BIFURCATION CHARACTERISTICS IN PARAMETRIC SYSTEMS WITH COMBINED DAMPING

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The damping or the damping forces represent certain speciality in the investigation of the internal dynamics of the transmission systems. The special significance has this damping parameter especially in the areas of impact effects in the high-speed light aircraft and mobile transmission constructions. On the example of gear mesh in one branch of forces-flow in the pseudoplanetary reducer deals the paper with some partial results of the analysis of the influence of combination damping, i.e. linear and non-linear damping, on the gear mesh dynamics.

Keywords: *non-linear dynamics, parametric systems, impact effects, linear and non-linear damping, gearing mechanisms*

1. Introduction

The investigation in the area of dynamics of systems particularly then in the area of internal dynamics of non-linear heteronymous systems with time variable terms, like are planetary systems with gears i.e. with split power flow, is constantly highly relevant and motivated above all with non-explanation of series of phenomena. The issue and existence of these phenomena are in many cases not entirely well known from the viewpoint of basic research of such complicated systems. Their methodical i.e. analytical, numerical and experimental mastering of qualitative and quantitative analysis of influence of parameters, e.g. of damping – linear, non-linear and combined, self-excitation, linear and non-linear stiffness terms of systems and so on, are fundamental condition for explanation of many phenomena particularly in area of light aeronautical structures with high revolutions, e.g. in transmission systems of turboprop units.

The matter of analysis of specification of differences among criterion of deterministic chaos for the first, second, . . . , n-th approximation by the analytical method, i.e. method of transformation of non-linear boundary problems into systems of integro-differential time heteronymous equations with solving kernels in the form of Green's resolvent, is similarly still unsolved problem in mechanical dynamical systems, whose motion is described with non-linear time heteronymous ordinary differential equations.

The contribution, which was presented at conference 'Dynamics of Machines 2009', continues previous research of influence of linear and non-linear damping in gear mesh of one branch, i.e. a specific case, of force flow in the pseudoplanetary system, i.e. the system with six degrees of freedom [1], [3], [4], [5]. As isn't known any deeper experimental investigation of damping in the area of gear mesh, was performed the theoretical attempt to find certain still unknown profounder rightfulness of damping influence both of gear mechanism material

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in the mesh and the lubricating oil film in tooth space at tooth profile contact bounce into the area of technological gear backlash – by impact effects – onto internal dynamics of gear mesh. The investigation of variations of linear, quadratic, cubic and combined damping with constant coefficients K, described by vector expression [1]

$$ {}_1\mathbf{K}(\beta, \delta_i, H)\,\mathbf{v}' + \sum_{K_1>1} {}_{K_1}\mathbf{K}(D, D_i, H)\,|\mathbf{w}'(\mathbf{v}')|^{K_1}\,\mathrm{sgn}[\mathbf{w}'(\mathbf{v}')] \quad \text{for} \quad K_1 = 2.3 \qquad (1) $$

aims to approximate analytically besides a experimental research most of all to still unknown reality.

The next still theoretically unknown area of damping in gear mesh is the influence of time damping variability by the lightening discs of cog wheels as occur in light aeronautical structures with circular or sector lightening holes. This time variation of stiffness – mass of discs – depends on the size, form and number of holes on a circumference of wheel. The resultant damping and its characteristics is then a function of size and width of actual gearing, thickness of gear rim, dimensions of discs, size, form and number of lightening holes, speed and meshing frequency of rotating wheels and gearing including a material, its technological processing and working temperature.

The real damping in gear mesh can be complicated time variable, i.e. heteronymous, function that influences not only with its own quantitative but also qualitative properties for example the phase shifts of amplitudes of relative motions in a gear mesh towards amplitudes of resultant time alternatively modifying stiffness functions of gearing and so can by decisive means influence the dynamics, i.e. the dynamic forces in these systems.

Such a time damping function of disc $k_\mathrm{d}(t) = k_\mathrm{d}(t+T)$ of a cog-wheel with period $T = 2\pi/\omega$ in the interval $(0; T)$ that qualifies as a explicitness and a finiteness with finite number of maximums and minimums and non-continuities (Dirichlet's conditions) can we elaborate on convergent Fourier's series in the form

$$ \begin{aligned} k_\mathrm{d}(t) = k_\mathrm{d}(t+T) = \frac{1}{2T}\int_0^{2T} k_\mathrm{d}(t)\,\mathrm{d}t + \sum_{m=1}^{\infty}\Bigg\{ \left[\frac{2}{T}\int_0^{T} k_\mathrm{d}(t)\,\cos p\,m\,\omega\,t\,\mathrm{d}t\right] \cos p\,m\,\omega\,t\,\mathrm{d}t \,+ \\ + \left[\frac{2}{T}\int_0^{T} k_\mathrm{d}(t)\,\sin p\,m\,\omega\,t\,\mathrm{d}t\right] \sin p\,m\,\omega\,t\,\mathrm{d}t \Bigg\}\,, \end{aligned} \qquad (2) $$

$\omega = \pi\,n/30$ is angular velocity of cog-wheel, n – number of revolutions of wheel, p – number of lightening holes in a disc of a cog wheel or generally a number of periods of courses of damping by influence of disc non-homogeneity.

The specialty of this paper including the symbolism is related to general research study of an internal dynamics of planetary systems. The analysis of a specific case, i.e. the one branch – the mesh of sun gear 2 with the one, i.e. $j = 1$, planet wheel $\bar{3}$, is closely connected with the general analysis of system with the number of satellites $j > 1$.

2. To the influence of combined damping in a gear mesh on dynamic characteristics

The paper continues previous studies [1], [4], [5] of influence of linear and non-linear damping both material and viscous, i.e. the damping in a oil boundary layer of tooth gap due to tooth profile contact bounce at impact effects. By reason that in such complicated parts of transmission systems, e.g. in the normal gear mesh ‘rolling – sliding’, are still unknown neither approximate data about damping properties of about damping patterns, is this study devoted a combined, i.e. linear and non-linear – quadratic and cubic – constant damping according to a expression (1).

The time variable damping $k_\mathrm{d}(t)$ according to (2) in the system of motion equations (3) of transmission system

$$\begin{aligned}\mathbf{M}\,\mathbf{v}'' + {}_1\mathbf{K}(\beta,\delta_i,H)\,\mathbf{v}' + \sum_{K_1>1} {}_{K_1}\mathbf{K}(D,D_i,H)\,|\mathbf{w}'(\mathbf{v}')|^{K_1}\,\mathrm{sgn}[\mathbf{w}'(\mathbf{v}')] + \\ + {}_1\mathbf{C}(\varepsilon,\kappa,Y_n,U_n,V_n,H,\tau)\,\mathbf{v} + \sum_{K>1} {}_K\mathbf{C}(\varepsilon,\kappa,I_n,H,\tau)\,\mathbf{w}^K(\mathbf{v}) = \mathbf{F}(a_n,b_n,\bar{\varphi},H,\tau)\end{aligned} \tag{3}$$

is not pursued here in this paper. In the equation (3) $\mathbf{v}$ means generally the m-dimensional vector ($m = 6$) of displacement of system vibration, $\mathbf{w}^K(\mathbf{v})$ K-th power of vector $\mathbf{v}$, which is defined by expression $\mathbf{w}^K(\mathbf{v}) = \mathbf{D}[\mathbf{w}(\mathbf{v})\,\mathbf{w}^{K-1}(\mathbf{v})]$. $\mathbf{D}[\mathbf{w}(\mathbf{v})]$ means the diagonal matrix, whose elements at the main diagonal are comprised by elements of vector $\mathbf{w}(\mathbf{v}) \equiv \mathbf{v}$. Fur-

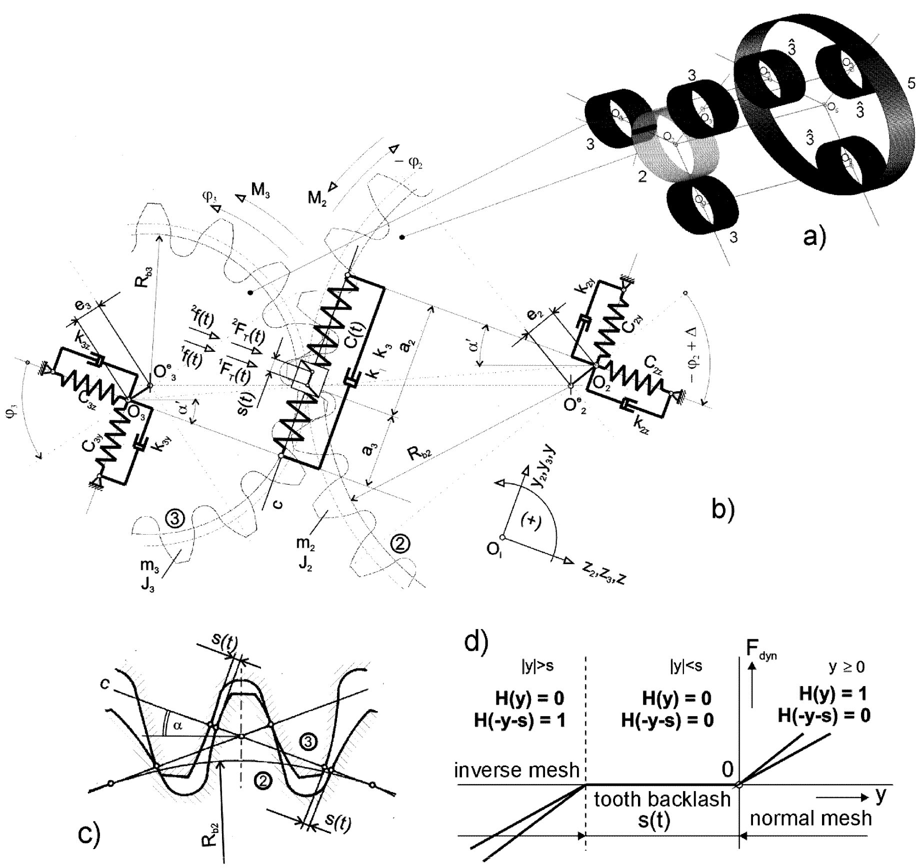

Fig.1: The substitutive mathematical – physical model of kinematic pair of gears of common differential planetary gear system with double planet wheels (a), the technological teeth backlash $s(t)$ and the values of Heaviside’s function H in the area of gear mesh with backlash (d)

thermore $\mathbf{M}$ is the matrix of mass and inertia forces, ${}_1\mathbf{K}$ and ${}_{K_1}\mathbf{K}$ are the matrix of linear and nonlinear damping forces, ${}_1\mathbf{C}$ and ${}_K\mathbf{C}$ are the matrix of linear and non-linear reversible forces and $\mathbf{F}(\tau)$ is the vector of non-potential external excitation with components a_n, b_n and with the phase angle $\bar{\varphi}$. H is the Heaviside's function, which allows to describe the motions – contact bounces – due to strongly non-analytical non-linearities, for example due to technological tooth backlash $s(\tau)$. Corresponding linear and non-linear coefficients of damping are denoted by β, δ_i, D, D_i linear parametric stiffness function by the symbols Y_n, U_n, V_n and non-linear parametric functions, so-called parametric non-linearities, by the symbol I_n. ε and κ are the coefficients of mesh duration and amplitude modulation of stiffness function ${}_1\mathbf{C}(\tau)$, see [1]. Derivative by non-dimensional time τ are denoted by dashes, $\tau = \omega_c\, t$, ω_c – mesh frequency, t – time.

	conser-vative system in gear mesh	linear damping in gear mesh – L			quadratic damping in gear mesh – Kv			cubic damping in gear mesh – Ku			Combinations of damping in gear mesh											
											L+Kv			L+Ku			L+Kv+Ku			Kv+Ku		
k_1	0	×		×							×		×	×		×	×		×			
k_2	0				×		×				×		×				×		×	×		×
k_3	0							×		×				×		×	×		×	×		×
$k_{1\mathrm{m}}$	0		×	×								×	×		×	×		×	×			
$k_{2\mathrm{m}}$	0					×	×					×	×					×	×		×	×
$k_{3\mathrm{m}}$	0								×	×					×	×		×	×		×	×
Symbol of marked solution in figures	○	□	•	×	□	•	×	□	•	×	□	•	×	□	•	×	□	•	×	□	•	×
Variation	a	b			c			d			e			f			g			h		

Note: $k_{1,2,3}$ – material damping in gear mesh; $k_{1\mathrm{m},2\mathrm{m},3\mathrm{m}}$ – viscous damping of lubricant mediums in the tooth backlash (without contemplation of temperature influence); 1 – linear, 2 – quadratic, 3 – cubic.

Tab.1: Combinations of damping in gear mesh

The particular combinations of of linear and non-linear constant quadratic and cubic damping in the gear mesh according to (1) are given in Tab. 1 and denoted with letters e, f, g, h. The particular combinations of damping in a material or in a lubricating ambient in the variation with symbols (○) for conservative homogeneous system, for the non-conservative homogeneous system with a symbols (□), (•), (×).

For the possibility of qualitative and quantitative comparison of solution results with the previous studies have been solved all resonance characteristics $\{\nu_s; t\}$ of given substitutive mathematical – physical model of pair of gears of pseudoplanetary system with double planet wheels with six degrees of freedom, see Fig. 1, for the parameters $\varepsilon = 1.569$; $\kappa = 0.5879$; $C_{\max} = 4{\times}10^5\,[\mathrm{N\,mm^{-1}}]$; $m_{\mathrm{red}} = 3.123{\times}10^{-3}\,[\mathrm{kg}]$. The values of material damping $k \equiv k_{1,2,3}$ both in the area of normal mesh and inverse one, as well as the values of viscous damping in the tooth backlash $k_{\mathrm{m}} \equiv k_{1\mathrm{m},2\mathrm{m},3\mathrm{m}}$ (for reason of simplicity we do not take into consideration the temperature influence) are considered in all next given examples of solution identical, i.e. $k = k_{\mathrm{m}} = 3.95$, which corresponds to proportional damping $\beta, \beta_{\mathrm{m}}, D, D_{\mathrm{m}} = 0.062$. The subscripts $1, 2, 3$ by coefficients both material k and viscous damping in the tooth backlash k_{m} mean in accordance with Tab. 1 the belonging to 1 – linear, 2 – quadratic and 3 – cubic term of dissipative function. The dimension of these coefficients

$k \equiv k_{1,2,3,\ldots,n}$; $k_{1\mathrm{m},2\mathrm{m},3\mathrm{m},\ldots,n\mathrm{m}}$ can we express generally in the form $k_n, k_{n\mathrm{m}}$ $[\mathrm{Ns}^n (10^3\mathrm{mm})^{-n}]$, n – degree of non-linear damping. The coefficients of material k alternatively k_m in the tooth space result on the ground of Lagrange's theory from the expressions for damping forces

$$\cdots + k_1\, y'(t) + \frac{3}{2}\, k_2\, y'^2(t)\ \mathrm{sgn}[y'(t)] + 2\, k_3\, y'^3(t) + \cdots +$$
$$+ k_{1\mathrm{m}}\, y'(t) + \frac{3}{2}\, k_{2\mathrm{m}}\, y'^2(t)\ \mathrm{sgn}[y'(t)] + 2\, k_{3\mathrm{m}}\, y'^3(t) + \cdots \ .$$

By the reason that system of equations (3) for the system from Fig. 1 leads for all here mentioned variations of combined damping (e)–(h) (see Tab. 1) on steady vibration already in the fifth revolution of gear wheels, are all resonance characteristics in following figures plotted for this fifth revolution.

The parametric, i.e. with time variable resultant – potential function $C(t)$ of cogs in gear mesh or its modification in the form $C(t)(H1 + H2)$, is the only exciting source of vibrations. With time variable damping $k_\mathrm{d}(t)$ in accordance with equation (2) is not in this study considered. The conservative system with $k = k_\mathrm{m} = 0$ is unstable in whole extent of wheel revolutions, see for example [2].

In Fig. 2 are for the fifth revolution illustrated the resonance characteristics for the frequency area of median cog stiffness C_s from the interval $\nu_\mathrm{s} \in \langle 0.6, 1.2\rangle$. Three areas of gear mesh are here colour-coded, i.e. with white colour the area of normal mesh, where $y(t) \geq 0$,

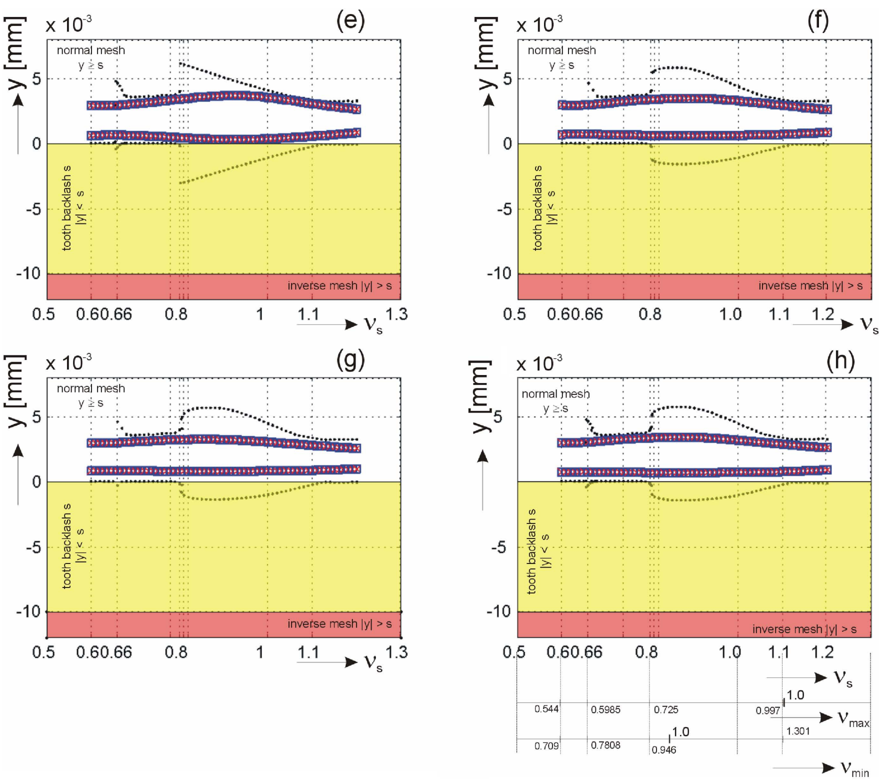

Fig.2: Resonance characteristics of relative motion $y(t)$ in frequency area of median cog stiffness C_s from the interval $\nu_\mathrm{s} \in \langle 0.6; 1.2\rangle$ for the combined damping (e)–(h), see Tab. 1, and for the fifth revolution of gear pair from Fig.1

with yellow colour the area of jump effects ($|y(t) < s(t)|$) and finally with red colour is determined the area of inverse mesh, i.e. $|y(t) \geq s(t)|$. By reason of comparison and easier analysis are plotted under figure three scales of tuning

– ν_s towards the median resultant cog stiffness C_s,
– ν_{max} for the frequency tuning of system towards maximal stiffness C_{max} and
– ν_{min} for the minimum level C_{min} of resultant parametric stiffness function $C(t)$ of gear mesh.

The mentioned frequency tuning are defined by relations

$$\nu_{max} = \omega_c \, \Omega_{max}^{-1} = \nu_s \, \Omega_s \, \Omega_{max}^{-1} \quad \text{and} \quad \nu_{min} = \omega_c \, \Omega_{min}^{-1} = \nu_s \, \Omega_s \, \Omega_{min}^{-1} \ ,$$

where $\Omega_{max}^2 = C_{max} \, m_{red}^{-1}$ and $\Omega_{min}^2 = C_{min} \, m_{red}^{-1}$.

The causation of occurance of sharp locations of discreteness in resonance characteristics that are accompanied by teeth contact bounces ($y(t) < 0$) with impact effects in the given moment consist in

– the given tuning of gear mesh,
– the time period of gear mesh on appropriate level of stiffness function $C(t)$ with the possibility of progress of amplitude of relative motion $y(t)$, see [2],

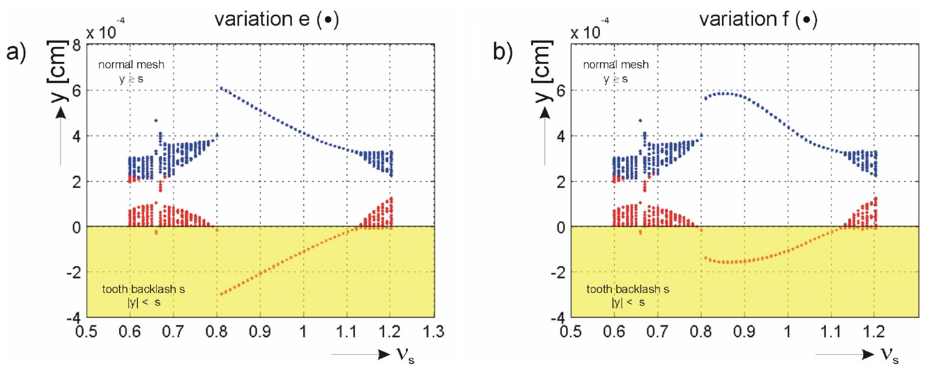

Fig.3: Bifurcation characteristics of relative motion $y(t)$ in frequency area $\nu_s \in \langle 0.6; 1.2\rangle$ for the combined damping (e) (•) and (f) (•), see Tab. 1, in the fifth revolution of gear pair from Fig.1

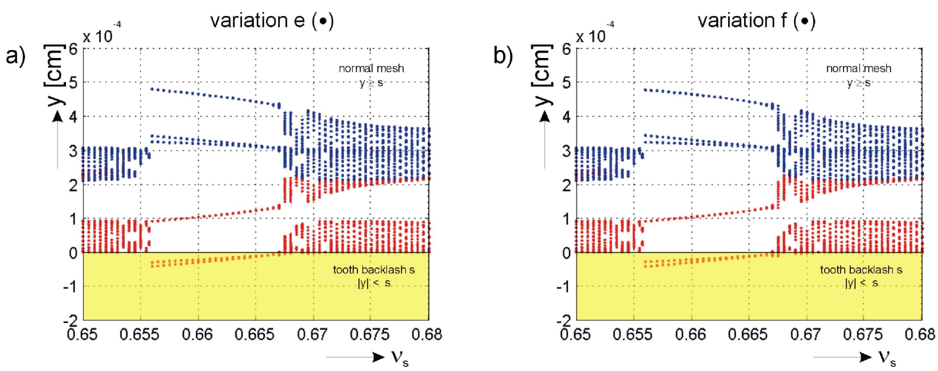

Fig.4: Bifurcation characteristics of relative motion $y(t)$ in frequency area $\nu_s \in \langle 0.65; 0.68\rangle$ for the combined damping (e) (•) and (f) (•), see Tab. 1

– the phase shift of relative motion $y(t)$ towards the resultant stiffness function $C(t)$ of gear mesh alternatively $C(t)(H1 + H2)$ by influence of damping action of system in non-conservative systems,
– the phase shift can happen in the non-conservative and conservative systems then in the area of teeth contact bounces and in the inverse motion towards modify stiffness function [2].

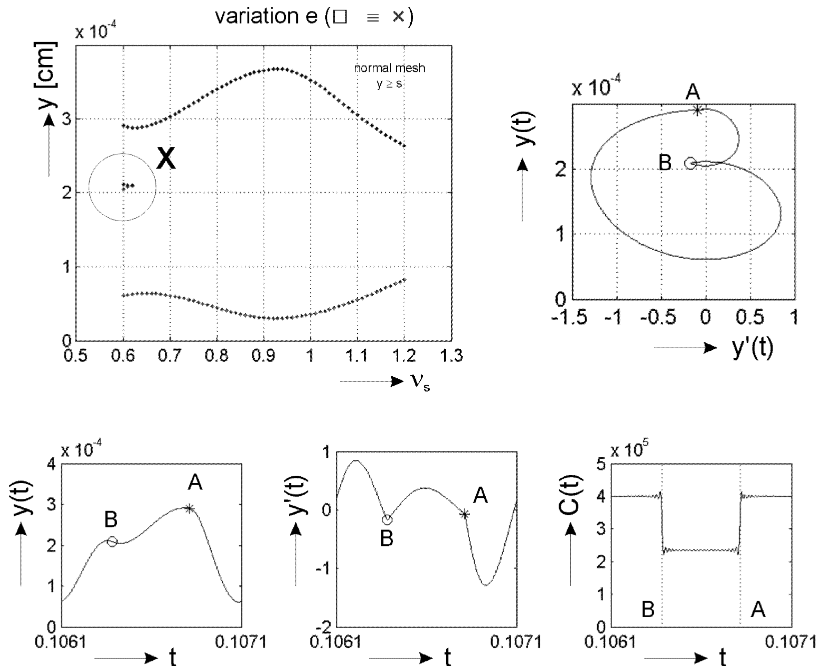

Fig.5: Bifurcation characteristics of variation (e) with the combination of damping $(\square) \equiv (\times)$ *in frequency area* $\nu_{\mathrm{s}} \in \langle 0.6; 1.2\rangle$ *and by the normal gear mesh*

In the Fig. 3 are given the resonance bifurcation characteristics of variants of combined damping (e) (•) and (f) (•), see Tab. 1, for the variation (e) and the combination (•) with $k_1, k_2, k_3, k_{3\mathrm{m}} = 0$ and with $k_{1\mathrm{m}}, k_{2\mathrm{m}} \neq 0$, for the variation (f) with the combination (•) with $k_1, k_2, k_3, k_{2\mathrm{m}} = 0$ and $k_{1\mathrm{m}}, k_{3\mathrm{m}} \neq 0$. The damping in both variations is considered linear + quadratic alternatively linear + cubic only in the tooth space, in the normal mesh is the solved system conservative with appropriate unstable course. The discreteness of courses in the tuning vicinity $\nu_{\mathrm{s}} \approx 0.66$ vibrates with the tuning on stiffness level $C_{\min}$ with frequency tuning $\nu_{\min} \approx 0.78$; the discreteness in the vicinity $\nu_{\mathrm{s}} \approx 0.8$ vibrates on stiffness level $C_{\min}$ with the frequency tuning $\nu_{\min} \approx 0.946$ therefore with quantitative greater and with qualitative different amplitude $y(t)$. The bifurcation on the lower branch of resonance characteristics of relative motion $y(t)$ are coded with red colour, with the blue colour are coded the bifurcation on upper branch of resonance characteristics.

Detailed bifurcation course of the discreteness in the frequency area $\nu_{\mathrm{s}} \in \langle 0.65; 0.68\rangle$ in the variations (e) and (f) is evident from Fig. 4.

Both the qualitative and the quantitative difference between resonance characteristics of both variations in the interval $\nu_{\mathrm{s}} \in \langle 0.65; 0.68\rangle$ are invisible. The noticeable difference is evident by the second discreteness, i.e. in the tuning vicinity $\nu_{\mathrm{s}} \approx 0.8$, by the change of quadratic damping onto cubic one in tooth space.

The different bifurcation course is in the case of the steady – stable solution of resonance characteristics of variation (e) with the combination of damping ($\square$) $\equiv$ ($\times$), as is evident from the Fig. 5. In addition to the smooth course of resonance characteristics $y_{\max}(t)$, $y_{\min}(t)$ occur in the tuning vicinity $\nu_{\mathrm{s}} \approx 0.6$ the small bifurcation deviation, see DETAIL X. This deviation is induced by looped course of graph in the phase plane $\{y'(t); y(t)\}$, illustrated in Fig. 5. This loop is formed by the character of course $y(t)$, $y'(t)$ and with the growing of ν_{s} decreases and converges into the point of discreteness B.

3. Closing remarks

On the ground of existing studies can yet note in fine, that the dynamic qualitative and quantitative analysis of the time in stiffness heteronymous non-linear systems and their resonance characteristics are markedly influenced above all by

- the value of the coefficient of mesh duration ε which determines at which stiffness level (potential level) $C_{\min}$ or $C_{\max}$ and in what time interval run the solution of relative motion in gear mesh and the value of the amplitude modulation $\kappa = C_{\min} C_{\max}^{-1}$, alternatively with the potential level of the stiffness $C = 0$ by modify resulting stiffness function $C(t)(H1 + H2)$,
- the resonance tuning of stiffness level at which the solution of relative motion just run, i.e. $\nu_{\min}$, $\nu_{\max}$ or $\nu = 0$,
- the phase shift of relative motion $y(t)$ towards stiffness function $C(t)$ alternatively $C(t)(H1 + H2)$ caused by linear and non-linear damping effects both the material in gear mesh and temperature dependent viscosity of environment in the area of tooth backlash.

The damping coeficients both material and viscous medium were selected, as an example, equal, because any experimental damping values in the problems of gear mesh of kinematic pairs still not exist.

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References

[1] Hortel M., Kratochvíl C., Škuderová A., Houfek L., Byrtus M.: Parazitní jevy, bifurkace a chaos v převodových soustavách s ozubenými koly, ISBN 978-80-214-3758-2, Brno 2008 (in Czech)

[2] Hortel M., Škuderová A.: K vlivu tlumení na rázové jevy v záběru ozubení, In Proceedings 22nd conference with international participation 'Computational mechanics 2006', Volume 1, ISBN 80-7043-477-5, s. 179–186 (in Czech)

[3] Hortel M., Škuderová A.: K některým dynamickým vlastnostem parametrických – heteronomních soustav s vnitřními kinematickými vazbami a větveným tokem výkonu, In: Sborník Colloquium Dynamika strojů 2007, 6.–7. 2. 2007, Praha, ISBN 978-80-87012-03-1, s. 85–94 (in Czech)

[4] Hortel M., Škuderová A.: K numerické analýze singularit v rezonančních charakteristikách nelineárních parametrických soustav s rázy v záběru ozubení. In: Sborník Inženýrská mechanika 2007, 14.–17. 5. 2007, Svratka, ISBN 978-80-87012-06-2, CD-ROM

[5] Hortel M., Škuderová A.: To the influence of nonlinear damping on the bifurcation phenomena in gear mesh of one branch of power flow of the pseudoplanetary gear system, In: Sborník Inženýrská mechanika 2008, 12.–15. 5. 2008, Svratka, ISBN 978-80-87012-11-6, CD-ROM

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