SYSTEMIC APPROACH TO MODELLING OF CONSTITUTIVE BEHAVIOUR OF VARIOUS TYPES OF MATTER
Part I – Basic and Simple Constitutive Models

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The paper presents a systemic overview of constitutive models, i.e. mathematical or graphical representations of responses of a matter initiated by its activation coming from its surroundings (especially stress- or strain-controlled loadings in mechanics). Various states of matter showing different behaviour are related with different distances among particles of the matter and their mutual movements. However, in opposite to the previous centuries, when different approaches and methods were developed and used for description of various types of matters (in solid mechanics, hydromechanics, thermodynamics etc.), recently more and more often solid mechanics meets materials showing some features of fluids (e.g. creep, flow), and interactions of matters in different states (e.g. solid-liquid) need to be solved as well. The presented paper, together with another consequent one (Part II), creates a set of two related articles aiming at facilitating you the orientation in various types of constitutive equations. It presents graphical representations of basic mechanical responses (stress as a function of strain magnitude and strain rate, creep, stress relaxation), as well as their simplified mathematical substantiation. Some more complex types of constitutive models will be presented in part II. On the base of these papers, the chapter on constitutive models was published in [1].

Keywords: *constitutive models, mechanical behaviour, state of matter, perfect solid, fluid, gas*

1. Introduction

The term 'mechanical behaviour of matter' can be defined as a set of responses of a matter initiated by its activation coming from its surroundings (especially stress or strain controlled loadings). Various states of matters show various types of behaviour; this is related to different distances among particles of the matter and their mutual movements. Therefore different methods have been used for their description in the history of mechanics what lead to birth of specialized branches such as solid mechanics, hydromechanics, aeromechanics or thermodynamics. Recently, however, more and more often solid mechanics meets materials (this term is commonly used for matters in the solid state) showing some features of fluids (e.g. creep, flow), as well as interactions of matters in different states (e.g. solid-liquid). The presented set of two related articles aims at facilitating you the orientation in various types of constitutive equations, i.e. mathematical models of constitutive behaviour of real matters.

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2. Systemization of constitutive dependencies and models

In a broader sense, constitutive models are mathematical descriptions of mutual dependencies between loads (forces etc.) and deformations (consisting of changes of shape and volume) or their rates, including time dependencies such as creep and relaxation responses. Constitutive dependencies can be defined as follows:

Constitutive dependencies *are causal dependencies between stress and strain tensors or among some alternative quantities derived from them by mathematical manipulations, including time dependencies.*

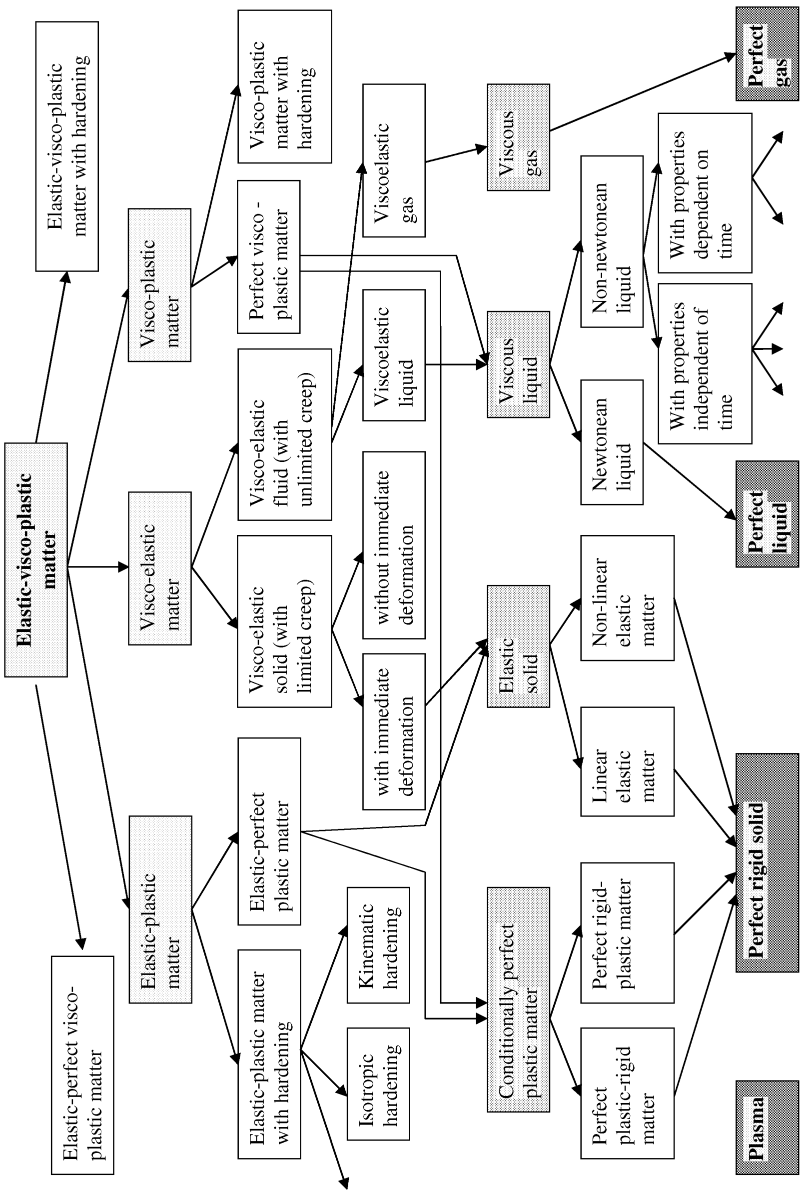


Fig.1: Systematic overview of isotropic constitutive relations of matters with presentation of mutual relations

Thus the simplest constitutive models are represented by the relations defining perfect matters in various states (perfect liquid, perfect gas, perfect rigid solid). On the other hand, some rather complex constitutive models have been formulated with various components of their behaviour (elastic, plastic, viscous), which can be used for derivation of the simpler ones as their special cases.

Therefore it is useful to divide the constitutive models into several hierarchical levels.

- **Basic constitutive models** – The mathematical relations and the geometrical dependencies will be presented here, relating substantial quantities contained in constitutive relations for **perfect rigid solid**, **perfect liquid** and **perfect gas**.
- **Simple constitutive models** – This term is introduced for such constitutive models that describe behaviour of matters differing from the 'perfect' ones by an only one certain property, e.g. perfect elastic matter, conditionally perfect plastic matter, viscous liquid.
- **Combined constitutive models** – These models can be created by a combination of two or more simple constitutive models. They exploite often the so-called rheological models to describe e.g. behaviour of the following types of matters: viscoelastic, elastic-plastic, viscoplastic and elastic-viscoplastic.

A schematic overview of constitutive models is presented in Fig. 1; this scheme presents isotropic models only, i.e. without taking any direction dependency of properties into account. This paper (part I) deals with the basic and simple constitutive models in detail, while the combined constitutive models will be analyzed in the consequential paper (part II).

2.1. Definition of basic constitutive models

These models characterize basic properties typical for substances in basic states of matter. Note that these perfect matters do not exist in fact, they represent certain models of the reality that are defined as follows:

- **Perfect rigid solid** – its feature is that it is non-deformable (in volume as well as in shape), i.e. its resistance against changes of shape and volume is infinite.
- **Perfect fluid** – it shows **zero resistance against changes of shape**. This term comprehends matters in liquid state as well as gases. A liquid differs from a gas by its volumetric part of deformation as follows:
  - **Perfect liquid** – its volumetric component of deformation equals zero under any loading conditions – it is incompressible (any processes are isovolumic only).
  - **Perfect gas** – its volumetric change is governed by the **state equation of gases**.

To differ among solids, liquids and gases, it is obviously necessary to **divide the deformation of the matter into its volumetric and shape (deviatoric) component**. This must be strictly kept at almost all constitutive models. Unfortunately, the only exception is probably the well known Hooke's law, by virtue of the principle of superposition at linear dependencies. When using combined rheological models, however, it is **strictly necessary to model deviatoric and volumetric components of stress and strain tensors separately***.

* As the volumetric change is relatively small under load in many technical matters (liquids as well as solids), **combined rheological models are often used for description of dependencies of deviatoric components of stress and strain tensors** only and the relation between the spherical components of stress and strain is described by linear elastic relations only or even the volumetric deformation is neglected.

2.2. Types of dependencies representing behaviour of matters

Behaviour of matters is comprehensively described (from the viewpoint of mechanics) by these dependencies :

1. **Constitutive dependencies** – In a narrower sense, we denote as constitutive the dependencies between stress and strain tensors. If they are expressed by mathematical relations, then simplified shapes of these formulas are often used for practical reasons, namely shapes valid for specific cases of stress states (biaxial, shear, or uniaxial stress states).

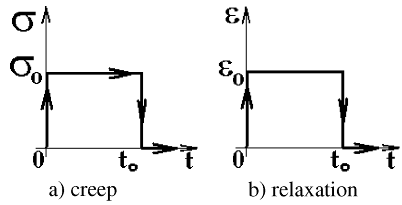


Fig.2: Time dependencies of input quantities for description of creep and relaxation

2. **Creep response** – This is the time dependency of the deformation commonly called creep; it is investigated usually under static load inducing uniaxial stress state given by the stress : $\sigma = \sigma_0\, H(t)$, or $\sigma = \sigma_0\, H(t) - \sigma_0\, H(t - t_0)$, where $H(t)$ is Heaviside function, given by relations : $H = 0$ for $t < 0$, $H = 1$ for $t > 0$ (see Fig. 2a).
3. **Relaxation reponse** – This is the time dependency of stress. Relaxation response is investigated usually under deformation state given by the following strain values : $\varepsilon = \varepsilon_0\, H(t)$, resp. $\varepsilon = \varepsilon_0\, H(t) - \varepsilon_0\, H(t - t_0)$, where $H(t)$ is the same Heaviside function (see fig. 2b).
4. **Speed response** – In matters showing time dependency of stress-strain response to loading, also the dependency of stress σ on the strain speed $\dot{\varepsilon}$ can be investigated (i.e. $\sigma - \dot{\varepsilon}$).

The presented models are based on [2] and completed with some additional formulations and with a comprehensive overview of constitutive responses of the models in question.

3. Basic constitutive models in mechanics

Using the above considerations, constitutive models of perfect rigid solid, perfect liquid and perfect gas are presented in this chapter. The systemized knowledge will be used in formulations of simple and combined constitutive models.

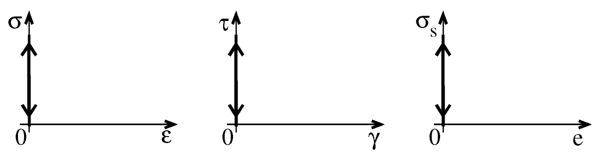


Fig.3: Dependencies of stress and strain components at a perfect rigid solid under uniaxial, shear and hydrostatic stress states conditions

3.1. Perfect rigid solid

As mentioned above, this matter is non-deformable in volume and shape; in other words, its deformation equals zero under any values of stress components.

If perfect solid matter is considered a special case of linear elastic matter, then both bulk K and shear G modules equal infinity. Here is the reason why the constitutive dependencies 'shear stress τ vs. shear strain γ' and 'middle stress σ_s vs. relative volumetric change e' are formally the same. Therefore the dependency of the normal stress σ on longitudinal strain ε under uniaxial tension test can also be described by the same linear formula, even if the stress state consists of the spherical $\sigma_s = \sigma_x/3$ as well as deviatoric $\mathbf{T}_{\sigma D}$ components of stress tensor (see Fig. 3).

Applications: analyses of movement at bodies showing negligible deformations, i.e. in kinematics and dynamics of solid (rigid) bodies.

3.2. Perfect liquid

In hydromechanics perfect liquid is defined as a matter incompressible in volume (i.e. with bulk modulus $K \to \infty$, see Fig. 4a) and with zero viscosity (i.e. with zero resistance against shape – isovolumic changes, see Fig. 4b).

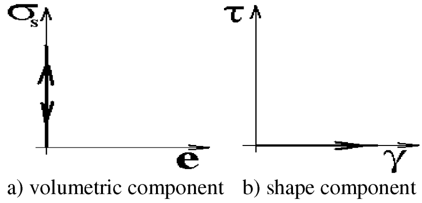


Fig.4: Dependencies of stresses on strains at perfect liquid

Applications: hydrostatics, mechanics of water and other liquids with low viscosity under low flow rate.

3.3. Perfect gas

In hydromechanics, aeromechanics and thermodynamics, a perfect gas is defined as a matter with zero resistance against shape changes that is compressible and the volumetric change of which is governed by the state equation of perfect gas in the form:

$$p\,v = R\,T\ ,$$

where p is gas pressure, v specific volume (per grammolecule), T absolute temperature, R universal gas constant.

The zero resistance against shape changes is a common property of perfect gas and perfect liquid. Characteristic feature of gases, as a direct consequence of the above state equation, is their extensibility, i.e. their ability to fill up all the disposable space.

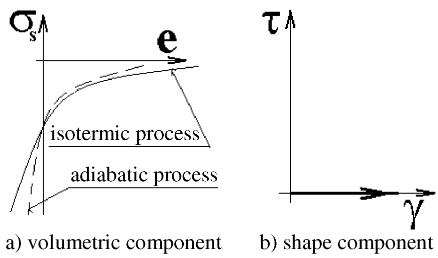


Fig.5: Dependencies of stresses on strains in perfect gas

From the viewpoint of dependency between strains and stresses, it can be stated:

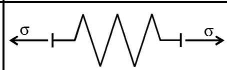
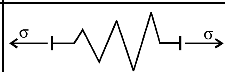
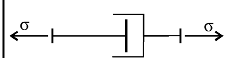
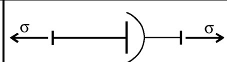
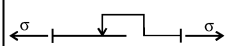
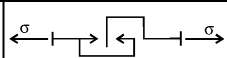
- the character of the dependency 'mean stress σ_s vs. relative volumetric change e' depends on the type of the thermodynamic conditions under which the perfect gas is compressed, Fig. 5a,
- the dependency 'shear stress τ vs. shear strain γ' is, in consequence of the zero resistance against the shape change, the same like at the above perfect liquid, i.e. shear stress equals zero under any magnitude, rate and history of shear strain, Fig. 5b.

Applications: mechanics of air and other gases under small flow velocities.

4. Simple constitutive models in mechanics

Simple constitutive models describe such matters that differ from the perfect matters described in the previous chapter in one substantial property. In this feature they differ from the **combined constitutive models** that 'model' more of these substantial properties. The combined models will be dealt with in the consequential paper.

The simple constitutive models represent the lowest level in the hierarchy of constitutive models of real (non-perfect) matters. They can be exploited as basic elements of constitutive models of higher levels. Although the solid mechanics aims mostly at matters in solid state, also some simple constitutive models of liquids will be analyzed here, because some real materials (in solid state) show some properties of liquids as well.

No.	Name	Modelled property	Scheme	Mathematical description
1	Linear spring (Hooke's element)	Linear elasticity		$\sigma = E\,\varepsilon$ $\tau = G\,\gamma$
2	Non-linear spring	Non-linear elasticity		$\sigma_{ij} = \partial W / \partial \varepsilon_{ij}$
3	Liquid linear damper (Newton's element)	Linear viscosity		$\sigma = \eta\,\dot{\varepsilon}$
4	Liquid non-linear damper (Norton's element)	Non-linear viscosity		$\sigma = \lambda\,\dot{\varepsilon}^{(1/N)}$
5	Skidding block	Rigidity up to a certain threshold value of stress		$\dot{\varepsilon} = 0$ for $\sigma \in (-\sigma_B, +\sigma_B)$
6	Stopping block	Rigidity above a certain threshold value of strain		$\sigma = 0$ for $\varepsilon \in (-\varepsilon_B, +\varepsilon_B)$

Tab.1: Overview of basic elements of rheological models

Explanations to the individual elements in the table:

- No.1 Linear dependency of stress σ on strain ε – this linear dependency is denoted as Hooke's law.
- No.2 Non-linear dependency of stress σ on strain ε, described by various forms of strain energy density functions W in the theory of hyperelasticity.
- No.3 Linear dependency of stress σ on strain rate $\dot{\varepsilon}$.
- No.4 Non-linear dependency of stress σ on strain rate $\dot{\varepsilon}$ – the non-linearity is expressed by means of the exponent $(1/N)$ at the strain rate $\dot{\varepsilon}$.

– No.5 If the stress σ inheres in the interval $\sigma \in (-\sigma_\mathrm{B}, +\sigma_\mathrm{B})$, then the element movement is blocked. If the stress value exceeds the defined limits (e.g. overcomes the yield stress σ_K), then the **element movement is enabled**.

– No.6 If the strain ε inheres in the interval $\varepsilon \int(-\varepsilon_\mathrm{B}, +\varepsilon_\mathrm{B})$, then the element movement (deformation) is stress-free. If the strain value reaches the defined limit value $\varepsilon = \pm\varepsilon_\mathrm{B}$, then the element movement is blocked at this value. The **blocking strain** ε_B can be e.g. the strain ε_L, corresponding to the straightening of the initialy wavy rigid fibres in the composite material structure.

An overview of 'mechanical schemes', which represent basic elements of rheological models of substantial properties of matters in the solid and liquid states, is presented in the table 1. This overview includes also verbal and mathematical descriptions of the modelled property or behaviour of the matter in question. Combined models can be created on the base of the elements from this table in serial or parallel, as well as combined orderings, in dependence on the type of the property to be modelled. The following formulas are valid for individual connections:

series connection $\varepsilon = \sum_i \varepsilon_i \;, \quad \sigma = \sigma_i \;,$ parallel connection $\sigma = \sum_i \sigma_i \;, \quad \varepsilon = \varepsilon_i \;.$

The formulas mean that, in series connection, strains of individual elements are summarized (while stresses are the same), and in parallel connection stresses are summarized (while strains in individual elements are the same).

Description with simple constitutive models is applicable especially for the following matters: perfect elastic matter linear or non-linear, conditionally perfect plastic matter, linear and non-linear viscous liquid. For these matters, the dependencies of stress on strain magnitude and rate, as well as time dependencies for creep and relaxation responses, are presented in the following chapters in their mathematical and graphical interpretations.

4.1. Perfect elastic matter

① **General characteristic:** This matter features by its stress-strain dependence along the same curve in loading and unloading, under both stress as well as strain controlled loadings. This curve is not linear in general, it must be monotonous only (increasing under loading and decreasing when unloaded). The mutual relation between stresses and strains is always unique what results in zero hysteresis under cyclic loading. Stress-strain response under changed load is instantaneous.

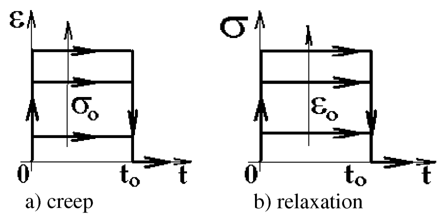


Fig.6: Responses of a perfect elastic matter

② **Creep:** From the thermodynamical viewpoint, loading and unloading of an elastic material is a sequence of equilibrium states so that $\varepsilon = \mathrm{const}$ for $\sigma = \mathrm{const}$. Therefore it can be concluded that there is no creep in a perfect elastic matter (see Fig. 6a).

③ **Relaxation:** Similarly it can be concluded that there is no relaxation in a perfect elastic matter. The time history of stress corresponds to the time history of strain, differing mutually only by the scales of the ordinates (see Fig. 6b). This holds under both stress and strain controlled loadings, for both linear and non-linear elastic matters.

Applications: metals in elastic strain range, rubbers and other elastomers, concrete.

4.1.1. Linear elastic (Hookean) matter

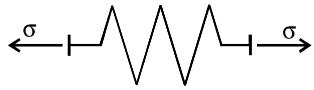


Fig.7: Hooke's element

① **General characteristic:** At the most general level, the linear elastic matter can be characterized as follows: it is a matter whose dependencies between components of stress tensor $\mathbf{T}_\sigma$ and strain tensor $\mathbf{T}_\varepsilon$ are linear (in both loading and unloading). In rheological models, this type of behaviour is represented by a cylindric spring and creates the so called Hooke's element, denoted as element No. 1 – **linear spring** in Table 1. Though this definition of a linear elastic matter is the most general, more notorious is the definition saying that, in a linear elastic matter, the dependence between stress and strain is given (in a limited range in fact) by the linear function $\sigma = E\,\varepsilon$. This formula is well-known as **Hooke's law**, it holds, however, for uniaxial stress states only and does not describe the fact that there exists a triaxial strain state under uniaxial tension (or compression). A complete formulation of Hooke's law for uniaxial tension is given by the following equations:

$$\sigma_\mathrm{x} = E\,\varepsilon_\mathrm{x}\ , \qquad \varepsilon_\mathrm{y} = -\mu\,\varepsilon_\mathrm{x}\ , \qquad \varepsilon_\mathrm{z} = -\mu\,\varepsilon_\mathrm{x}\ .$$

For a general, i.e. triaxial stress state, the so-called **generalized Hooke's law** is formulated; it is based on the superposition principle (valid for linear dependencies only) and one of its possible shapes is as follows:

$$\begin{aligned}
\varepsilon_\mathrm{x} &= (\sigma_\mathrm{x} - \mu\,\sigma_\mathrm{y} - \mu\,\sigma_\mathrm{z})/E\ , & \gamma_\mathrm{yz} &= \tau_\mathrm{yz}/G\ ,\\
\varepsilon_\mathrm{y} &= (\sigma_\mathrm{y} - \mu\,\sigma_\mathrm{x} - \mu\,\sigma_\mathrm{z})/E\ , & \gamma_\mathrm{xz} &= \tau_\mathrm{xz}/G\ ,\\
\varepsilon_\mathrm{z} &= (\sigma_\mathrm{z} - \mu\,\sigma_\mathrm{x} - \mu\,\sigma_\mathrm{y})/E\ , & \gamma_\mathrm{xy} &= \tau_\mathrm{xy}/G\ .
\end{aligned}$$

The **inverse shape of the Hooke's law**, with components of stress tensor formulated explicitely, can be obtained after some mathematical manipulations in the following shape:

$$\begin{aligned}
\sigma_\mathrm{x} &= 2\,G\,\varepsilon_\mathrm{x} + \lambda\,(\varepsilon_\mathrm{x} + \varepsilon_\mathrm{y} + \varepsilon_\mathrm{z})\ , & \tau_\mathrm{yz} &= G\,\gamma_\mathrm{yz}\ ,\\
\sigma_\mathrm{y} &= 2\,G\,\varepsilon_\mathrm{y} + \lambda\,(\varepsilon_\mathrm{x} + \varepsilon_\mathrm{y} + \varepsilon_\mathrm{z})\ , & \tau_\mathrm{xz} &= G\,\gamma_\mathrm{xz}\ ,\\
\sigma_\mathrm{z} &= 2\,G\,\varepsilon_\mathrm{z} + \lambda\,(\varepsilon_\mathrm{x} + \varepsilon_\mathrm{y} + \varepsilon_\mathrm{z})\ , & \tau_\mathrm{xy} &= G\,\gamma_\mathrm{xy}\ .
\end{aligned}$$

where $G = E/[2(1+\mu)]$ is shear modulus, $\lambda = E\,\mu/[(1+\mu)\,(1-2\,\mu)]$ is called Lamé's constant.

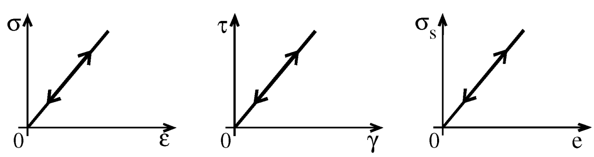


Fig.8: Basic characteristics of a linear elastic matter

It is useful to remember now that this model is the only one (with the exception of the perfect rigid matter) that does not need to distinguish between the volumetric and shape components of deformation, because the behaviour of both parts is the same (see Fig. 8). Due to this fact, the Young's modulus of elasticity E could be formulated and used in description of multiaxial stress states as well. For all the combined constitutive models, however, it is necessary to **separate the volumetric and shape components of deformation**, and, consequently, to rewrite the Hooke's law as dependence of stress tensor components on the deviatoric and spherical (volumetric) components of strain tensor, expressed e.g. in the following form:

$$\begin{aligned}
\sigma_x &= 2\,G\,T_{\varepsilon xxD} + K\,(\varepsilon_x+\varepsilon_y+\varepsilon_z) = 2\,G\,T_{\varepsilon xxD} + K\,e\ , && \tau_{yz} = 2\,G\,T_{\varepsilon yzD}\ ,\\
\sigma_y &= 2\,G\,T_{\varepsilon yyD} + K\,(\varepsilon_x+\varepsilon_y+\varepsilon_z) = 2\,G\,T_{\varepsilon yyD} + K\,e\ , && \tau_{xz} = 2\,G\,T_{\varepsilon xzD}\ ,\\
\sigma_z &= 2\,G\,T_{\varepsilon zzD} + K\,(\varepsilon_x+\varepsilon_y+\varepsilon_z) = 2\,G\,T_{\varepsilon zzD} + K\,e\ , && \tau_{xy} = 2\,G\,T_{\varepsilon xyD}\ .
\end{aligned}$$

where G, K are shear and bulk modules, respectively, $T_{\varepsilon ij\mathrm{D}}$ is a component of strain deviator and e is the relative volumetric change, i.e. the volumetric part of strain tensor. The volumetric part of the strain tensor is given by three principal strain components of the equal magnitude $\varepsilon_s = (\varepsilon_x+\varepsilon_y+\varepsilon_z)/3$, while the deviatoric part of strain tensor $T_{\varepsilon ij\mathrm{D}}$ is given by subtraction of the spherical part from the strain tensor.

If we denote the axes by numbers 1, 2, 3 instead of x, y, z to enable the use of general tensorial mathematical formulation (notation of the corresponding stress and strain components can be found in Table 2), we can obtain the Hooke's law in the following form:

$$\sigma_{ij} = 2\,G\,T_{\varepsilon ij\mathrm{D}} + \delta_{ij}\,K\,e\ ,$$

where Kronecker's symbol δ_{ij} (given by $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$) is used for simplification.

Technical notation	σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{xz}	ε_x	ε_y	ε_z	γ_{xy}	γ_{yz}	γ_{xz}
Tensorial 2-subscript notation	σ_{11}	σ_{22}	σ_{33}	σ_{12}	σ_{23}	σ_{13}	ε_{11}	ε_{22}	ε_{33}	ε_{12}	ε_{23}	ε_{13}
Tensorial 1-subscript notation	σ_1	σ_2	σ_3	σ_6	σ_4	σ_5	ε_1	ε_2	ε_3	ε_6	ε_4	ε_5

② **Creep:** Creep is treated as loading with a stepwise controlled stress increase up to the value of σ_0 in the time $t=0$ and a stepwise stress decrease back to the zero value in the time $t=t_0$, as shown in Fig. 2. The response has a form of the strain time dependence $\varepsilon(t)$ under the above loading. Here holds (in the range of the validity of Hooke's law) $\sigma = E\,\varepsilon$, and a general solution is based on the analysis of strain rate $\mathrm{d}\varepsilon/\mathrm{d}t$. As $E=$ constant, it holds:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma_0}{E}\right) = \frac{1}{E}\,\frac{\mathrm{d}\sigma}{\mathrm{d}t}\ .$$

If the stress change velocity is zero, also the velocity of deformation (strain rate) equals zero. This is a correct mathematical evidence that there is no creep in a linear elastic matter.

③ **Relaxation:** Relaxation is solved for the case of a deformation controlled loading, corresponding to the stepwise strain increase up to the value of σ_0 in the time $t=0$ and a stepwise strain decrease back to the zero value in the time $t=t_0$, as shown in Fig. 2; for the strain the same function is used here with the stress in the case of the creep description. The analysis is based then on the same mathematical formulas. As the strain ε is constant in the time interval in question as defined above, its time derivative equals zero, i.e. $\mathrm{d}\varepsilon/\mathrm{d}t = 0$. Then it holds:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma}{E}\right) = 0 \quad\rightarrow\quad \frac{1}{E}\,\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 0 \quad\rightarrow\quad \sigma = \text{const.}$$

So it is confirmed mathematically that there is **no relaxation** in a linear elastic matter. The response to a change of load is instantaneous, an immediate attainment of equilibrium state is supposed after any change. No creep or stress relaxation occurs, also there is no hysteresis under cyclic loading.

Applications: metals in elastic strain range, concrete.

Note – use of proper symbols

As it is necessary to distinguish between the spherical and deviatoric components of stress and strain tensors in most constitutive models, as mentioned above, this should be respected in the interpretation of the mathematical symbols used in the constitutive models as follows:

- Only in the case where the constitutive model is used for description of the spherical (volumetric) part of stress and strain tensors, the symbols σ can be understood as normal stresses and ε as longitudinal strains, namely their mean values σ_{s} a ε_{s}, and modulus of elasticity as bulk modulus (never Young's modulus). Otherwise, the symbols σ a ε should be understood as components of the deviatoric part of stress and strain tensors, e.g. shear stress and angular (shear) strain.
- In the case that the constitutive model is used for description of shape (deviatoric) part of deformation, common symbols for shear stress and strain and shear modulus are often used in technical literature, i.e. τ, γ and G, respectively. Some of these symbols are used in the text below.

For example, the stiffness of the spring representing Hooke's element (element No. 1) should not be denoted by E symbol (as Young's modulus) but either G (shear modulus), or K (bulk modulus), if the deviatoric or spherical parts of strain is described by the model.

4.1.2. Non-linear elastic matter

① **General characteristic:** A matter with this characteristic shows a mutually unique (unambigous) but non-linear dependence $\sigma = f(\varepsilon)$, see Fig. 10. Conical spring (see Fig. 9) or spring with an arrow could be used as symbol of this type of behaviour in rheological models but it is not very common. Mathematical description can have e.g. the form of the formula $\sigma = E\,\varepsilon^{1/N}$, where modulus of elasticity E and inverse exponent N are elastic parameters of the model. This formula, however, holds for uniaxial stress state only, similarly to the simplest shape of Hooke's law $\sigma = E\,\varepsilon$ that can be understood as its special case with $N = 1$.

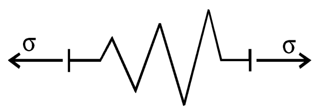


Fig.9: Non-linear elastic element

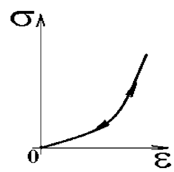


Fig.10: Non-linear dependence $\sigma = f(\varepsilon)$

A generalization of the above constitutive dependence for multiaxial stress states brings substantial problems, because the principle of superposition is not valid for non-linear relations. These problems are also associated with the multilinear elastic constitutive model. Priority should be given to hyperelastic models, based on the postulate of existence of a strain energy density function.

Multilinear elastic model can be obtained by replacing the stress-strain curve $\sigma = f(\varepsilon)$ with a number of linear parts described by the Hooke's law. The stress-strain analysis is then carried out using increment method and the modulus of elasticity is changed after having achieved a limit value of strain. This rheological model (Fig. 11) should be, however,

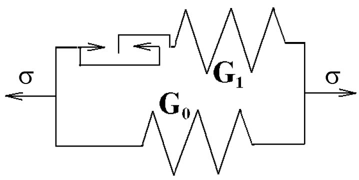


Fig.11: Multilinear elastic model

classified as a combined model, because it consists of several linear springs (elements No. 1 in Table 1) and one or more stopping blocks (elements No. 6).

This model of the non-linear elastic matter shows linear elastic behaviour defined by the modulus of elasticity G_0 up to the limit strain value ε_{B} of the stopping block, above which the modulus of elasticity increases stepwise by the value of G_1 and the dependency is linear again. Such a behaviour can occur in structures with wavy reinforcing fibres (with negligible bending stiffness) that do not show their load-bearing ability below a certain threshold strain value, at which they can straighten. More complex multilinear elastic constitutive characteristics can be obtained by use of a higher number of the basic elements.

The family of hyperelastic models is based on various formulations of the strain energy density function being a scalar function of all components of deformation tensor. By differentiation of this function with respect to the components of deformation tensor, the corresponding stress components can be calculated. These constitutive models are used especially for materials showing large elastic strains causing a non-linear material behaviour by themselves. Then it is necessary to distinguish among various formulations of stress and strain tensors; there values differ very substantially, whether the elementary force (deformation) is related to the initial (undeformed) or final (deformed) geometry, or (in icrements) to the actual deformed geometry. Models by Mooney-Rivlin, Ogden, Arruda-Boyce are the most frequent among them. The theory of hyperelastic models exceeds substantially the extent of this article so that the reader is referred e.g. to [3] for more information.

Applications: rubber and other elastomers, soft biological tissues.

4.2. Conditionally perfect plastic matter

This concept is not a standard one in the professional terminology. We would like to offer, however, a comprehensive overview of models of behaviour of matters, so that the term '**conditionally perfect plastic matter**' is required to be introduced. Such a matter is perfectly plastic under certain conditions and perfectly rigid under some other conditions. Its **ability to be plastic** is conditioned by some circumstances.

From the simple constitutive models, rigid-plastic models used in moulding belong to this category, as well as the less frequent matters plastic-rigid. These models are used exclusively for modelling of the deviatoric strain component $T_{\varepsilon \mathrm{D}}$. Their individual properties, under the conditions of creep and relaxation, are analyzed below.

4.2.1. Perfect rigid-plastic matter

① **General characteristic:** This matter behaves like perfectly rigid up to a certain limit value of deviatoric stress tensor component $\mathbf{T}_{\varepsilon D}$. If this value has been reached, a perfect plastic flow without any viscosity or stiffening occurs, i.e. there is a perfect plastic deformation under constant stress. Such a behaviour corresponds to the element No. 5 in Table 1, the so called **skidding block**. If the stress value σ is inside the interval $\sigma \in (-\sigma_{\mathrm{B}}, +\sigma_{\mathrm{B}})$ than the element movement is blocked. If the stress value σ reaches σ_{B}, then the **movement becomes unblocked** (see Fig. 12a). The blocking stress σ_{B} can equal e.g. the yield stress σ_{K}.

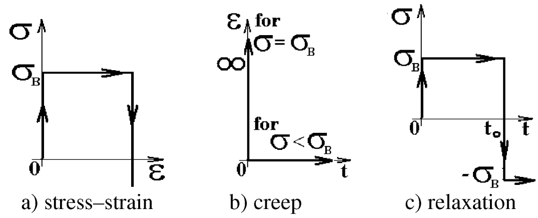


Fig.12: Basic characteristics of perfect rigid-plastic matter

② **Creep:** With respect to the existence of skidding block, the following two cases need to be taken into account:

- for $\sigma < \sigma_B$: The element deformation is blocked in this case so that the strain equals zero. This situation is represented by a straight line identical with the abscissa (t-axis) in the graph $\varepsilon(t)$ (Fig. 12b), what means no creep.
- for $\sigma = \sigma_B$: The element is unblocked but the stress meets $\sigma = \sigma_B$ for any value of loading force (stress) so that no equilibrium state can be reached and strain increases without any limitation. In the graph $\varepsilon(t)$ this situation is represented by a straight line identical with the ordinate (ε-axis) (Fig. 12b), what means the strain ε increases to infinity.

③ **Relaxation:** Stress corresponding to the limit value σ_B (e.g. yield stress σ_K) occurs under any non-zero deformation because the deformation is blocked at all lower stress values. This stress is time-independent for any deformation. If the deformation is required to return to its initial (zero) value, an opposite stress is induced in this way, with its magnitude corresponding to the limit value σ_B again because the deformation is blocked at lower stress magnitudes. Therefore no stress relaxation occurs here (Fig. 12c).

Applications: soil mechanics, metal moulding analyses etc.

4.2.2. Perfect plastic-rigid matter

① **General characteristic:** In this matter, a perfect flow without viscosity and stiffening, i.e. perfect plastic deformation occurs up to a certain limit value ε_B. After having achieved the value of ε_B, the further deformation is blocked. If the load increase continues, the matter behaves like perfect rigid, i.e. the deformation does not increase under any stress increase (Fig. 13a). This behaviour corresponds to element No. 6, i.e. stopping block.

- If the strain value ε lies within the interval $\varepsilon \in (-\varepsilon_B, +\varepsilon_B)$, then the element movement is fully free, it deforms without any stress.
- If the strain achieves the limit value of ε_B, the **element movement becomes blocked**. The blocking strain ε_B can be given by straightening of the initially wavy reinforcing fibres or macromolecular chains in the material structure.

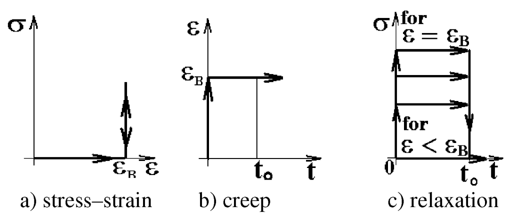


Fig.13: Basic characteristics of perfect plastic-rigid matter

② **Creep:** Here a deformation with the magnitude corresponding to the limit strain ε_B occurs under any loading force inducing a non-zero stress. Then the deformation is blocked independently of the stress magnitude. This strain is independent of time for a steady stress.

After the body having been unloaded, the strain remains without any change, if no load in opposite sense occurs. It can be stated that there is no creep in a perfect plastic-rigid matter (Fig. 13b).

③ **Relaxation:** With respect to the existence of the stopping block, the following two cases should be taken into account:

- for $\varepsilon < \varepsilon_\mathrm{B}$: In this case the deformation is fully free, it occurs under zero stress value. This situation is represented by a straight line identical with the abscissa (t-axis) in the graph $\sigma(t)$ (Fig. 13c).
- for $\varepsilon = \varepsilon_\mathrm{B}$: A higher deformation cannot be achieved, the deformation is blocked at this value. The stress value is not given by the deformation but by a static equilibrium and is invariable in time. When the deformation returns back to the zero value, the stress vanishes as well. This means no relaxation occurs.

Applications: fibre composites with wavy (in the initial state) reinforcing fibres, being several orders stiffer than the matrix.

4.3. Viscous liquid

Viscous liquid is a matter in which the shear stress depends on the strain rate. It means that only the deviatoric component of strain is described by the constitutive models below. The volumetric strain component is either neglected (rather often), or described by a linear elastic model (bulk modulus). In this paper the linear viscous (so called Newtonian) liquid and non-linear viscous (non-Newtonian) liquid are analysed. Liquids with time-dependent viscosity are not dealt with.

4.3.1. Linear viscous liquid (Newtonian liquid)

① **General characteristic:** Shear stress is driven by Newton's law of viscosity saying that shear stress is proportional to the shear strain rate (or velocity gradient). The **liquid linear damper** (element No. 3 in Table 1, Newton's element) is used to represent this type of behaviour in the rheological schemes. Using a general tensorial notation, Newton's law of viscosity can be written as follows:

$$\sigma_{ij} = \eta \, \frac{\partial \varepsilon_{ij}}{\partial t} \ . \tag{a}$$

The dependence of stress on the strain rate $\dot{\varepsilon}$ is then linear with viscosity η being its parameter (Fig. 14b). In opposite stress is independent of strain magnitude ε, so that it is constant if the specimen is loaded with a constant strain rate (Fig. 14a).

② **Creep:** In this case the time response (change) of strain is investigated after a stepwise change of stress. The member $\partial\varepsilon/\partial t$ is expressed from the relation (a) and then the equation is integrated. For the simplest case of uniaxial stress state this can be manipulated as follows:

$$\varepsilon = \int \frac{\sigma_0}{\eta} \, \mathrm{d}t = \frac{\sigma_0}{\eta} \, t + c \ .$$

The initial condition can be written as: for $t = 0$ it holds $\varepsilon = 0$, so that $c = 0$. It means that the strain is a linear function of time, this case is called **unlimited creep** (Fig. 14c).

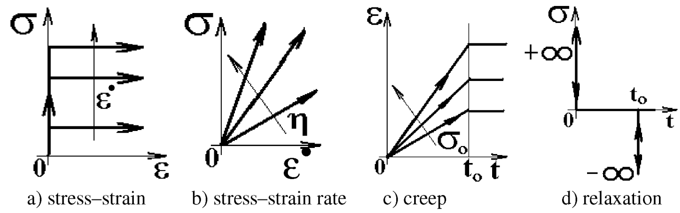


Fig.14: Basic characteristics of linear viscous liquid

③ **Relaxation:** The character of stress decrease in consequence of relaxation can be derived from the relation (a) as well. Let's remember that the stress relaxation is analysed here in the case of a strain-controlled loading, given by the stepwise change in strain $\varepsilon(t) = \varepsilon_0\, H(t)$, where $H(t)$ is Heaviside function; at the time $t = 0$ a stepwise change of the H function occurs and therefore a stepwise change in strain ε as well, so that the derivative $\mathrm{d}\varepsilon/\mathrm{d}t$ is limitless in the instant of this change (Dirac function). This means that also the stress σ is limitless for $t = 0$, according to relation (a). Strain ε is invariable for time interval $0 < t < t_o$. This means that $\mathrm{d}\varepsilon/\mathrm{d}t = 0$, so that also the stress equals zero, according to relation (a). Another stepwise change in strain to its zero value occurs at the time instant $t = t_\mathrm{o}$, what induces unlimited stress, negative in this case. The shape of the dependence $\sigma(t)$ under relaxation is presented in Fig. 14d. It can be concluded that the shear stress relaxes theoretically instantaneously to zero in a (linear) viscous liquid.

4.3.2. Non-linear viscous liquid (non-Newtonian liquid)

① **General characteristic:** This liquid shows a non-linear dependence between shear stress and shear strain rate (Fig. 15a). In the rheological schemes, it is represented by the **liquid non-linear damper** (element No. 4 in Table 1, Norton element). In the case of uniaxial stress state, behaviour of a non-linear viscous liquid can be described by the following relation:

$$\left(\frac{\sigma}{\lambda}\right)^N = \dot{\varepsilon} \qquad \text{or} \qquad \sigma = \lambda\, \dot{\varepsilon}^{(1/N)} \; . \tag{b}$$

The dependence between strain rate $\dot{\varepsilon}$ and stress σ is then described by a relation with several parameters, while the parameters (λ, N) are independent of the strain rate or stress.

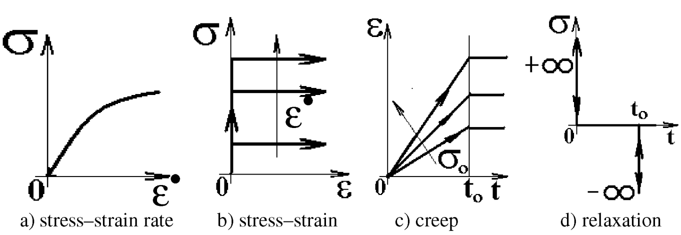


Fig.15: Basic characteristics of non-linear viscous liquid

② **Creep:** Similar to the chapter 4.3.1, the derivative $\mathrm{d}\varepsilon/\mathrm{d}t$ can be calculated from the relation (b), and the obtained relation then integrated:

$$\mathrm{d}\varepsilon = \int \left(\frac{\sigma_0}{\lambda}\right)^N \mathrm{d}t \quad \rightarrow \quad \varepsilon = \left(\frac{\sigma_0}{\lambda}\right)^N t + c \; .$$

The initial condition has also the same shape ($\varepsilon = 0$ for $t = 0$, i.e. $c = 0$). Then strain is a linear function of time; an **unlimited creep** exists again (Fig. 15c). Also the graphical representation of the dependence is the same (cf. Fig. 14c and 15c) but the slope of the straight lines in the non-constant part is different.

③ **Relaxation:** After similar manipulations like at the linear viscous liquid, the same conclusions can be drawn, as presented in Fig. 15d. The stepwise change in deformation causes an instantaneous limitless stress value, which relaxes to zero immediately (Dirac function).

5. Concluding remarks

The presented overview does not take the influence of temperature into account. The temperature can:

- influence the constitutive parameters of individual models which do not remain constant then, but they become functions of temperature (or even of other influencing quantities),
- change the type of constitutive behaviour significantly – change of elastic behaviour into viscoelastic or viscoplastic with increasing temperature,
- evoke a quite specific type of behaviour not mentioned in Fig. 1 (thermoelasticity, thermoplasticity).

Neither the presented overview nor the scheme in Fig. 1 does take the dependency of properties on directions into account. All the above models are therefore described as isotropic in their simplest variants but they can also include various types of anisotropy (e.g. orthotropic or transversal isotropic models). If they are linear, rheological models and principal of superposition can be used in creation of their constitutive relations, while in the case of non-linear behaviour, special constitutive models need to be created (e.g. anisotropic hyperelastic material); a higher number of constitutive parameters is typical for those models.

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