COMPARISON OF TWO PROCEDURES FOR DETERMINATION OF STABILITY BOUNDARY

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Analysis of two methods used for ascertaining stability of dynamical systems was carried out on a mathematical model of experimental stand of aerodynamic bearings. It was proofed that method of direct eigenvalue calculation gives practically the same results, which are obtained from the solution of equations of motion by means of integral Runge Gutta procedure.

Keywords: *identification of dynamic model, stability, linear system, comparison of numerical methods, eigenvalue problem, Runge Kutta solution*

1. Introduction

Gas lubricated bearings are modern elements in machinery and are often investigated by numerical methods [1–4]. Stiffness and damping properties for different types of aerostatic journal bearings were identified in the project of Czech Grant Agency during the recent years. Investigated types of bearings and experimental equipment are briefly described in [6–9]. The results of identification were used for the verification of computational methods for this type of bearings and for analysis of the influence of non-diagonal elements of stiffness and damping matrices on spectral and modal properties. Stability of motion was carried out as a function of non-diagonal elements of damping matrix using mathematical model corresponding to bearing No 2 (diameter $D = 30$ mm, ratio $l/D = 1$, radial clearance 0.025 mm, inlet pressure 0.2 MPa) using both investigated methods (the eigenvalue problem and Runge-Kutta procedure). Limits of stability were calculated numerically and verified graphically.

2. Numerical analysis

The first attempt of mathematical description of stiffness and damping properties of aerostatic bearings of experimental stand Rotor Kit Bently Nevada was given in paper [4] presented on conference 'Computational Mechanics' in November 2008. Basic assumption for there elaborated theoretical model was the exact axially symmetrical arrangement of test stand and axially symmetrical flow inlet of compressed air. Because of this assumption, linear parts of reaction forces of aerostatic bearing without rotation were symmetrical and diagonal; with shaft rotation they were anti-symmetrical with non-diagonal elements proportional to the square of angular velocity of shaft rotation.

Results of measurements and identifications of dynamic properties of aerostatic bearing on special adapted Rotor Kit Bently Nevada give full matrices of stiffness **K** and dam-

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ping $\mathbf{B}$, together with diagonal mass matrix $\mathbf{M}$. Values of one of measured cases for bearing No 2 – [5] (air pressure 0.2 MPa, rotor revolution $\omega_\mathrm{r} = 0$) are

$$\mathbf{K} = \begin{bmatrix} 2162649 & 65224 \\ 4935 & 1213610 \end{bmatrix} \text{[N/m]} , \qquad \mathbf{B} = \begin{bmatrix} 822 & -71 \\ -99 & 1173 \end{bmatrix} \text{[Ns/m]} , \tag{1}$$

and

$$\mathbf{M} = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.2 \end{bmatrix} \text{[kg]} .$$

Two methods for determination of stability boundary are most frequently used in technical practice:

a) Ascertaining zero value of real part of one of roots in eigenvalue problem.

b) Check in long time behaviour of motion solution calculated numerically by means of Runge-Kutta procedure.

The former treatment can be used only for a linear system or for a nonlinear system when nonlinear functions are linearized. The latter treatment is more general, with no restrictions on nonlinearities and/or on small displacements, but it is less accurate then the eigenvalue method.

An example of ascertaining of accuracy of the method b) based on numerical solution of time history is presented in this article using the real values (1) identified on the system of aerostatic bearing.

In the differential equation

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{B}\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} = \mathbf{F}(t) \tag{2}$$

motion is expressed by components x, y (vertical and horizontal motion of mass) :

$$\mathbf{q} = [x, y]^\mathrm{T} . \tag{2a}$$

For the ideal conditions – precise forms of bearing shell and shaft, symmetrical contribution of inner air pressure, etc. – the matrices $\mathbf{K}$ and $\mathbf{B}$ should have zero non-diagonal elements. However, in the real conditions these elements are non-zero. The aim of this paper is to analyze influence of these non-diagonal elements on the eigenvalues of studied system and its response on excitation by a small external harmonic force $\mathbf{F}(t)$ which, according to experiments, has amplitude 10 N. In numerical solution we assume forms

$$\begin{aligned} &\text{a)} \quad \mathbf{F}_1(t) = \cos\omega\, t\,[10, 0]^\mathrm{T} , \\ &\text{b)} \quad \mathbf{F}_2(t) = \cos\omega\, t\,[0, 10]^\mathrm{T} , \\ &\text{c)} \quad \mathbf{F}_3(t) = 10\,[\cos\omega\, t,\, \sin\omega\, t]^\mathrm{T} . \end{aligned} \tag{3}$$

Let us first study the influence of non-diagonal elements B(1,2) and B(2,1) in damping matrix. System response on exponential increasing excitation rotating force

$$[1 - \exp(-2\,t)]\,\mathbf{F}_3(t) \quad \text{[N]} \tag{4}$$

at identified values

$$B(1{,}2) = -71\,\mathrm{kg\,s^{-1}} , \qquad B(2{,}1) = -99\,\mathrm{kg\,s^{-1}}$$

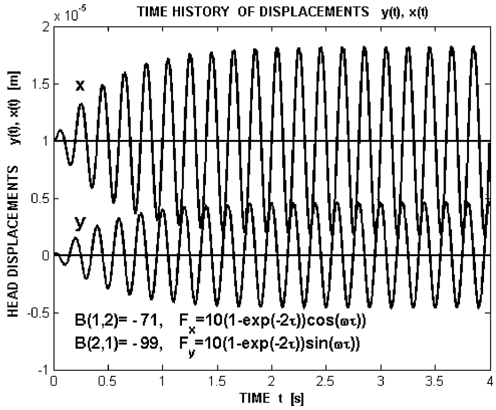

Fig.1: Time history of bearing head motion

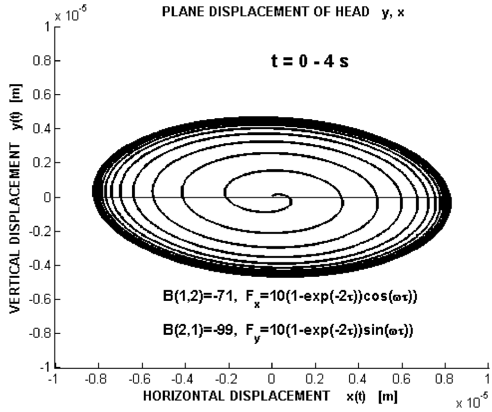

Fig.2: Plane trajectory of bearing head motion

and at angular frequency of excitation $\omega = 31.4\,\mathrm{s}^{-1}$ are in Fig. 1 and corresponding plane trajectory in Fig. 2. Smoothly exponentially increasing rotating force (4) was selected for suppressing initial shock in $t = 0$.

Motion is well stabilized as seen from the quick decay of disturbance motion $x(t)$, $y(t)$ of free system ($F(t) = 0$) recorded in short time interval in Fig. 3 after setting the initial conditions $x(0) = y(0) = 0.01\,\mathrm{m}$, $\dot{x}(0) = \dot{y}(0) = 0$.

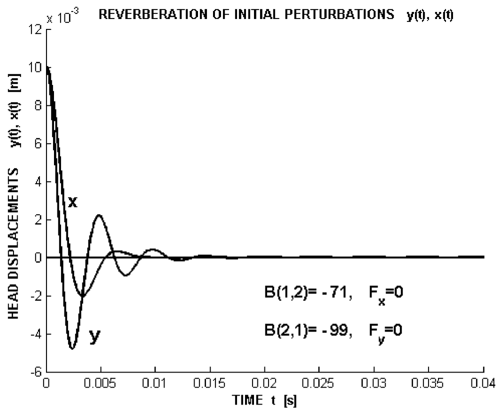

Fig.3: Free system response on initial perturbation

Corresponding eigenvalue problem is defined by the equation

$$(\mathbf{K} + s\,\mathbf{B} + s^2\,\mathbf{M})\,\mathbf{x} = \mathbf{0}$$

and after the transformation to double-dimension state space $\mathbf{q} = [\dot{x}, \dot{y}, x, y]^{\mathrm{T}}$ eigen-problem of the matrix

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}\,\mathbf{B} & -\mathbf{M}^{-1}\,\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} ,$$

solved by MATLAB procedure $eig(\mathbf{A})$ gives spectral matrix

$$\mathbf{S} = \begin{bmatrix} -337.8 + 1297.6\,\mathrm{i} & 0 & 0 & 0 \\ 0 & -337.8 - 1297.6\,\mathrm{i} & 0 & 0 \\ 0 & 0 & -493.5 + 887.6\,\mathrm{i} & 0 \\ 0 & 0 & 0 & -493.5 - 887.6\,\mathrm{i} \end{bmatrix} ,$$

where the smallest real part of roots is $\mathrm{Re}(\lambda) = -337.8$ and corresponding imaginary part is $\mathrm{Im}(\lambda) = 1297.6$. This was calculated for the non-diagonal damping elements $B(1{,}2) = -71$, $B(2{,}1) = -99$ identified from dynamic experiments.

Let us investigate the influence of these coefficients on the value of real part $\mathrm{Re}(\lambda)$ determining the stability properties. Pairs of the same values $B(1{,}2) = B(2{,}1)$ is supposed for simplicity in the following analysis. Variation of $B(1{,}2) = B(2{,}1)$ in the range $\langle 0, -1200\rangle$ causes rise of $\mathrm{Re}(\lambda)$ as seen in Fig. 4.

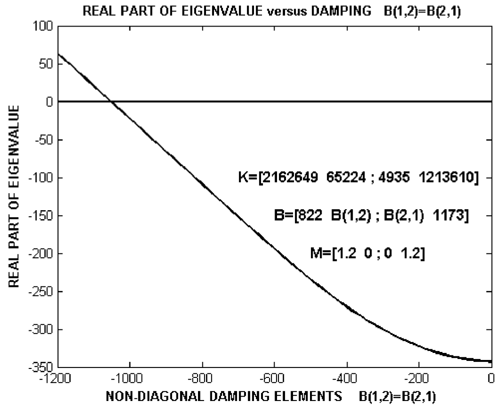

Fig.4: Variation of real part eigenvalue with decrease of damping elements

If the non-diagonal damping elements $B(1{,}2)$, $B(2{,}1)$ decrease to the value $B(1{,}2) = B(2{,}1) = -1050.807$, the real part of one of eigenvalues goes to zero (exactly 1.6698 e-13) and the roots turn into purely imaginary $\lambda = \pm 1227.657\,\mathrm{i}$. Smaller non-diagonal damping elements $B(1{,}2) \le -1050.807$ give instability, greater ones correspond to stable system.

The influence of variation of non-diagonal damping elements was investigated also by means of response on the external force (3). We shall try, whether the stability boundary at $B(1{,}2) = -1050.807$ will be valid also for numerical solution of motion by means of Runge Kutta method of fourth order [11] at sufficient small step $\Delta t = 0.0003\,\mathrm{s}$. There is a danger that due to the great differences among different coefficients of stiffness and damping, the solution of motion solved by Runge Kutta procedure for these values $B(1{,}2)$ could cause an unstable motion.

Results of numerically solved motion of equation (2) near the stability boundary, but on the side of instability, i.e. for $B(1{,}2) = B(2{,}1) = -1053$, is shown in Fig. 5* and 6 in time interval $t \in (0, 6)\,\mathrm{s}$ corresponding to approx. 1200 periods of natural vibrations and 20 000 integration steps with $\Delta t = 0.0003\,\mathrm{s}$. The external force was very low, $F_0 = 0.01\,\mathrm{N}$ in order to show the rise of unstable components more clearly. Graphically recorded motion up to 2 s seems to be stable containing only forced component with low frequency $\omega = 30\,\mathrm{s}^{-1}$, but then due to the overstepping of stability boundary ($B(1{,}2) = B(2{,}1) = -1050.807$) the oscillations with high frequency exponentially increase. Plane trajectory x, y of this motion is in Fig. 6.

* The graph $x(t)$ is displaced upward by value $2{\times}10^{-7}$ for the clarity. Similar shifts of x-records are used also in Fig. 7, 9, 11.

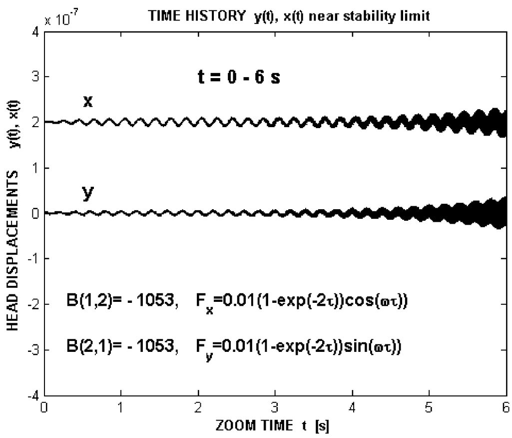

Fig.5: Unstable motion at $B(1{,}2) = B(2{,}1) = -1053$

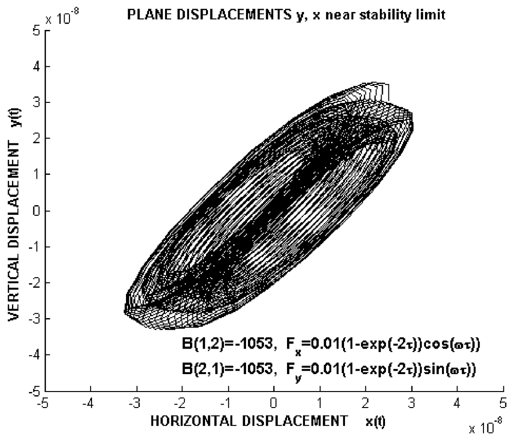

Fig.6: Plane trajectory of unstable motion at $B(1{,}2) = B(2{,}1) = -1052$

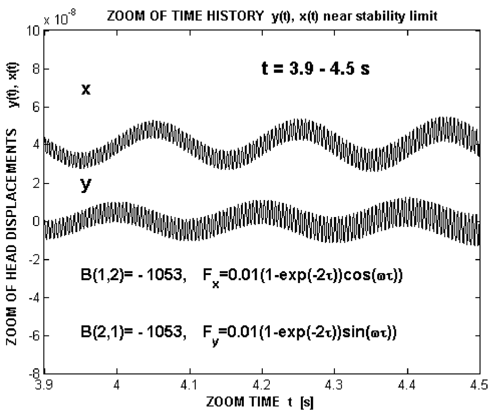

Fig.7: ZOOM of unstable motion versus time

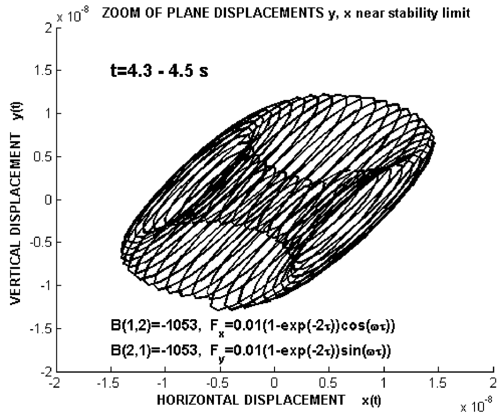

Fig.8: Plane trajectory of unstable motion for $t = 4.3–4.5\,s$

ZOOM pictures of this oscillations in a smaller time range $t = 3.9–4.5$ s is shown in Fig. 7. The small increase of high frequency oscillations corresponding to the imaginary part $\pm 1227\,\mathrm{i}$ of eigenvalue with nearly zero real part is clearly seen. The low frequency oscillations is response on external force $F(t)$ with small amplitude 0.01 N.

Plane trajectory of head bearing motion during approximately one period of excitation is in Fig. 8.

Further approaching to the stability boundary limit given by $B(i,j) = -1050.807$ for non-diagonal damping elements $B(1{,}2) = B(2{,}1) = -1052$ is shown in Fig. 9 and 10, where the time history and plane trajectory in time interval $t \in (5, 6)$ s, i.e. in the interval of integration steps $(16\,667, 20\,000)$ is plotted. The unstable high frequency component is here much lower than in previous figures 7 and 8 in spite of recorded time t is higher.

Let us solve motion of system with values $B(1{,}2) = B(2{,}1) = -1050.807$ identical with the stability boundary ascertained by eigenvalue method. Time histories $x(t)$ and $y(t)$, recorded in Fig. 11 seem to be stable, without any high frequency distortion. However the plane trajectory recorded for time interval $t \in \langle 5.5, 6\rangle$ s in Fig. 12 at approx. 6-times greater

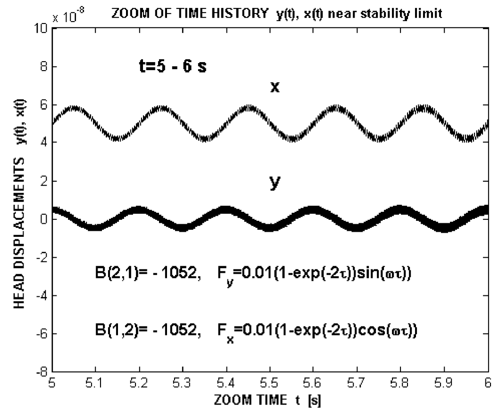

Fig.9: Parts of unstable motion at $B(1{,}2) = B(2{,}1) = -1052$

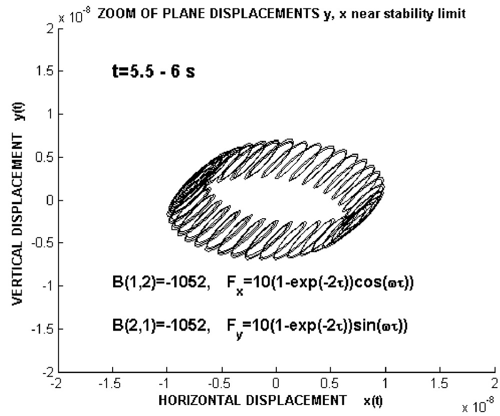

Fig.10: Plane trajectory at $B(1{,}2) = B(2{,}1) = -1052$

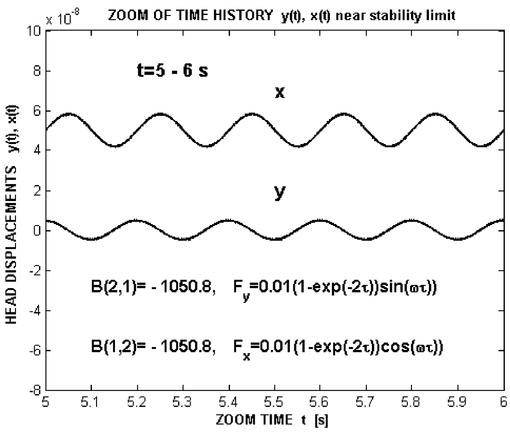

Fig.11: Apparent stable motion at $B(1{,}2) = B(2{,}1) = -1050.8$

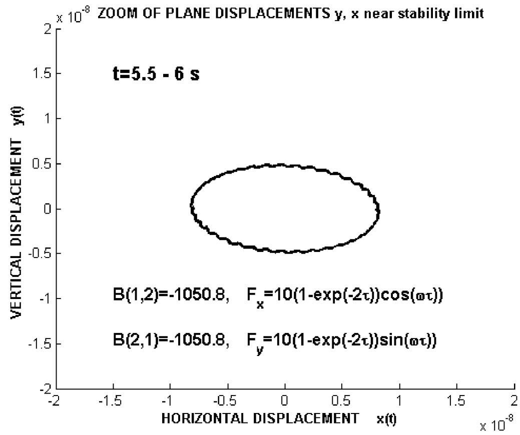

Fig.12: Plane trajectory with small higher frequency component

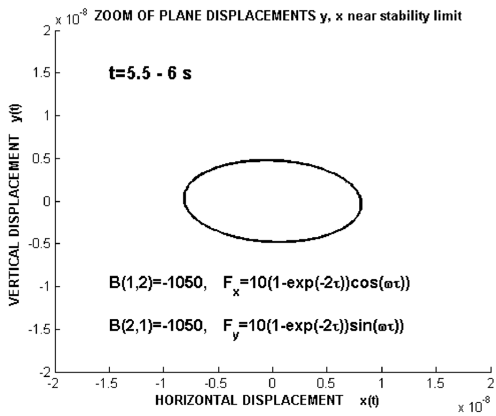

Fig.13: Apparent stable plane trajectory at $B(1{,}2) = B(2{,}1) = -1050$

scale contains small frequency vibrations that will exponentially grow with increasing time. The calculated motion gained by numerical solution of differential equations is therefore again unstable.

Alternation of non-diagonal damping coefficients to $B(1{,}2) = B(2{,}1) = -1050$ seems to be sufficient for stabilization of investigated linear system, as the plane trajectory in Fig. 13 is in comparison with Fig. 12 smooth, but the further enlargement of a part of trajectory in Fig. 14 uncovered a small content of high frequency component.

Until of non-diagonal damping coefficients are set on values $B(1{,}2) = B(2{,}1) = -1049$, the numerical solution of differential equations is stable as can be deduced from the smooth trajectory in Fig. 15.

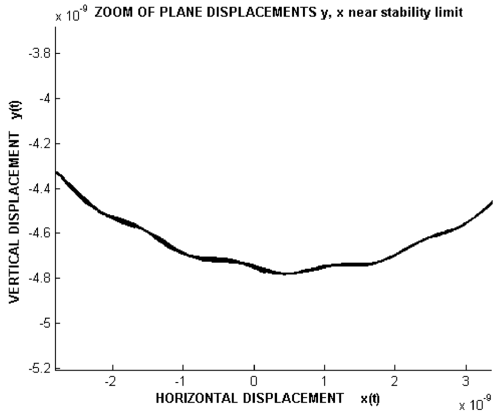

Fig.14: High frequency component at $B(1{,}2) = B(2{,}1) = -1050$ *and at great enlargement*

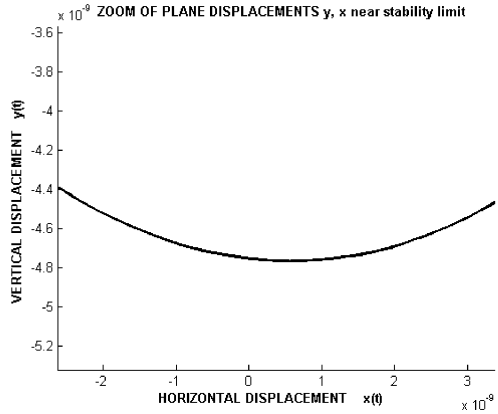

Fig.15: Plane trajectory without high frequency component at $B(1{,}2) = B(2{,}1) = -1049$

It was shown on this simple case, that numerical solution of differential equations of motion does not give exact boundary of stability.

The difference of non-diagonal damping elements of mathematical linear model of aerostatic bearing determined by two methods is approximately only 2 ‰. However in some other cases it can reach much higher values. Therefore the verification of applied numerical method by comparison with exact analytical solution is recommended.

3. Conclusion

Analysis of non-diagonal elements influence of damping matrix of linearized dynamical system on the response of experimental stand with aerostatic bearing was utilized to compare two methods commonly used for ascertaining the stability boundary.

It was shown that the limit values of non-diagonal elements $B(1{,}2) = B(2{,}1)$ of damping matrix calculated by a direct eigenvalue problem in the state space and by a Runge-Kutta solution of time history of corresponding dynamical system are very close, nevertheless the numerical solution of differential equations gives less stable properties than they really are.

Results of this analysis carried out on the linear system are important also for general types of non-linear systems, where the 'eig' Matlab procedure cannot be used and only the integral solutions are possible.

The exactness of numerical determination of stability boundary of nonlinear system, where the bifurcations, stability and chaotic phenomena are investigated, is recommended always to verify on a similar linear system (of the same range of time, number of steps, length of step dt, number of DOF).

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