VERIFICATION OF THE CALCULATION OF NATURAL VIBRATION CHARACTERISTICS OF LINEAR UNDAMPED ROTATIONALLY PERIODIC STRUCTURES

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The article presents results of the verification calculation of the method for the calculation of natural frequencies and mode shapes of a linear undamped rotationally periodic systems considering the possibility of the elimination of degrees of freedom. As the test example a thin circular plate was chosen. The method can be applied e.g. for the calculation of the natural vibration characteristics of the steam turbine bladed disks.

Keywords : *rotational periodicity, natural vibration characteristics, subsystem, circular plate*

1. Introduction

The approach towards the calculation of the natural vibration characteristics (i.e. natural frequencies and mode shapes) of the rotationally periodic structures (e.g. steam turbine bladed disks) utilizing their rotational periodicity was solved in [1], the approach without considering the possibility of elimination of degrees of freedom of the structure computational model was given in [2], [3], the possibility of elimination of degrees of freedom was presented in [4], [5]. In [5] and this article the verification of the method on the test example of the thin circular plate is given. The motivation for introducing this article is a logical continuation of the article [2].

Description of various methods that use specific properties of the structure for the calculation of its investigated characteristics or behaviour under the given conditions can be found in the appropriate literature relatively often. The periodicity of the system is used to solve various problems. There is a large amount of publications dealing with using the periodicity of the system for the calculation of its dynamic properties (their description in more detail is presented e.g. in [1], [6], [7]).

The method used in this article for the calculation of the natural vibration characteristics of a thin circular plate was derived on the basis of [8]. In industrial practice the method was already applied in the calculation of the natural vibration characteristics of the steam turbine bladed disk with the blades of 220Z-1085 type connected with the continuous zig-zag binding [9], [10], [11] and the steam turbine bladed disk with the blades of ZN340-2 type with the continuous binding in the form of shrouding and tie-boss [12], [13].

In the computational model for determination of the natural vibration characteristics based on rotational periodicity of the system it is considered that the structure can be divided

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into the certain number of identical parts – subsystems. The subsystem discretization will be performed in such a way that the subsystem may be coupled in the equal number of points in the same degrees of freedom to their left-side and right-side adjacent subsystems. After the mathematical formulation of the problem it is possible to derive relations for calculation of natural frequencies and mode shapes of the whole system [1] using the theory of solution of the matrix difference equations [14]. Compared with solution of the system as a whole the main advantage of the mentioned method (utilizing the rotational periodicity) consists in the fact that the order of stiffness and mass matrices does not increase (the manner of their assembling follows from the process of solution) but it is at most double than in case of considering a single subsystem. Thus the solution is less demanding for the computer operating memory and computing time. Performing a lower number of numerical operations and thus reducing the probability of a computing error is the result of a lower number of solved equations.

2. Relations for calculation of the natural vibration characteristics of a linear undamped rotationally periodic system

In order to perform accurate dynamical analyses of real mechanical systems their mathematical models can be very large and complex. Creating mathematical models based on the elimination of degrees of freedom is important mainly due to reducing the computing time and reducing the computing memory requirements. Without the elimination of degrees of freedom it is not possible to perform e.g. the optimization of complex real mechanical systems.

Detailed derivation of the mathematical relations for calculation of the natural vibration characteristics of a linear undamped rotationally periodic system is performed in [1]. Relations for calculation of these characteristics with the consideration of the possibility of elimination of the subsystem internal degrees of freedom are given in [4]. Due to the understandability of the article the relations for calculation of the natural vibration characteristics with considering the possibility of elimination of degrees of freedom are reminded in brief.

Let the finite periodic system be composed of the definite number M of identical parts – subsystems. The subsystem discretization will be performed in such a way, that it may be coupled in identical number N of points in identical degrees of freedom to their left-side and right-side adjacent subsystems (see Fig. 1). There are no requirements imposed on internal points of the subsystem.

Mathematically, this approach to the problem formulation leads to assembling and solving the matrix difference equations.

By introducing the condition that the whole system is linear and the motion of the k-th subsystem is investigated during its harmonic vibration it is possible to formulate the displacement vector $\mathbf{u}_k(t)$ of the k-th subsystem in the form

$$\mathbf{u}_k(t) = \mathbf{U}_k \, \mathrm{e}^{\mathrm{i}\,\omega\, t} \; , \tag{1}$$

where $\mathbf{U}_k$ is the vector of displacement amplitudes of the k-th subsystem, ω is the angular frequency, t is the time and i is the imaginary unit. The vector of generalized forces $\mathbf{q}_k(t)$ acting on the k-th subsystem can be formulated in the form

$$\mathbf{q}_k(t) = \mathbf{Q}_k \, \mathrm{e}^{\mathrm{i}\,\omega\, t} \; , \tag{2}$$

where $\mathbf{Q}_k$ is the vector of amplitudes of generalized forces acting on the k-th subsystem.

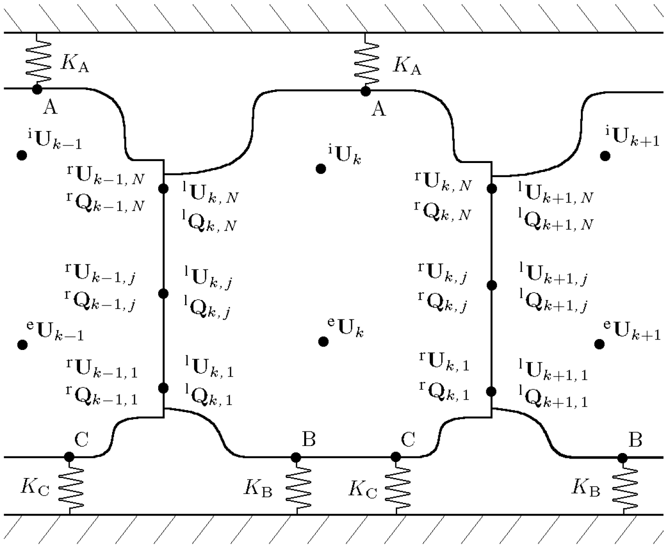

Fig.1: The k-th subsystem and its couplings with the adjacent subsystems

Thus it is supposed that the vector of generalized forces $\mathbf{q}_k(t)$ is proportional to the instantaneous state of the subsystem and it changes proportionally to its displacements. The relation between the amplitude of the subsystem displacements and the amplitude of generalized forces acting on it is described in a matrix equation

$$\mathbf{D}(\omega)\,\mathbf{U}_k = \mathbf{Q}_k \ , \tag{3}$$

where $\mathbf{D}(\omega)$ is the frequency dependent dynamic stiffness matrix identical for all subsystems $(k = 1, 2, \ldots, M)$.

The vector of displacements amplitudes $\mathbf{U}_k$ of the k-th subsystem can be partitioned into subvectors $^\mathrm{l}\mathbf{U}_k$ corresponding to the degrees of freedom, in which the k-th subsystem is coupled with the subsystem $k-1$, $^\mathrm{r}\mathbf{U}_k$ corresponding to the degrees of freedom, in which it is coupled with the subsystem $k+1$ (note: in comparison with previous papers, e.g. [2], superior index denoting right-side adjacent subsystems p was changed to the more logical index r), $^\mathrm{i}\mathbf{U}_k$ corresponding to the internal degrees of freedom and $^\mathrm{e}\mathbf{U}_k$ corresponding to the internal degrees of freedom to be eliminated (see Fig. 1). The vector of the amplitudes of generalized forces $\mathbf{Q}_k$ can be partitioned in the same way (see Fig. 1).

In the points of coupling the k-th subsystem with the adjacent subsystems the compatibility conditions hold

$$\begin{aligned} {}^\mathrm{r}\mathbf{U}_{k-1} &= {}^\mathrm{l}\mathbf{U}_k \ , \\ {}^\mathrm{r}\mathbf{U}_{k} &= {}^\mathrm{l}\mathbf{U}_{k+1} \end{aligned} \tag{4}$$

and the conditions of equilibrium of the generalized coupling forces hold

$$\begin{aligned} {}^\mathrm{r}\mathbf{Q}_{k-1} &= -{}^\mathrm{l}\mathbf{Q}_k \ , \\ {}^\mathrm{r}\mathbf{Q}_{k} &= -{}^\mathrm{l}\mathbf{Q}_{k+1} \end{aligned} \tag{5}$$

The compatibility conditions (4) and the conditions of equilibrium of the generalized coupling forces (5) are given in the general form. When formulating the concrete compatibility conditions and the concrete conditions of equilibrium of the generalized coupling forces for cyclic systems it is necessary to respect the influence of the subsystem geometry, of the geometry of the whole periodic system and of the option of the local coordinate system, in which the coordinates of the subsystem points are determined. Derivation of the concrete compatibility conditions and the concrete conditions of equilibrium of the generalized coupling forces and form of transformation matrices in the rotationally periodic system composed of the subsystems of a sector shape (the case of circular plates and steam turbine bladed disks) are given in [1], [2] or [3].

The dynamic stiffness matrix $\mathbf{D}(\omega)$ of the subsystem can be written using the submatrices corresponding to the individual groups of degrees of freedom. When introducing the condition that the generalized forces act on the subsystem only in the points common with the adjacent subsystems and that they do not act on the subsystem internal points, the matrix equation (3) can be written in the form

$$\begin{bmatrix} {}^{\mathrm{ll}}\mathbf{D} & {}^{\mathrm{li}}\mathbf{D} & {}^{\mathrm{lr}}\mathbf{D} & {}^{\mathrm{le}}\mathbf{D} \\ {}^{\mathrm{il}}\mathbf{D} & {}^{\mathrm{ii}}\mathbf{D} & {}^{\mathrm{ir}}\mathbf{D} & {}^{\mathrm{ie}}\mathbf{D} \\ {}^{\mathrm{rl}}\mathbf{D} & {}^{\mathrm{ri}}\mathbf{D} & {}^{\mathrm{rr}}\mathbf{D} & {}^{\mathrm{re}}\mathbf{D} \\ {}^{\mathrm{el}}\mathbf{D} & {}^{\mathrm{ei}}\mathbf{D} & {}^{\mathrm{er}}\mathbf{D} & {}^{\mathrm{ee}}\mathbf{D} \end{bmatrix} \begin{bmatrix} {}^{\mathrm{l}}\mathbf{U}_k \\ {}^{\mathrm{i}}\mathbf{U}_k \\ {}^{\mathrm{r}}\mathbf{U}_k \\ {}^{\mathrm{e}}\mathbf{U}_k \end{bmatrix} = \begin{bmatrix} {}^{\mathrm{l}}\mathbf{Q}_k \\ \mathbf{0} \\ {}^{\mathrm{r}}\mathbf{Q}_k \\ \mathbf{0} \end{bmatrix} . \tag{6}$$

Note: If the subsystems were coupled to the inertial frame, as it is illustrated in Fig. 1, by springs of stiffness K_{A}, K_{B} and K_{C} in nodes A, B and C these stiffnesses would be included in the submatrices ${}^{\mathrm{li}}\mathbf{D}$, ${}^{\mathrm{ri}}\mathbf{D}$, ${}^{\mathrm{ii}}\mathbf{D}$ or ${}^{\mathrm{ei}}\mathbf{D}$ of the dynamic stiffness matrix $\mathbf{D}(\omega)$.

By eliminating the subvector of displacements amplitudes ${}^{\mathrm{e}}\mathbf{U}_k$ the matrix equation for the partially reduced subsystem is obtained:

$$\begin{bmatrix} {}^{\mathrm{ll}}\tilde{\mathbf{D}} & {}^{\mathrm{li}}\tilde{\mathbf{D}} & {}^{\mathrm{lr}}\tilde{\mathbf{D}} \\ {}^{\mathrm{il}}\tilde{\mathbf{D}} & {}^{\mathrm{ii}}\tilde{\mathbf{D}} & {}^{\mathrm{ir}}\tilde{\mathbf{D}} \\ {}^{\mathrm{rl}}\tilde{\mathbf{D}} & {}^{\mathrm{ri}}\tilde{\mathbf{D}} & {}^{\mathrm{rr}}\tilde{\mathbf{D}} \end{bmatrix} \begin{bmatrix} {}^{\mathrm{l}}\mathbf{U}_k \\ {}^{\mathrm{i}}\mathbf{U}_k \\ {}^{\mathrm{r}}\mathbf{U}_k \end{bmatrix} = \begin{bmatrix} {}^{\mathrm{l}}\mathbf{Q}_k \\ \mathbf{0} \\ {}^{\mathrm{r}}\mathbf{Q}_k \end{bmatrix} . \tag{7}$$

where ${}^{\mathrm{ll}}\tilde{\mathbf{D}}$, ${}^{\mathrm{li}}\tilde{\mathbf{D}}$, ${}^{\mathrm{lr}}\tilde{\mathbf{D}}$, ${}^{\mathrm{il}}\tilde{\mathbf{D}}$, ${}^{\mathrm{ii}}\tilde{\mathbf{D}}$, ${}^{\mathrm{ir}}\tilde{\mathbf{D}}$, ${}^{\mathrm{rl}}\tilde{\mathbf{D}}$, ${}^{\mathrm{ri}}\tilde{\mathbf{D}}$ and ${}^{\mathrm{rr}}\tilde{\mathbf{D}}$ are submatrices of the reduced dynamic stiffness matrix $\tilde{\mathbf{D}}(\omega)$.

The equation obtained by multiplying the first row of matrix equation (7) and transcribed for the subsystem $k+1$ is, after introducing the second compatibility condition (4) and the second condition of generalized coupling forces equilibrium (5), of the form

$${}^{\mathrm{ll}}\tilde{\mathbf{D}}\,{}^{\mathrm{r}}\mathbf{U}_k + {}^{\mathrm{li}}\tilde{\mathbf{D}}\,{}^{\mathrm{i}}\mathbf{U}_{k+1} + {}^{\mathrm{lr}}\tilde{\mathbf{D}}\,{}^{\mathrm{r}}\mathbf{U}_{k+1} = -{}^{\mathrm{r}}\mathbf{Q}_k \ . \tag{8}$$

Using the first compatibility condition (4) the following relations are obtained from the second and the third row of matrix equation (7):

$${}^{\mathrm{il}}\tilde{\mathbf{D}}\,{}^{\mathrm{r}}\mathbf{U}_{k-1} + {}^{\mathrm{ii}}\tilde{\mathbf{D}}\,{}^{\mathrm{i}}\mathbf{U}_k + {}^{\mathrm{ir}}\tilde{\mathbf{D}}\,{}^{\mathrm{r}}\mathbf{U}_k = \mathbf{0} \ , \tag{9}$$

$${}^{\mathrm{rl}}\tilde{\mathbf{D}}\,{}^{\mathrm{r}}\mathbf{U}_{k-1} + {}^{\mathrm{ri}}\tilde{\mathbf{D}}\,{}^{\mathrm{i}}\mathbf{U}_k + {}^{\mathrm{rr}}\tilde{\mathbf{D}}\,{}^{\mathrm{r}}\mathbf{U}_k = {}^{\mathrm{r}}\mathbf{Q}_k \ . \tag{10}$$

Equations (8), (9) and (10) are the initial difference equations for calculation of the subvectors of displacements amplitudes ${}^{\mathrm{r}}\mathbf{U}_k$ and ${}^{\mathrm{i}}\mathbf{U}_k$ and the subvector of the amplitudes of generalized coupling forces ${}^{\mathrm{r}}\mathbf{Q}_k$.

Suitable procedure, which can be used for the derivation of relations for the calculation of natural frequencies and mode shapes of a periodic system, leads to solution of the system of homogeneous linear difference equations of the second type [14] (the variable in these equations is k; $k = 1, 2, \ldots, M$). Using this procedure the subvector of the amplitudes of generalized coupling forces ${}^{\mathrm{r}}\mathbf{Q}_k$ is eliminated from the difference equations (8), (9) and (10) and only the subvectors of displacements amplitudes ${}^{\mathrm{r}}\mathbf{U}_k$ and ${}^{\mathrm{i}}\mathbf{U}_k$ are determined.

The subvector of the amplitudes of generalized coupling forces ${}^{\mathrm{r}}\mathbf{Q}_k$ is eliminated by the summation of the equations (8) and (10):

$$ {}^{\mathrm{rl}}\tilde{\mathbf{D}}\,{}^{\mathrm{r}}\mathbf{U}_{k-1} + \left({}^{\mathrm{ll}}\tilde{\mathbf{D}} + {}^{\mathrm{rr}}\tilde{\mathbf{D}}\right){}^{\mathrm{r}}\mathbf{U}_k + {}^{\mathrm{ri}}\tilde{\mathbf{D}}\,{}^{\mathrm{i}}\mathbf{U}_k + {}^{\mathrm{lr}}\tilde{\mathbf{D}}\,{}^{\mathrm{r}}\mathbf{U}_{k+1} + {}^{\mathrm{li}}\tilde{\mathbf{D}}\,{}^{\mathrm{i}}\mathbf{U}_{k+1} = \mathbf{0}\ . \qquad (11) $$

Matrix equations (9) and (11) can be expressed using one matrix equation

$$ \begin{bmatrix} {}^{\mathrm{rl}}\tilde{\mathbf{D}} & \mathbf{0} \\ {}^{\mathrm{il}}\tilde{\mathbf{D}} & \mathbf{0} \end{bmatrix} {}^{\mathrm{r,i}}\mathbf{U}_{k-1} + \begin{bmatrix} {}^{\mathrm{ll}}\tilde{\mathbf{D}} + {}^{\mathrm{rr}}\tilde{\mathbf{D}} & {}^{\mathrm{ri}}\tilde{\mathbf{D}} \\ {}^{\mathrm{ir}}\tilde{\mathbf{D}} & {}^{\mathrm{ii}}\tilde{\mathbf{D}} \end{bmatrix} {}^{\mathrm{r,i}}\mathbf{U}_k + \begin{bmatrix} {}^{\mathrm{lr}}\tilde{\mathbf{D}} & {}^{\mathrm{li}}\tilde{\mathbf{D}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} {}^{\mathrm{r,i}}\mathbf{U}_{k+1} = \mathbf{0}\ , \qquad (12) $$

where

$$ {}^{\mathrm{r,i}}\mathbf{U}_k = \begin{bmatrix} {}^{\mathrm{r}}\mathbf{U}_k \\ {}^{\mathrm{i}}\mathbf{U}_k \end{bmatrix}\ . $$

In the process of solution (see [1] or [2]) the condition that the periodic system is undamped is introduced and cyclic condition ${}^{\mathrm{r,i}}\mathbf{U}_{k+M} = {}^{\mathrm{r,i}}\mathbf{U}_k$ (where $k = 1, 2, \ldots, M$) is used.

After solving the matrix difference equations (see [1] or [2]; the course of solving complies with the conditions given in [14]) the vector of displacements amplitudes ${}^{\mathrm{r,i}}\mathbf{U}_k$ for cyclic periodic systems is dependent on the optional parameter β (for the mode shapes of the rotationally periodic systems the parameter β means the number of nodal diameters). Two forms of equations are obtained for the calculation of natural frequencies and mode shapes of the whole periodic system in dependence on the parameter β value: the first form of equations (see relations from (13) to (16)) holds for $\beta = 0$ and in addition in case of even M $\beta = M/2$, the second form of equations (see relations from (17) to (21)) holds for $\beta = 1, 2, \ldots, M{-}1$ and when M is even with the condition $\beta \neq M/2$. As it was already mentioned, detailed derivation of the mathematical relations for calculation of the natural vibration characteristics of a linear undamped rotationally periodic system is performed in [1] or in [2].

For $\beta = 0$ and in addition in case of even M $\beta = M/2$ the vector of displacement amplitudes ${}^{\mathrm{r,i}}_{\beta}\mathbf{U}_k$ can be calculated from the relation

$$ {}^{\mathrm{r,i}}_{\beta}\mathbf{U}_k = C\,\cos({}_{\beta}\alpha\,k)\,{}_{\beta}\mathbf{a}\ , \qquad (13) $$

where ${}_{\beta}\alpha = 2\pi\,\beta/M$, C is the optional constant, ${}_{\beta}\mathbf{a}$ is the characteristic vector.

The characteristic vector ${}_{\beta}\mathbf{v} = {}_{\beta}\mathbf{a}$ (for $\beta = 0$ and in addition in case of even M $\beta = M/2$) can be determined (for specifically chosen β) from equation

$$ {}_{\beta}\mathbf{H}\,{}_{\beta}\mathbf{v} = {}_{\beta}\mathbf{H}_1\,{}_{\beta}\mathbf{a} = \mathbf{0}\ , \qquad (14) $$

where

$$ {}_{\beta}\mathbf{H} = {}_{\beta}\mathbf{H}_1 = \begin{bmatrix} {}^{\mathrm{ll}}\tilde{\mathbf{D}} + {}^{\mathrm{rr}}\tilde{\mathbf{D}} & {}^{\mathrm{ri}}\tilde{\mathbf{D}} \\ {}^{\mathrm{ir}}\tilde{\mathbf{D}} & {}^{\mathrm{ii}}\tilde{\mathbf{D}} \end{bmatrix} + \left(\begin{bmatrix} {}^{\mathrm{rl}}\tilde{\mathbf{D}} & \mathbf{0} \\ {}^{\mathrm{il}}\tilde{\mathbf{D}} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} {}^{\mathrm{lr}}\tilde{\mathbf{D}} & {}^{\mathrm{li}}\tilde{\mathbf{D}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \cos{}_{\beta}\alpha\ . \qquad (15) $$

Natural frequencies of the whole periodic system can be determined from the condition of a nontrivial solution:

$$\det {}_\beta\mathbf{H} = \det {}_\beta\mathbf{H}_1 = 0 \ . \tag{16}$$

Mode shapes of the whole periodic system (for $\beta = 0$ and in addition in case of even M $\beta = M/2$) can be calculated by substituting the characteristic vectors ${}_\beta\mathbf{a}$ into equation (13). Mode shapes are calculated for the k-th subsystem in its local frame of reference and then they must be transformed to the global one.

When $\beta = 1, 2, \ldots, M{-}1$ and when M is even with the condition $\beta \neq M/2$ the vector of displacement amplitudes ${}^{\mathrm{r,i}}_{\beta}\mathbf{U}_k$ can be calculated from the relation

$${}^{\mathrm{r,i}}_{\beta}\mathbf{U}_k = C_1\left[\cos({}_\beta\alpha\, k)\, {}_\beta\mathbf{a} + \sin({}_\beta\alpha\, k)\, {}_\beta\mathbf{b}\right] + C_2\left[\cos({}_\beta\alpha\, k)\,(-{}_\beta\mathbf{b}) + \sin({}_\beta\alpha\, k)\, {}_\beta\mathbf{a}\right] \ , \tag{17}$$

where ${}_\beta\alpha = 2\pi\,\beta/M$, C_1 and C_2 are the optional constants, ${}_\beta\mathbf{a}$ and ${}_\beta\mathbf{b}$ are the characteristic vectors.

Characteristic vectors ${}_\beta\mathbf{a}$ and ${}_\beta\mathbf{b}$ (for $\beta = 1, 2, \ldots, M{-}1$ and when M is even with the condition $\beta \neq M/2$) can be determined (for the specifically chosen β) from the relation

$${}_\beta\mathbf{H}\, {}_\beta\mathbf{v} = \begin{bmatrix} {}_\beta\mathbf{H}_1 & {}_\beta\mathbf{H}_2 \\ -{}_\beta\mathbf{H}_2 & {}_\beta\mathbf{H}_1 \end{bmatrix} \begin{bmatrix} {}_\beta\mathbf{a} \\ {}_\beta\mathbf{b} \end{bmatrix} = \mathbf{0} \ , \tag{18}$$

where

$${}_\beta\mathbf{H}_1 = \begin{bmatrix} {}^{\mathrm{ll}}\tilde{\mathbf{D}} + {}^{\mathrm{rr}}\tilde{\mathbf{D}} & {}^{\mathrm{ri}}\tilde{\mathbf{D}} \\ {}^{\mathrm{ir}}\tilde{\mathbf{D}} & {}^{\mathrm{ii}}\tilde{\mathbf{D}} \end{bmatrix} + \left(\begin{bmatrix} {}^{\mathrm{rl}}\tilde{\mathbf{D}} & \mathbf{0} \\ {}^{\mathrm{il}}\tilde{\mathbf{D}} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} {}^{\mathrm{lr}}\tilde{\mathbf{D}} & {}^{\mathrm{li}}\tilde{\mathbf{D}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \cos {}_\beta\alpha \ , \tag{19}$$

$${}_\beta\mathbf{H}_2 = \left(\begin{bmatrix} {}^{\mathrm{lr}}\tilde{\mathbf{D}} & {}^{\mathrm{li}}\tilde{\mathbf{D}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} {}^{\mathrm{rl}}\tilde{\mathbf{D}} & \mathbf{0} \\ {}^{\mathrm{il}}\tilde{\mathbf{D}} & \mathbf{0} \end{bmatrix} \right) \sin {}_\beta\alpha \tag{20}$$

and natural frequencies of the whole periodic system (for $\beta = 1, 2, \ldots, M{-}1$ and when M is even with the condition $\beta \neq M/2$) can be determined from the condition of nontrivial solution:

$$\det {}_\beta\mathbf{H} = \det \begin{bmatrix} {}_\beta\mathbf{H}_1 & {}_\beta\mathbf{H}_2 \\ -{}_\beta\mathbf{H}_2 & {}_\beta\mathbf{H}_1 \end{bmatrix} = 0 \ . \tag{21}$$

Mode shapes of the whole periodic system can be calculated by substituting the characteristic vectors ${}_\beta\mathbf{a}$ and ${}_\beta\mathbf{b}$ into equation (17). Mode shapes are calculated, as well as in case of equation (13), for the k-th subsystem in its local frame of reference and then they must be transformed to the global one. Constants C_1 and C_2 when enumerating mode shapes ${}^{\mathrm{r,i}}_{\beta}\mathbf{U}_k$ can be selected arbitrarily. It follows from both the decision procedure itself and the definition of mode shapes, which must form a linearly independent basis. During the visualization of the particular mode shapes of the real steam turbine bladed disk [1], [12] conditions $C_1 = 1$ and $C_2 = 0$ are applied similarly as in [8].

Generally, characteristic vectors ${}_\beta\mathbf{a}$ and ${}_\beta\mathbf{b}$ are different for identical β. It can be shown that with certain types of the subsystem symmetry it holds ${}_\beta\mathbf{a} = c\, {}_\beta\mathbf{b}$ (where c is a constant; $\beta = 1, 2, \ldots, M{-}1$ and when M is even with the condition $\beta \neq M/2$) and relations for calculation of natural frequencies and mode shapes of the rotationally periodic system are simpler [8].

As the periodic system is considered undamped, the dynamic stiffness matrix $\mathbf{D}(\omega)$ of the subsystem can be put in the form [1], [3]

$$\mathbf{D}(\omega) = \mathbf{K} - \omega^2\,\mathbf{M}\ , \tag{22}$$

where $\mathbf{K}$ is the stiffness matrix of the subsystem and $\mathbf{M}$ is the mass matrix of the subsystem.

In the case in which the dynamic stiffness matrix $\mathbf{D}(\omega)$ is considered in the form of (22), it is possible to use e.g. the Guyan reduction (e.g. [1], [15]) for assembling the submatrices ${}^{\mathrm{ll}}\tilde{\mathbf{D}}$, ${}^{\mathrm{li}}\tilde{\mathbf{D}}$, ${}^{\mathrm{lr}}\tilde{\mathbf{D}}$, ${}^{\mathrm{il}}\tilde{\mathbf{D}}$, ${}^{\mathrm{ii}}\tilde{\mathbf{D}}$, ${}^{\mathrm{ir}}\tilde{\mathbf{D}}$, ${}^{\mathrm{rl}}\tilde{\mathbf{D}}$, ${}^{\mathrm{ri}}\tilde{\mathbf{D}}$, ${}^{\mathrm{rr}}\tilde{\mathbf{D}}$.

When assembling the equations for calculation of the natural vibration characteristics of a linear undamped rotationally periodic system according to the described method the solution of the matrix equation (14) or (18) leads to the generalized eigenvalue problem [1], [2] or [3]. The equation of the following type will be solved (in accordance with equation (22); $\beta = 0, 1, \dots, M{-}1$) :

$$_{\beta}\mathbf{H}\,_{\beta}\mathbf{v} = \left(_{\beta}\mathbf{H}_{\mathrm{K}} - {}_{\beta}\omega^2\,_{\beta}\mathbf{H}_{\mathrm{M}}\right)\,_{\beta}\mathbf{v} = \mathbf{0}\ . \tag{23}$$

The subspace iteration method (e.g. [16]), which enables to determine the selected number q of the lowest eigenvalues $_{\beta}\omega^2$ and corresponding natural vectors, is a very efficient numerical method. This method requires the matrices $_{\beta}\mathbf{H}_{\mathrm{K}}$ and $_{\beta}\mathbf{H}_{\mathrm{M}}$ to be symmetric and in addition matrix $_{\beta}\mathbf{H}_{\mathrm{K}}$ to be positive definite (at the same time those conditions guarantee the eigenvalues $_{\beta}\omega^2$ of equation (23) to be real positive). The proof that matrices $_{\beta}\mathbf{H}_{\mathrm{K}}$ and $_{\beta}\mathbf{H}_{\mathrm{M}}$ comply with the stated conditions is carried out in [1] ($\beta = 0, 1, \dots, M{-}1$).

After determination of the values $_{\beta}\omega_j^2$ ($j = 1, 2, \dots, q$) the natural frequencies (in [Hz]) of the whole periodic system can be calculated from the relation

$$_{\beta}f_j = \frac{\sqrt{_{\beta}\omega_j^2}}{2\pi}\ , \quad j = 1, 2, \dots, q\ . \tag{24}$$

The PERCOK in-house software [12] was created in the Compaq Visual Fortran programming language on the basis of the relations for calculation of the natural vibration characteristics of the linear undamped rotationally periodic systems with the subsystems of a sector shape. The Gyuan reduction for the elimination of degrees of freedom is used. The PERCOK software uses the subsystem stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$ assembled applying the COSMOS/M FEM software. The PERG in-house software [12], in the Compaq Visual Fortran programming language as well, was created to visualize the mode shapes of the rotationally periodic systems.

3. The test example of the circular plate

The homogeneous thin circular plate was chosen as the test example for the verification of the proposed method, which is aimed at the calculation of natural frequencies and mode shapes of a linear undamped rotationally periodic systems considering the possibility of elimination of degrees of freedom. The chosen circular plate of uniform thickness is clamped at its inner radius. This example was chosen due to the geometric similarity and analogical character of mode shapes as disks of steam turbines have – e.g. [17]. In addition it is a simple rotationally periodic system, in which it is possible to create the model of the whole periodic system in the FEM software and calculate its natural frequencies and mode shapes. When thin plates of uniform thickness are considered it is possible to compare the results with the natural vibration characteristics calculated analytically or numerically – e.g. [17], [18], [19], [20], [21].

The investigated circular plate is of an inner radius $r = 0.1\,\mathrm{m}$, outer radius $R = 1\,\mathrm{m}$ and thickness $h = 0.03\,\mathrm{m}$. Material characteristics of the plate correspond to steel (Poisson's ratio $\nu = 1/3$, Young's modulus $E = 2.1\times10^{11}\,\mathrm{N\,m^{-2}}$ and density $\varrho = 7900\,\mathrm{kg\,m^{-3}}$).

A ten-degree sector of the plate discretized into 360 eight-node three-dimensional elements SOLID (see Fig. 2) is considered to be a subsystem. The subsystem is coupled in 135 nodes (in 405 degrees of freedom) to right-side adjacent subsystems, the same couplings are to its left-side adjacent subsystems, the number of internal nodes is 405 (with 1215 degrees of freedom).

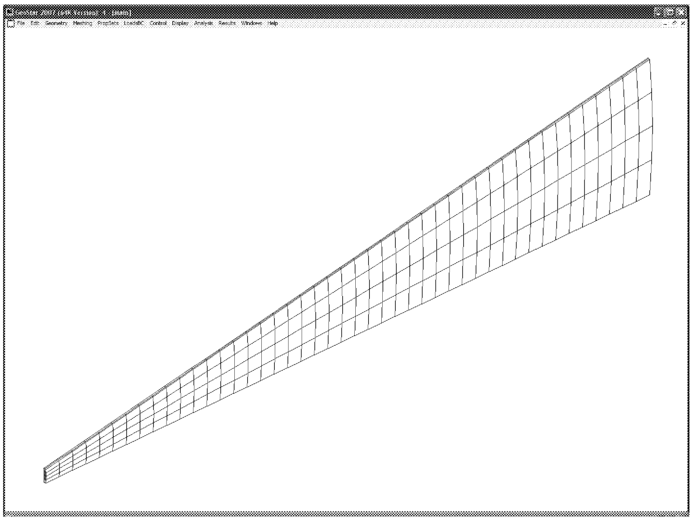

Fig.2: Finite element mesh of the subsystem of the thin circular plate (the COSMOS/M FEM software)

Natural frequencies of the investigated circular plate calculated applying the method utilizing the rotational periodicity of the system without the elimination of degrees of freedom are given in Tab. 1, natural frequencies calculated applying the method utilizing the rotational periodicity of the system with the elimination of degrees of freedom are given in Tab. 2. The natural vibration characteristics of the FEM model of the whole plate calculated in the COSMOS/M software are the same as the natural vibration characteristics calculated applying the method utilizing the rotational periodicity of the system without the elimination of degrees of freedom (see Tab. 1). This coincidence is given by the fact that the mass and the stiffness matrices of the subsystem were assembled applying the COSMOS/M software, and the finite element mesh of the subsystem was the same as the finite element mesh of the sector of the whole plate model.

Visualization of the third bending mode shape with three nodal diameters of the circular plate (without degrees of freedom elimination) in the PERG in-house software is given in Fig. 3.

To compare the results in Tab. 3 there are given the calculated values of natural frequencies corresponding to the bending mode shapes of undamped plate clamped at its inner

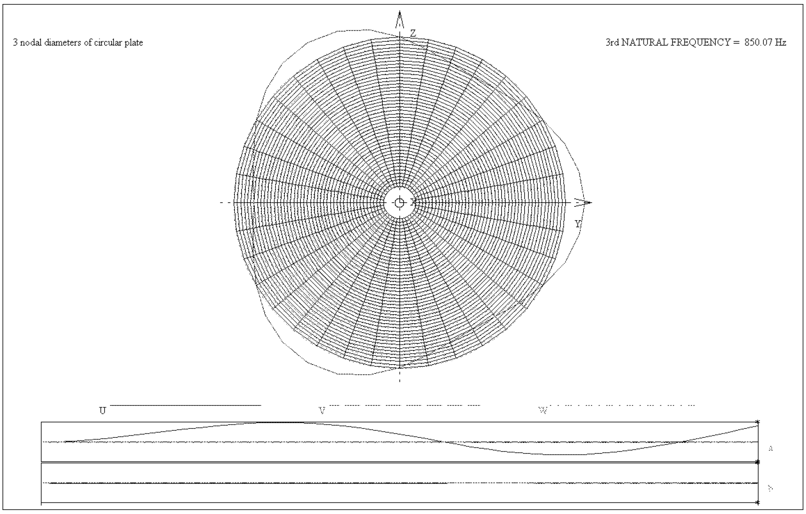

*Fig.3: Visualization of the third bending mode shape with three nodal diameters of the circular plate in the PERG in-house software (**a** and **b** – characteristic vectors $_{\beta}\mathbf{a}$ and $_{\beta}\mathbf{b}$ from the relations (17) and (18); U – bending deflections, V – torsional deflections, W – radial deflections)*

Natural frequencies calculated without degrees of freedom elimination or using the whole plate (clamped at its inner radius) FEM model [Hz]				
Number of nodal diameters	0	1	2	3
1 nodal circle	32.20	26.32	41.58	92.15
2 nodal circles	191.57	209.36	278.20	399.75
3 nodal circles	559.37	585.78	679.74	850.07
4 nodal circles	1110.73	1144.15	1256.35	1461.18
5 nodal circles	1849.27 (torsional, 1 nodal circle, 1440.08)	1887.66	2012.69	2240.76

Tab.1: Natural frequencies of the circular plate clamped at its inner radius calculated applying the method utilizing the rotational periodicity of the system without degrees of freedom elimination or using the whole plate FEM model

Natural frequencies calculated with all internal degrees of freedom elimination [Hz] (plate clamped at its inner radius)				
Number of nodal diameters	0	1	2	3
1 nodal circle	32.21	26.32	41.58	92.16
2 nodal circles	191.59	209.38	278.25	399.87
3 nodal circles	559.66	586.12	680.25	851.06
4 nodal circles	1112.75	1146.36	1259.22	1465.54
5 nodal circles	1857.71 (torsional, 1 nodal circle, 1440.11)	1896.60	2023.31	2254.87*

* For the reason of stability of the numerical solution three internal degrees of freedom (in one internal point) were not eliminated.

Tab.2: Natural frequencies of the circular plate clamped at its inner radius calculated applying the method utilizing the rotational periodicity of the system with all internal degrees of freedom elimination

Natural frequencies calculated using the formula (25) given in [18] [Hz] (plate clamped at its inner radius)				
Number of nodal diameters	0	1	2	3
1 nodal circle	32.33	23.89	40.80	93.26
2 nodal circles	192.54	206.07	276.03	400.31

Tab.3: Natural frequencies of the circular plate clamped at its inner radius calculated using the formula (25) given in [18]

Natural frequencies calculated using the formula (26) [Hz] (plate clamped at its centre)				
Number of nodal diameters	0	1	2	3
1 nodal circle	28.28	13.51	39.65	92.26
2 nodal circles	157.68	154.67	265.77	398.88
3 nodal circles	461.29	451.19	632.39	838.92
4 nodal circles	909.01	896.95	1160.76	1447.94
5 nodal circles	1507.33	1493.92	1829.33	2191.13

Tab.4: Natural frequencies of the circular plate clamped at its centre calculated using the formula (26)

radius using the formula given in [18]

$$f_{\beta\,j} = \frac{\lambda_{\beta\,j}}{2\pi\,R^2}\sqrt{\frac{E\,h^2}{12\,\varrho\,(1-\nu^2)}}\;, \tag{25}$$

where $\lambda_{\beta\,j}$ are the coefficients dependent on the ratio between the inner radius r and outer radius R of the plate, on the number of nodal diameters β and on the number of nodal circles j (order of appropriate natural frequency corresponding to the bending mode shape at the same β). Coefficients $\lambda_{\beta\,j}$ are given in [18] only for $\beta = 0, 1, 2, 3$ and for $j = 1, 2$. The same formula is presented in [19]. In comparison with [18], in [19] the values of the coefficients $\lambda_{\beta\,j}$ differ only slightly.

Further, to compare the results in Tab. 4 there are given the calculated values of natural frequencies corresponding to the bending mode shapes of undamped plate clamped at its centre (i.e. $r = 0\,\mathrm{m}$) using the formula given in [17]

$$f_{\beta\,j} = \frac{\kappa_{\beta\,j}}{2\pi}\sqrt{\frac{E\,h^2}{3\,\varrho\,R^4\,(1-\nu^2)}}\;, \tag{26}$$

where coefficients $\kappa_{\beta\,j}$ are dependent on the boundary conditions and on the ratio between the inner radius r and outer radius R of the plate, on the number of nodal diameters β and on the number of nodal circles j. Coefficients $\kappa_{\beta\,j}$ are reported in [17]. The formula (26) holds at neglecting shear deformations (which holds for fairly thin plates). It is evident, that the results calculated using the formula (26) only get near to the real natural frequencies of the investigated plate because the inner radius is considered to be zero (deviation of the natural frequencies values should not be too large however).

The formula for the calculation of the natural frequencies corresponding to the bending mode shapes of the plate clamped at its centre ($r = 0\,\mathrm{m}$) presented in [20] or [21]

$$f_{\beta\,j} = \frac{\mu^2_{\beta\,j}}{2\pi}\sqrt{\frac{E\,h^2}{12\,\varrho\,R^4\,(1-\nu^2)}}\;, \tag{27}$$

where the coefficients $\mu_{\beta\,j}$ are dependent on the number of nodal diameters β and on the number of nodal circles j, does not give the comparable results.

Note: 'Uncommented' natural frequencies given in Tabs. 1 to 4 correspond to the bending mode shapes.

4. Conclusion

The article presents the results of the verification calculation of the method for the calculation of natural frequencies and mode shapes of a linear undamped rotationally periodic systems considering the possibility of elimination of degrees of freedom. As a test example a thin circular plate was chosen. The method can be applied e.g. for calculation of the natural vibration characteristics of the steam turbine bladed disks.

It is possible to conclude from the results given in Tabs. 1 to 4 that the method for the calculation of the natural vibration characteristics of a linear undamped rotationally periodic systems is functional and gives very good results. When the result calculated using the Gyuan reduction for the elimination of internal degrees of freedom and results calculated without the elimination of degrees of freedom are compared the best coincidence is in the lowest natural frequencies for any number of nodal diameters. With the increasing order of natural frequency the differences between the results without degrees of freedom elimination and with degrees of freedom elimination increase. Using the Gyuan reduction for the elimination of internal degrees of freedom has only a minimal influence on the values of calculated natural frequencies at a simple structure as the thin circular plate of a uniform thickness. At calculating the natural vibration characteristics of a more complicated structure, e.g. a steam turbine bladed disk, it is necessary to choose the eliminated degrees of freedom with more attention.

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