangle) + \mathcal{F}_{\mathrm{GLS}}(V_\triangle) , \tag{20}
$$

holds for all $V_\triangle = (\mathbf{z}^{n+1}, q) \in \mathcal{X}^{n+1}_\triangle \times \mathcal{Q}^{n+1}_\triangle$.

The parameters δ_K and τ_K are defined by

$$
\tau_K = \nu\left(1 + Re^{\mathrm{loc}} + \frac{h_K^2}{\nu\,\Delta t}\right) , \qquad \delta_K = \frac{h_K^2}{\tau_K} ,
$$

where the local Reynolds number Re^{loc} is

$$
Re^{\mathrm{loc}} = \frac{h_K\,\|\overline{\mathbf{w}}^{n+1}\|_K}{2\,\nu} ,
$$

and h_K denotes the local element length measured in the direction of the transport velocity $\overline{\mathbf{w}}^{n+1}$.

Solution of the nonlinear problem

The nonlinear discrete problem (20) is solved on each time level t_{n+1} with the aid of the linearized Oseen iterative process

$$
\begin{aligned}
a(U^{(\ell)}_\triangle, U^{(\ell+1)}_\triangle, V_\triangle) + \mathcal{L}_{\mathrm{GLS}}(U^{(\ell)}_\triangle, U^{(\ell+1)}_\triangle, V_\triangle) + \mathcal{P}_\triangle(U^{(\ell+1)}_\triangle, V_\triangle) =&\\
= f(V_\triangle) + \mathcal{F}_{\mathrm{GLS}}(V_\triangle) \quad \text{for all} \quad V_\triangle \in \mathcal{X}_\triangle \times \mathcal{Q}_\triangle &,
\end{aligned}
\tag{21}
$$

where we start from the initial approximation $U^{(0)}_\triangle = (\widehat{\mathbf{v}}^n, \widehat{p}^n)$ or $U^{(0)}_\triangle = (2\widehat{\mathbf{v}}^n - \widehat{\mathbf{v}}^{n-1}, 2\widehat{p}^n - \widehat{p}^{n-1})$. It is usually enough to compute 5–8 Oseen iterations on each time level.

4.1. Coupling of fluid-structure models

The coupling of fluid and structure problems is given by Equation (7) and the boundary condition (5b). For a given value of α^{n+1}, h^{n+1} the ALE mapping is constructed with the aid of linear elasticity analogy, cf. [13]. Then the domain velocity $\mathbf{w}_\mathrm{D}^{n+1}$ at the time instant t_{n+1} is approximated with the aid of the two step second order backward difference formula

$$\mathbf{w}_\mathrm{D}^{n+1}(x) = \frac{3\,x^{n+1} - 4\,x^n + x^{n-1}}{2\,\Delta t} \;, \qquad x^{n+1} \in \Omega_{n+1} \;,$$

where $x^{n+1} = \mathcal{A}_{t_{n+1}}(\xi)$, $x^n = \mathcal{A}_{t_n}(\xi)$, $x^{n-1} = \mathcal{A}_{t_{n-1}}(\xi)$ denote the locations of a moving point x^{n+1} with a given fixed reference $\xi \in \Omega_0$ at time instants t_{n+1}, t_n, and t_{n-1}.

The aerodynamic lift force $L(t)$ and moment $M(t)$ are computed from the approximations of fluid velocity and pressure with the aid of the weak formulation of the problem. The Navier-Stokes equations in the ALE form discretized with respect to time at instant $t := t_{n+1}$ can be expressed component-wise as

$$\begin{aligned} \rho\,\frac{3\,v_i^{n+1} - 4\,\widehat{v}_i^n + \widehat{v}_i^{n-1}}{2\,\Delta t} + \rho\left((\mathbf{v}^{n+1} - \mathbf{w}_\mathrm{D}^{n+1})\cdot\nabla\right) v_i^{n+1} = \\ = \sum_{j=1}^{2} \frac{\partial \tau_{ij}}{\partial x_j} \quad \text{in} \quad \Omega_{n+1} \;, \quad i = 1,2 \;. \end{aligned} \tag{22}$$

Let us set $\Omega_{\Gamma_\mathrm{Wt}} = \cup\{K \in \mathcal{T}_\triangle^{n+1};\ K \cap \Gamma_\mathrm{Wt} \neq \emptyset\}$, which represents a one layer strip around the airfoil formed by finite elements. We shall use the function $\varphi \in \mathcal{W}_\triangle$ such that $\varphi(x) = 1$ for $x \in \Gamma_\mathrm{Wt}$ and $\varphi(x) = 0$ outside the set $\Omega_{\Gamma_\mathrm{Wt}}$, see Fig. 3.

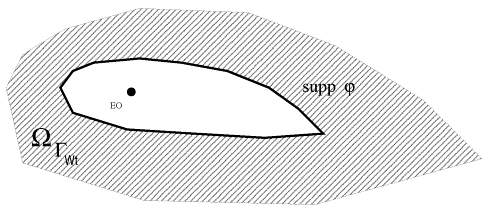


Fig.3: The example of the one layer strip $\Omega_{\Gamma_\mathrm{Wt}}$ around Γ_Wt

Multiplying equation (22) with $i = 2$ by the function φ, integrating over $\Omega_{\Gamma_\mathrm{Wt}}$, applying Green's theorem to the terms with τ_{ij} and, finally, writing the already known finite element approximations $\mathbf{v}_\triangle^{n+1}$, $\mathbf{v}_\triangle^n$ and $\mathbf{v}_\triangle^{n-1}$ instead of the functions $\mathbf{v}^{n+1}$, $\mathbf{v}^n$ and $\mathbf{v}^{n-1}$, respectively, we arrive at the representation of the force L:

$$\begin{aligned} L = -l \int\limits_{\Omega_{\Gamma_\mathrm{Wt}}} \Bigg\{ \rho \left(\frac{3v_{\triangle 2}^{n+1} - 4\,\widehat{v}_{\triangle 2}^n + \widehat{v}_{\triangle 2}^{n-1}}{2\,\Delta t} + \left((\mathbf{v}_\triangle^{n+1} - \mathbf{w}_\mathrm{D}^{n+1})\cdot\nabla \right) v_{\triangle 2}^{n+1} \right) \varphi - \\ - \sum_{j=1}^{2} \tau_{2j}\,\frac{\partial \varphi}{\partial x_j}, \Bigg\}\,\mathrm{d}x \;. \end{aligned} \tag{23}$$

(Here we use the notation $\mathbf{v}_\triangle = (v_{\triangle 1}, v_{\triangle 2})$, $\widehat{\mathbf{v}}_\triangle^n = (\widehat{v}_{\triangle 1}^n, \widehat{v}_{\triangle 2}^n)$, etc.)

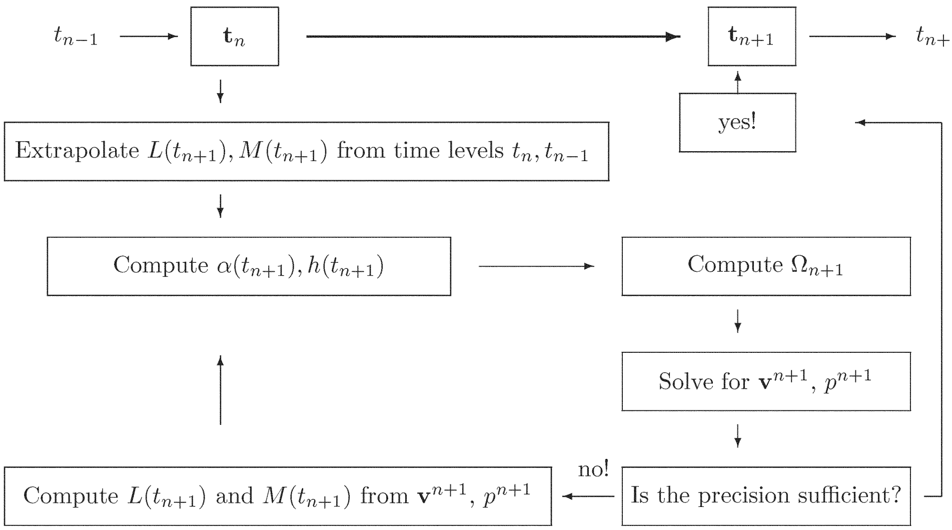
Similarly, if we use the vector-valued function $\mathbf{v}^\mathbf{ort} = (v_1^{ort}, v_2^{ort}) = \varphi\,(r_1^\mathrm{ort}, r_2^\mathrm{ort})$, where the functions $r_1^\mathrm{ort}, r_2^\mathrm{ort}$ are defined by (8), we can derive the formula

$$\begin{aligned} M = -l \int\limits_{\Omega_{\Gamma_\mathrm{Wt}}} \Bigg\{ \rho \left(\frac{3\,\mathbf{v}_\triangle^{n+1} - 4\,\widehat{\mathbf{v}}_\triangle^n + \widehat{\mathbf{v}}_\triangle^{n-1}}{2\,\Delta t} + \left((\mathbf{v}_\triangle^{n+1} - \mathbf{w}_\mathrm{D}^{n+1})\cdot\nabla \right) \mathbf{v}_\triangle^{n+1} \right) \cdot \mathbf{v}^\mathbf{ort} \Bigg\}\,\mathrm{d}x - \\ - \int\limits_{\Omega_{\Gamma_\mathrm{Wt}}} \sum_{i,j=1}^{2} \tau_{ij}\,\frac{\partial v_i^{ort}}{\partial x_j}\,\mathrm{d}x \;. \end{aligned} \tag{24}$$

The components τ_{ij} are computed from (8), where $\mathbf{v}_{\triangle}^{n+1}$ and $p_{\triangle}^{n+1}$ are substituted for $\mathbf{v}$ and p respectively.

Coupling algorithm

The coupling of fluid and structure problems is given by Equations (23), (24) and the boundary condition (5b). The approximate solution should satisfy both conditions Equations (23), (24) as well as boundary condition (5b). In the presented computations the strong coupling algorithm shown in next table is used:



5. Numerical examples

In this section we shall apply the developed ALE method to an aeroelastic problem for velocities nearby the flutter limit. In order to compare the ALE conservative and non-conservative formulations we shall test these formulations on a parabolic problem.

5.1. Model problem

We test the stability of the above developed technique on a simplified problem from [8]. The deformation of the reference domain $\Omega_0 = \{\xi = (\xi_1, \xi_2) :\ 0 < \xi_i < 1\}$ is given by

$$\mathcal{A}_t : \Omega_0 \mapsto \Omega_t \ , \qquad x = \mathcal{A}_t(\xi) \ , \qquad x_i = \xi_i \, (2 - \cos(20\,\pi\, t)) \ ,$$

for $t \in [0, T]$. We consider the equation

$$\frac{\partial u}{\partial t} - \nu \, \triangle u = 0 \quad \text{in} \quad \Omega_t \ , \tag{25}$$

equipped with the Dirichlet boundary condition $u = 0$ on $\partial\Omega_t$ and the initial condition

$$u(\xi, 0) = 1600\, \xi_1 \, (1 - \xi_1) \, \xi_2 \, (1 - \xi_2) \quad \text{in} \quad \Omega_0 \ .$$

We shall use two ALE formulations of problem (25). The ALE non-conservative form of equation (25) reads

$$\frac{\mathrm{D}^{\mathcal{A}} u}{\mathrm{D}t} - (\mathbf{w}_{\mathrm{D}} \cdot \nabla u) - \nu \, \triangle u = 0 \quad \text{in} \quad \Omega_t \ . \tag{26}$$

In the ALE conservative formulation (25) has the form

$$\frac{1}{\mathcal{J}}\,\frac{\mathrm{D}^{\mathcal{A}}}{\mathrm{D}t}\Big(\mathcal{J}\,u\Big) - \mathrm{div}\,(\mathbf{w}_{\mathrm{D}}\,u) - \nu\,\triangle u = 0 \quad \text{in} \quad \Omega_t\ . \tag{27}$$

Weak formulation

In order to introduce the weak formulation of problems (26), (27) we multiply equations by a test function v, integrate over the domain Ω_t and apply Green's theorem. Similary as in Section 2 we consider a test function $v = v(x,t)$ in the form $v = \hat{v}\circ\mathcal{A}_t^{-1}$, which means that $v(x,t) = \hat{v}(\xi)$, where $x = \mathcal{A}_t(\xi)$ for all $\xi \in \Omega_0$, and $\hat{v} \in H_0^1(\Omega_0)$. The space of all such test functions $v = v(x,t)$ will be denoted by $\widetilde{\mathcal{Z}}$. We multiply by a test function $v \in \widetilde{\mathcal{Z}}$ equation (27), integrate over Ω_t, use Green's theorem and relation (9). Then the weak formulation of (27) reads: Find $u = u(x,t)$ such that $u(x,t) = 0$ for $x \in \partial\Omega_t$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t} u\,v\,\mathrm{d}x - \int\limits_{\Omega_t} \nabla\cdot(\mathbf{w}_{\mathrm{D}}\,u)\,v\,\mathrm{d}x + \nu\int\limits_{\Omega_t}\nabla u\cdot\nabla v\,\mathrm{d}x = 0\ , \tag{28}$$

for all $v \in \widetilde{\mathcal{Z}}$.

Weak formulation of equation (26) reads: Find $u = u(x,t)$ such that $u \in H_0^1(\Omega_t)$ for any $t \in [0,T]$ and

$$\int\limits_{\Omega_t}\left(\frac{D^{\mathcal{A}_t}u}{Dt}\,v - (\mathbf{w}_{\mathrm{D}}\cdot\nabla u)\,v\right)\mathrm{d}x + \nu\int\limits_{\Omega_t}\nabla u\cdot\nabla v\,\mathrm{d}x = 0\ , \tag{29}$$

holds for any $v \in \widetilde{\mathcal{Z}}$.

Apriori estimate

Both formulations (29) and (28) are equivalent. We shall proof the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\,\|u(t)\|^2_{L^2(\Omega_t)} + \nu\,\|\nabla u\|^2_{\mathbf{L}^2(\Omega_t)} = 0\ , \tag{30}$$

which shows that the quantity $E(t) = \|u(t)\|^2_{L^2(\Omega_t)}$ is strictly decreasing in time.

Proof. In order to estabilish (30) we first prove the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\frac{1}{2}\,|u|^2\,\mathrm{d}x = \int\limits_{\Omega_t}\frac{\mathrm{D}^{\mathcal{A}_t}u}{\mathrm{D}t}\,u + \frac{1}{2}\,|u|^2\,(\nabla\cdot\mathbf{w}_{\mathrm{D}})\,\mathrm{d}x\ , \tag{31}$$

which follows from the use of the substitution theorem

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\frac{1}{2}\,|u|^2\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_0}\widehat{\mathcal{J}}(\xi,t)\,\frac{1}{2}\,|\hat{u}(\xi,t)|^2\,\mathrm{d}\xi = \\ &= \int\limits_{\Omega_0}\frac{\partial}{\partial t}\left[\widehat{\mathcal{J}}\,\frac{1}{2}\,|\hat{u}|^2\right]\mathrm{d}\xi = \int\limits_{\Omega_0}\frac{1}{2}\,\frac{\partial\widehat{\mathcal{J}}}{\partial t}\,|\hat{u}|^2 + \widehat{\mathcal{J}}\,\frac{\partial\hat{u}}{\partial t}\,u\,\mathrm{d}\xi = \\ &= \int\limits_{\Omega_t}\left[\frac{1}{\mathcal{J}}\,\frac{\mathrm{D}^{\mathcal{A}}\mathcal{J}}{\mathrm{D}t}\,\frac{1}{2}\,|u|^2 + \frac{\mathrm{D}^{\mathcal{A}}u}{\mathrm{D}t}\,u\right]\mathrm{d}x = \int\limits_{\Omega_t}\left[(\nabla\cdot\mathbf{w}_{\mathrm{D}})\,\frac{1}{2}\,|u|^2 + \frac{\mathrm{D}^{\mathcal{A}}u}{\mathrm{D}t}\,u\right]\mathrm{d}x\end{aligned}$$

where equation (3) was used. Further the application of Green's theorem yields

$$\frac{1}{2}\int\limits_{\Omega_t} u^2(\nabla\cdot\mathbf{w}_\mathrm{D})\,\mathrm{d}x = -\int\limits_{\Omega_t}(\mathbf{w}_\mathrm{D}\cdot\nabla u)\,u\,\mathrm{d}x\ , \tag{32}$$

and with the use of equations (32) in (31) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\frac{1}{2}\,|u|^2\,\mathrm{d}x = \int\limits_{\Omega_t}\frac{\mathrm{D}^{\mathcal{A}_t}u}{\mathrm{D}t}\,u - (\mathbf{w}_\mathrm{D}\cdot\nabla u)\,u\,\mathrm{d}x = -\nu\int\limits_{\Omega_t}|\nabla u|^2\,\mathrm{d}x\ ,$$

where the last equation follows from equation (29) with the choice $v = u$. □

Time discretization

The resulting problem is discretized in time with the aid of either the implicit Euler method or by the second order two step backward difference formula (BDF2).

The numerical results were obtained for the viscosity value of $\nu = 0.01$, the time step 0.01 and for the the time $t \in [0, 0.4]$. The results are shown in Figure 4. In the case of the time discretized problem on the fixed domain $\Omega_t \equiv \Omega$ the quantity $E(t_k)$ is decreasing. This is not the case for the moving domain solution as shown in Figure 4. Nevertheless, the numerical results are stable for both conservative as well as non-conservative formulation. Moreover, the conservative formulation is observed to preserve the monotone behaviour at least for the implicit Euler method.

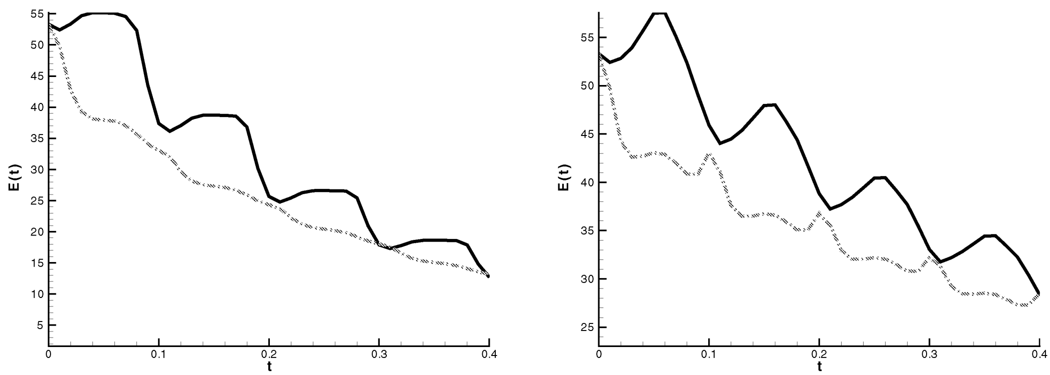


Fig.4: The numerical approximation of $E(t)$ for problem (25) with the implicit Euler formula (left) and the second order two step backward difference formula (right) : —— non-conservative formulation, $-\cdot-$ conservative formulation

5.2. Aeroelastic simulations

Now we present the numerical simulation of the coupled aeroelastic problem of flow induced vibrations of the airfoil NACA 0012. We consider the post-critical far field velocity $U_\infty = 40\,\mathrm{m\,s^{-1}}$. In that case the divergence type aeroelastic instability occurs.

We studied the response of the system in dependence on the initial values of α and h. The results are shown in Figures 5–8. In all figures a typical divergence type instability can be observed. The different behaviour of the aeroelastic system can be seen because of the different initial conditions. In Fig. 8 the response of the aeroelastic model reminds a combination of the divergence type instability and flutter type instability. The velocity magnitude distribution and pressure distribution during three time instants are shown in Fig. 9.

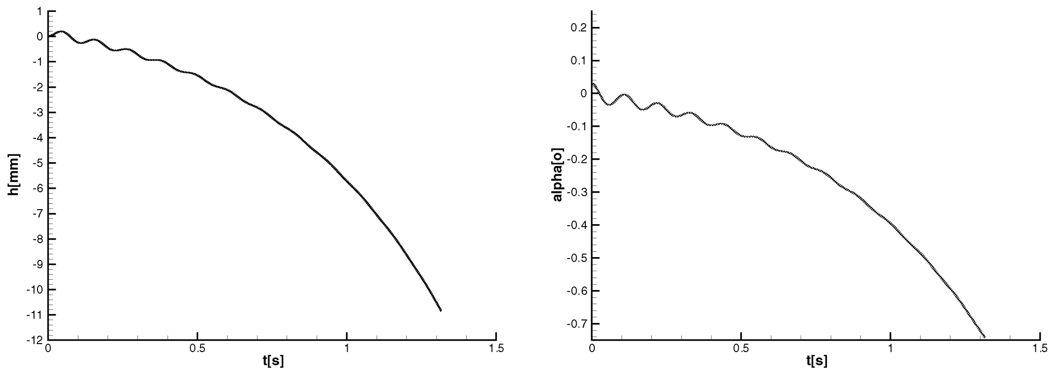


Fig.5: The aeroelastic response of the airfoil NACA 0012 for $U_\infty = 40\,m/s$ with the initial condition $\alpha(0) = 0.03^\circ$ and $h(0) = 0\,mm$

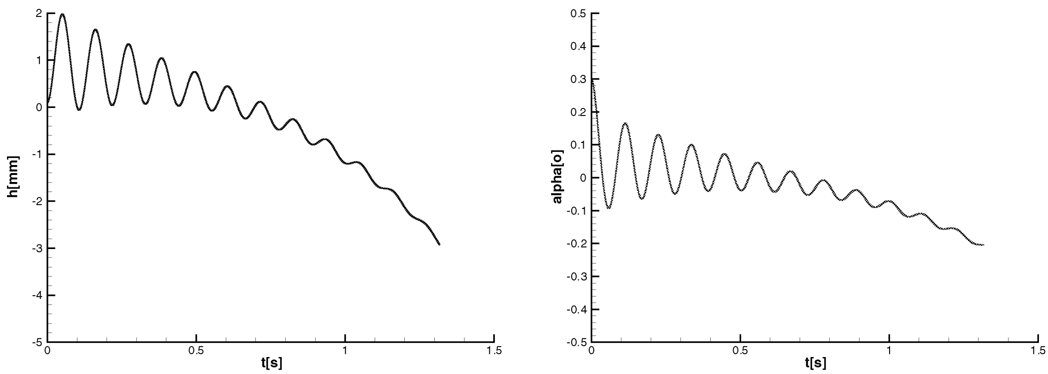


Fig.6: The aeroelastic response of the airfoil NACA 0012 for $U_\infty = 40\,m/s$ with the initial condition $\alpha(0) = 0.3^\circ$ and $h(0) = 0\,mm$

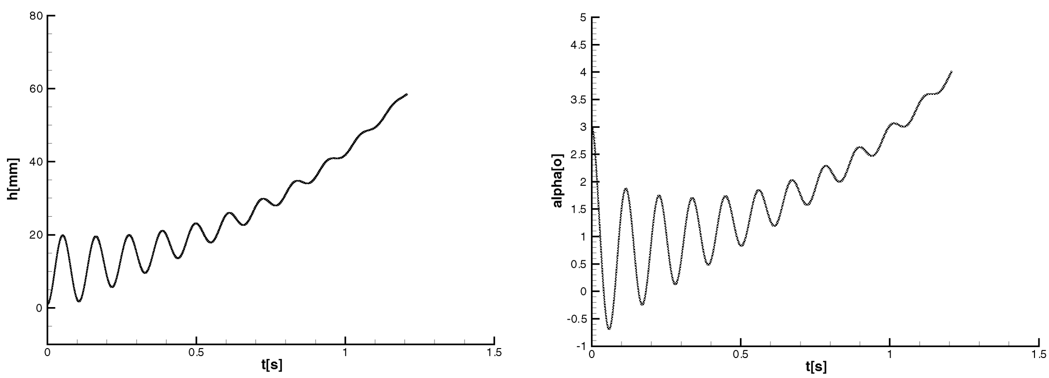


Fig.7: The aeroelastic response of the airfoil NACA 0012 for $U_\infty = 40\,m/s$ with the initial condition $\alpha(0) = 3^\circ$ and $h(0) = 0\,mm$

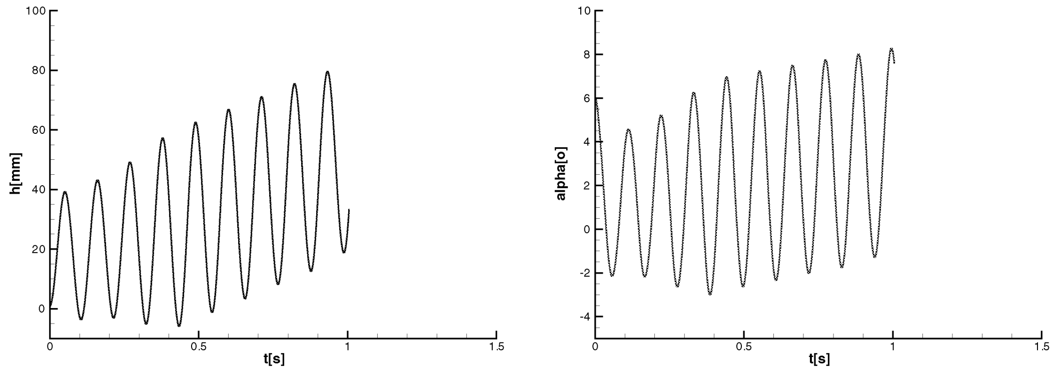


Fig.8: The aeroelastic response of the airfoil NACA 0012 for $U_\infty = 40\,m/s$ with the initial condition $\alpha(0) = 6^\circ$ and $h(0) = 0\,mm$

6. Conclusion

The robust finite element method for the numerical simulation of interaction of incompressible flow and a vibrating airfoil is presented. It is based on the combination of the Arbitrary Lagrangian Eulerian (ALE) conservative formulation of the Navier-Stokes equations, time discretization, and the finite element method stabilized by the Galerkin-Least Squares method. The comparison of ALE conservative and non-conservative formulation is presented and the influence of the initial condition is numerically studied.

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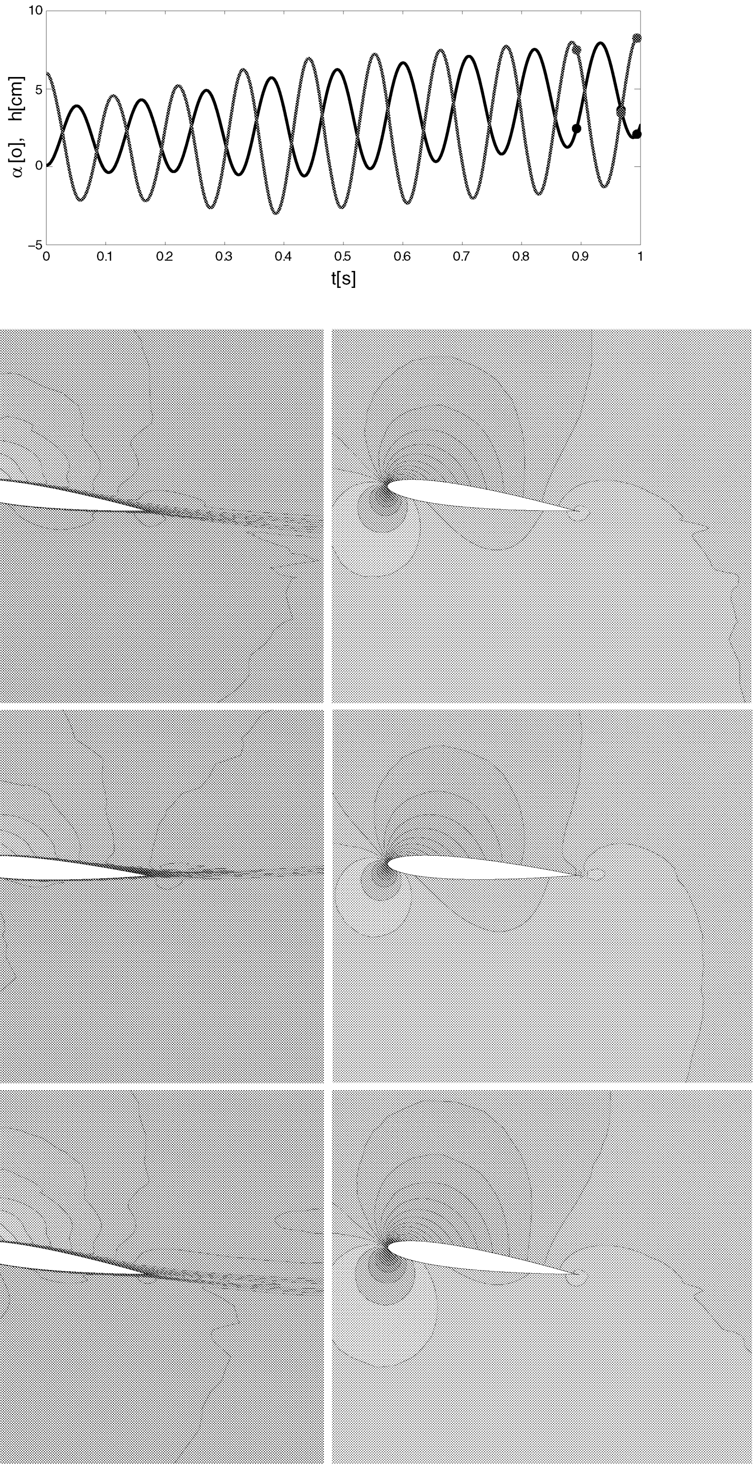

Fig.9: Velocity magnitude isolines on the left and pressure isolines on the right around the moving airfoil NACA 0012 for post-critical velocity $U_\infty = 40\,\mathrm{m/s}$ at three different time instants (shown in the first figure, numerical results of aeroelastic computation shown in Fig.8)

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