SYSTEMIC APPROACH TO MODELLING OF CONSTITUTIVE BEHAVIOUR OF VARIOUS MATTERS
Part II – Combined Constitutive Models in Solid Mechanics

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The paper continues the description of constitutive behaviour of matters, the overview of which was presented in the previous part of this paper (Part I – basic and simple constitutive models). The definition and systemization of constitutive models was presented there and basic and simple models were described in detail. In the same systemic approach, combined constitutive models of materials (solid matters) are presented in this paper (Part II). It analyzes the more complex types of constitutive behaviour and presents a comprehensive overview of their responses in standard tension tests (stresses as functions of strain magnitude and strain rate), as well as their creep and relaxation responses. Graphical as well as the simplest mathematical interpretations of viscoelastic, elastic-plastic, viscoplastic and elastic-viscoplastic matters are presented, except for various types of anisotropic materials and their constitutive models (all the models are presented as isotropic only). On the base of both of these papers, the chapter on constitutive models was published in [1].

Keywords: *mechanical behaviour, constitutive models, viscoelastic, viscoplastic, elastic-viscoplastic*

1. Introduction

In the first part of this paper the concept of constitutive dependencies and models was defined; constitutive models in mechanics were divided into three cathegories: 1) basic, 2) simple, and 3) combined constitutive models. Basic and simple constitutive models were analyzed in the first part of the paper; the second part presented here deals with the combined constitutive models in detail, analyzes their constitutive behaviour in standard tension test (stress as a function of strain magnitude and strain rate), as well as creep and relaxation behaviour. Graphical as well as the simplest mathematical interpretations of these constitutive models are presented. The presented overview is based on [2] and completed with some additional constitutive models and formulations and with a comprehensive overview of graphical constitutive representations of the models in question.

2. Specification of constitutive relations of combined models

Combined constitutive models are used for such matters, the response of which (under mechanical activation – either stress or strain controlled loading) cannot be described by the simple constitutive models. In the wider sense, constitutive relations comprehend not only

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the dependency of components σ_{ij} of the stress tensor $\mathbf{T}_\sigma$ on components ε_{ij} of the strain tensor $\mathbf{T}_\varepsilon$, but the dependency of stress on strain rate, the time dependency of strain under stress controlled loading (creep) and the time dependency of stress under strain controlled loading (relaxation) as well; the time history of the controlled quantities has been specified in chapter 2.2 (Part I). Rheological schemes of the combined constitutive models consist of two or more basic elements presented in Table 1 (Part I).

Note 1 – the most general constitutive relations

The most general constitutive relations can be formulated for the behaviour of an **elastic-visco-plastic** matter with hardening (cf. Fig. 1, Part I). The individual attributes in this term have the following meanings:

- **Elastic behaviour** means that the loading and unloading processes follow the same uniaxial stress-strain curve, i.e. there exists a unique dependence between strain and stress values.
- **Viscous behaviour** means that stress is a function of strain rate.
- **Plastic behaviour** means that a permanent deformation occurs and this remains preserved also after a stress decrease to zero.

Thus the general **elastic-visco-plastic model** of constitutive behaviour assumes that uniaxial stress is function of: 1) strain magnitude, 2) strain rate, and 3) history of loading. Whenever a dependency of stress on strain rate is displayed below, conditions of standard tension tests (constant strain rate) are assumed.

In the following text ε means mostly **angular (shear) strain** and σ means **shear stress** (i.e. components of strain and stress deviators). The volumetric (mean) components of strain and stress states can also be described with these models; however, the volumetric component of deformation is several orders lower than the deviatoric one at most technical materials, so that the non-elastic or non-linear component of volumetric deformation uses to be negligible.

3. Linear viscoelastic matters

At this constitutive model, the traditional division of matters into solids and fluids vanishes; all of them represent only some special cases of the viscoelastic matter. The distiction between solid and liquid is not unique (unambiguous) even under the given values of thermodynamic state quantities; it depends on the time scale of the investigated processes. We can say a matter is solid if it is able to be in the statical equilibrium under a general load. In opposite, a liquid shows flow under load, if the stress state includes a non-zero deviatoric component.

In principle, however, any loading process disturbs the thermodynamical equilibrium and its renewal requires a certain time. Nevertheless it is difficult to distinguish between an infinitely slow flow (a permanent non-equilibrium state) and the equilibrium reached in an infinite time. So the distinction between solid and fluid depends on the relation between the time dimension of the process in question and the time necessary to reach the steady (equilibrium) state, so that it is neither unique nor objective. Factually:

- If the time of the investigated loading process is substantially longer than the duration of the transient processes running in the matter, any matter seems to be liquid (flow of icebergs, creep of aluminium or steel under higher temperatures, etc.).

– If the time of the investigated loading process is relatively short (and, consequently, there is a high strain rate), a normally liquid matter can behave like an elastic one (a high speed impact against water level – if realized under a low angle, the water level drives the body back similarly to an elastic solid).

Note – viscoelastic responses

Let's remember that typical representations of viscoelasticity are as follows: **1) flow (creep)**, i.e. deformation increase under stress controlled loading (usually in a steady stress state), **2) relaxation**, i.e. stress decrease under strain controlled loading (usually in a constant strain state), and **3) hysteresis** (different stress-strain characteristics in loading and unloading). In dependence of the types of responses, the following basic types of viscoelastic behaviour can be defined:

- **According to the response under stress controlled loading**, we can distinguish:
  - **unlimited creep** – the deformation of the body increases to infinity under constant stress state, no equilibrium can be reached.
  - **limited creep** – the deformation tends asymptotically to a certain finite value, under which the equilibrium is reached.
- **According to the response under strain controlled loading**, we can distinguish:
  - matters **enabling an instantaneous stepwise deformation**,
  - matters, at which **an instantaneous stepwise deformation is disabled** (the response in the form of an instantaneous stress grows up to infinity).

3.1. Linear viscoelastic matter with unlimited creep (fluid)

① **General characteristic:** A fluid (liquid or gas) can be characterized as a matter showing permanent flow without tendency to equilibrium if deviatoric component of stress is present. (Note that an exact definition of fluid cannot be quite simple, as well as the definitions of liquid and gas below). To comprehend all the strain components in a fluid, both volume and shape changes are dealt with below. **Volume change** can be either

- **zero** – this is the case at a **perfect liquid**,
- **small** and approximately linear elastic – this holds for a **real liquid**, or
- **large**, described by the state equation of gases – then the matter is modelled as a perfect or **real gas**.

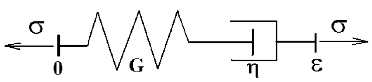
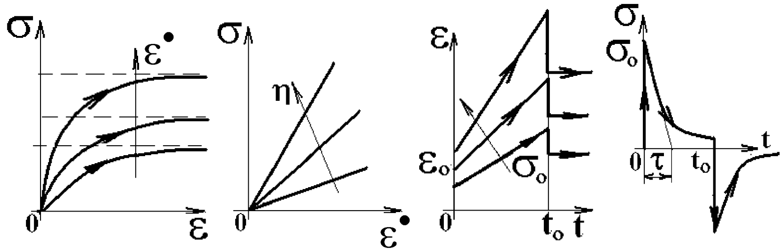


Fig.1: Maxwell rheological model

Shape changes, as a response to a loading process, can be simply described by **Maxwell rheological model** at these matters. This model consists of elements No. 1 and No. 3 from Table 1 (Part I), i.e. Hooke's a Newton's elements, in serial arrangement. Here we analyze the problems of creep and relaxation at this model.

In both of the above cases, the analysis is based on the fact that the resulting strain equals the sum of the strains of the individual elements in serial arrangement (see Part I). The same summation is valid also for strain rates. In opposite, there are equal stresses in both elements. Using this conditions, we can obtain the differential equation of the model in the following shape:

$$\dot{\varepsilon} = \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma}{G}\right) + \frac{\sigma}{\eta} \qquad \text{after integration:} \qquad \varepsilon = \frac{\sigma}{G} + \int\limits_0^t \frac{\sigma(t)}{\eta}\,\mathrm{d}t\,. \tag{a}$$



a) stress–strain b) stress–strain rate c) creep d) relaxation

Fig.2: Basic characteristics of viscoelastic fluid with unlimited creep

For a linear viscoelastic matter with unlimited creep, the dependence between stress and strain is non-linear (Fig. 2a), with strain rate being its parameter. The dependence of stress on strain rate is linear with viscosity being its parameter (Fig. 2b).

② **Creep:** In the standard creep test, the specimen is loaded by a stress controlled load, which is constant during the time interval t_0, so that it holds $\sigma(t) = \sigma_0$. By substituting this condition into the equation (a) we obtain:

$$\varepsilon = \frac{\sigma_0}{G} + \frac{\sigma_0}{\eta}\, t + c \qquad \Rightarrow \qquad \varepsilon = \varepsilon_0 + \frac{\sigma_0}{\eta}\, t + c\ . \tag{b}$$

This general solution containing an integration constant c can be solved for the following two types of boundary conditions:

a) For the initial condition $t = 0 \rightarrow \varepsilon = \varepsilon_0$ we obtain $c = 0$. Then it results from the eq. (b) that the liquid shows a linear increase of strain ε in time (unlimited creep – Fig. 2c).

b) For the condition $t = t_0$ (i.e. $\sigma \rightarrow 0$) $\rightarrow \varepsilon = \sigma_0\, t_0/\eta$, we obtain $c = \sigma_0\, t_0/\eta$, and also $\varepsilon_{(t>t_0)} = \sigma_0\, t_0/\eta$, so that the deformation remains constant in time $t > t_0$; this means there is no creep in this stage (Fig. 2c).

③ **Relaxation:** In the standard relaxation test, the specimen is loaded by a strain controlled load, which is constant during time interval t_0, so that $\varepsilon = \varepsilon_0 =$ const., and $\dot{\varepsilon} = 0$. Using the relation (a) we obtain:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma}{G}\right) + \frac{\sigma}{\eta} = 0 \quad \rightarrow \quad \frac{1}{G}\,\frac{\mathrm{d}\sigma}{\mathrm{d}t} + \frac{\sigma}{\eta} = 0 \quad \rightarrow \quad \frac{\mathrm{d}\sigma}{\sigma} + \frac{G}{\eta}\,\mathrm{d}t = 0$$

$$\text{after integration:} \qquad \ln\sigma + \frac{G}{\eta}\, t + c = 0\ .$$

The relevant initial condition is: for $t = 0 \rightarrow \sigma = \sigma_0$, while $\sigma_0 = G\,\varepsilon_0$. After substitution we obtain a relation for the integration constant. Then we introduce a new variable τ, given by the relation $\tau = \eta/G$, so that it holds subsequently:

$c = -\ln\sigma_0$ and after substitution

$$\ln\sigma - \ln\sigma_0 + \frac{G}{\eta}\, t = 0 \quad \rightarrow \quad \sigma = \sigma_0\, \mathrm{e}^{-\frac{G}{\eta}t} \quad \rightarrow \quad \sigma = \varepsilon_0\, G\, \mathrm{e}^{-\frac{t}{\tau}}\ .$$

The introduced variable $\tau = \eta/G$ with the dimension of time is called **relaxation time**. It determines the time of transient processes, these having an exponential character. Graphically this time is given by the intersection point of a tangent line to the transient curve in its origine with the asymptote of this curve for $t \rightarrow \infty$ (see Fig. 2d). After a time corresponding to four relaxation times, the transient process is practically finished, and the stress deviation from its final value (in time $t \to \infty$) decreases under 2 % of the initial stress value.

Aplications: thermoplastic polymers near their melting temperature, fresh concrete, many metals near their melting temperature.

3.2. Viscoelastic matter with limited creep

This group comprehends matters considered commonly as solids (rubber, organic polymers, wood under low load levels). Behaviour of matters showing limited creep (deformation under stress controlled steady load tends asymptotically to a certain finite value) can be divided into two groups depending on the fact whether the matter:

– enables an instantaneous elastic deformation (Kelvin model),
– disables an instantaneous elastic deformation (Voigt model).

Both of these models are analysed in detail below from the viewpoint of creep as well as relaxation.

3.2.1. Voigt model

① **General characteristic:** Behaviour of a matter showing limited creep and disabling an instantaneous elastic deformation (there is a delay between loading force and deformation) can be described by **Voigt rheological model** (Fig. 3). It consists of elements No. 1 and No. 3 from Table 1 (Part I), i.e. Hooke's and Newton's elements in a parallel arrangement.

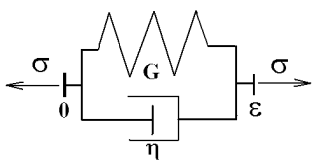


Fig.3: Voigt rheological model

It holds for parallel arrangement of basic elements that the resulting force as well as stress are given by summarization of forces (stresses) acting in the individual elements, while their deformation (strains) are equal (see Part I). The stress in Hooke's element σ_{No1} is proportional to the strain ε, the stress in Newton element σ_{No3} is proportional to the strain rate $\mathrm{d}\varepsilon/\mathrm{d}t$. It holds then:

$$\sigma = \sigma_{\mathrm{No1}} + \sigma_{\mathrm{No3}} = G\,\varepsilon + \eta\,\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\ . \tag{c}$$

This relation results in a linear stress-strain characteristic (see Fig. 4a).

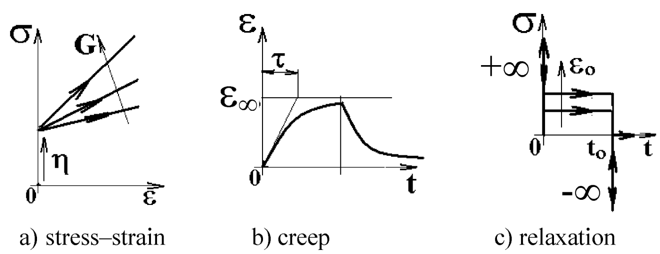


Fig.4: Characteristics of Voigt model of viscoelastic matter

② **Creep:** In standard creep test, the strain response ε is recorded under constant stress $\sigma = \sigma_0$. The time derivative $\mathrm{d}\varepsilon/\mathrm{d}t$ is eliminated from the above relation and the equation is consequently integrated. The shape of the initial condition is $t = 0 \rightarrow \sigma = \sigma_0$, so that it holds:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\sigma_0}{\eta} - \frac{G\,\varepsilon}{\eta} \qquad \text{resp.} \qquad \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\sigma_0}{\eta}\left(1 - \frac{G\,\varepsilon}{\sigma_0}\right) \,,$$
$$\int_0^{\varepsilon} \frac{\mathrm{d}\varepsilon}{1 - \dfrac{G\,\varepsilon}{\sigma_0}} = \int_0^{t} \frac{\sigma_0}{\eta}\,\mathrm{d}t \quad \rightarrow \quad -\frac{\sigma_0}{G}\ln\left|1 - \frac{G\,\varepsilon}{\sigma_0}\right| = \frac{\sigma_0}{\eta}\,t \,. \tag{d}$$

After some manipulations we can obtain the resulting relation in the following form:

$$-\frac{\sigma_0}{G}\ln\left|1 - \frac{G\,\varepsilon}{\sigma_0}\right| = \frac{\sigma_0}{\eta}\,t \quad \rightarrow \quad \ln\left|1 - \frac{G\,\varepsilon}{\sigma_0}\right| = -\frac{G}{\eta}\,t \quad \rightarrow$$
$$\rightarrow \quad 1 - \frac{G\,\varepsilon}{\sigma_0} = \mathrm{e}^{-\frac{Gt}{\eta}} \quad \rightarrow \quad \varepsilon = \frac{\sigma_0}{G}\left(1 - \mathrm{e}^{-\frac{Gt}{\eta}}\right) \,.$$

Similar to the previous type of matter, we use a general symbol G for modulus instead of E used for Young modulus and we introduce the variable τ. Here this variable (with a dimension of time) has the meaning of **retardation time** (it characterizes the delay of deformation). Then the resulting relation is:

$$\varepsilon = \frac{\sigma_0}{G}\left(1 - \mathrm{e}^{-\frac{t}{\tau}}\right) \,.$$

Constant τ determines the duration of the transient process and is given graphically by the point of intersection of the tangent line to the exponential transition curve in its origin with the asymptote of this curve for $t \rightarrow \infty$ (see Fig. 4b where the asymptote is denoted by the symbol ε_∞). After the time corresponding to four relaxation times, the transient process is practically finished, and the strain deviation from its final value ε_∞ decreases below 2 % of this value ($\varepsilon_\infty = \sigma_0/G$ in time point $t \rightarrow \infty$). During the subsequent unloading in time $t = t_0$, the strain decreases to zero and the solution to the differential equation (hommogeneous in this case) is given by an exponential function with its asymptote at zero strain value.

③ **Relaxation:** The time dependency of the stress under relaxation can be also derived from the differential equation of Voigt model. We substitute $\mathrm{d}\varepsilon/\mathrm{d}t \rightarrow \infty$ for the derivative in the equation (c) in the time instant of the stepwise strain change and set $\mathrm{d}\varepsilon/\mathrm{d}t = 0$ at any other times, because the relaxation is analyzed for strain controlled loading with $\varepsilon = \text{const}$. The analysis of this relations results in the following facts:

- As the strain rate $\dot{\varepsilon} \rightarrow \infty$ in time point $t = 0$, the stress approaches infinity in this moment (Dirac function).
- The strain rate equals zero in time interval $t \in (0, t_0)$, therefore the stress is constant and equal to $\sigma = G\,\varepsilon$.
- The strain rate approaches a negative infinite value ($\dot{\varepsilon} \rightarrow -\infty$) in time point t_0, so that the stress is also given by a (negative) Dirac function.
- The strain rate as well as strain magnitude equal zero for all times higher than t_0, so that also the stress equals zero (see Fig. 4c). It means there is no relaxation at this matter, or in other words, the relaxation finishes in an infinitesimal time interval.

Aplications: organic polymers.

3.2.2. Kelvin model

① **General characteristic:** Behaviour of a matter showing limited creep and non-zero instantaneous deformation can be described by the **rheological model**, denoted usually as **Kelvin model** (or **standard linear solid**). Its structure is created by parallel arrangement of Maxwell model with a linear spring. For this serial-parallel arrangement of the individual elements, the differential equation can be derived in the following form:

$$\sigma + \tau_\varepsilon \frac{\mathrm{d}\sigma}{\mathrm{d}t} = G_0 \left(\varepsilon + \tau_\sigma \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \right) , \qquad \text{where} \qquad \tau_\varepsilon = \frac{\eta}{G_1} , \quad \tau_\sigma = \frac{\eta}{G_0} \left(1 + \frac{G_0}{G_1} \right) . \tag{e}$$

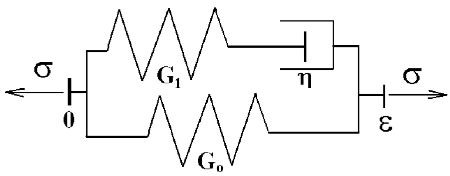


Fig.5: Kelvin rheological model

② **Creep:** The relation for creep could be derived in the similar way like at the Voigt model. It means to eliminate the derivative $\mathrm{d}\varepsilon/\mathrm{d}t$ from the above equation and to integrate the relation subsequently. For the initial condition $\varepsilon_{(t=0)} = \sigma_0/(G_0 + G_1)$ the following relation can be obtained:

$$\varepsilon(t) = \frac{\sigma_0}{G_0} \left[1 - \left(1 - \frac{\tau_\varepsilon}{\tau_\sigma} \right) \mathrm{e}^{-\frac{t}{\tau_\sigma}} \right] = \frac{\sigma_0}{G_0} \left[1 + \left(\frac{G_1}{G_0 + G_1} \right) \mathrm{e}^{-\frac{t}{\tau_\sigma}} \right] .$$

In the case of a matter modelled by Kelvin model it can be stated that the deformation increases instantaneously (stepwise) from zero to the value $\varepsilon_0 = \varepsilon_{(t=0)} = \sigma_0/(G_0 + G_1)$ and then increases exponentially up to the value $\varepsilon_\infty = \varepsilon_{(t\to\infty)} = \sigma_0/G_0$ (Fig. 6b), what is typical for limited creep. After unloading in the time point t_0, the deformation decreases stepwise and approaches then zero asymptotically.

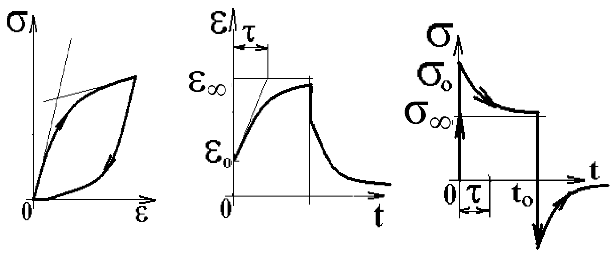


Fig.6: Basic characteristics of Kelvin model

③ **Relaxation:** The relation describing the time dependency of stress can be expressed from the condition $\mathrm{d}\varepsilon/\mathrm{d}t$, like at the Voigt model. The relation is as follows:

$$\sigma(t) = \varepsilon_0 \, G_0 \left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\varepsilon} \right) \mathrm{e}^{-\frac{t}{\tau_\sigma}} \right] = \varepsilon_0 \, G_0 \left[1 + \left(\frac{G_1}{G_0} \right) \mathrm{e}^{-\frac{t}{\tau_\sigma}} \right] .$$

Analysis of this relation concludes that the stress decreases exponentially from the value $\sigma_0 = \sigma_{(t=0)} = \varepsilon_0/(G_0+G_1)$ to the value $\sigma_\infty = \sigma_{(t\to\infty)} = \varepsilon_0\, G_0$ (Fig. 6c). After unloading in the time t_0 (a stepwise return to the original geometrical configuration) the stress changes stepwise down to negative values and then its magnitude approaches to zero.

Aplications: rubber, soft living tissues, wood (if its load is not too high).

4. Elastic-plastic matters

This group comprehends matters with elastic behaviour up to a certain limit value of stress; after having reached this, the behaviour becomes plastic. While the elastic behaviour uses to be modelled as linear, plastic behaviour can be modelled either perfect (without any stiffening) or with a (mostly linear) stiffening.

With respect to various types of plastic behaviour, we can create several constitutive models of these matters, e.g. perfect elastic-plastic matter, or elastic-plastic matter with linear or multilinear stiffening. These three models are analyzed in detail below.

4.1. Perfect elastic-plastic matter

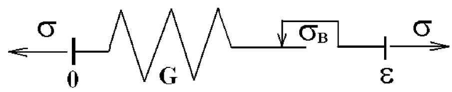


Fig. 7: Saint-Vénant model

① **General characteristic:** It is characteristic for this matter that it behaves linear elastic under loads up to a certain threshold value of stress σ (blocking stress σ_B). After having reached $\sigma = \sigma_B$, the stress remains constant and equal $\sigma = \sigma_B$; this holds for any strain value ε. This behaviour can be described by **Saint-Vénant rheological model**, created by elements No. 1 and No. 5 (Table 1, Part I), i.e. Hooke's element and skidding block, in serial arrangement (see Fig. 7). The threshold stress value σ_B equals usually to the yield stress σ_K. The dependence of stress on strain is shown in Fig. 8a.

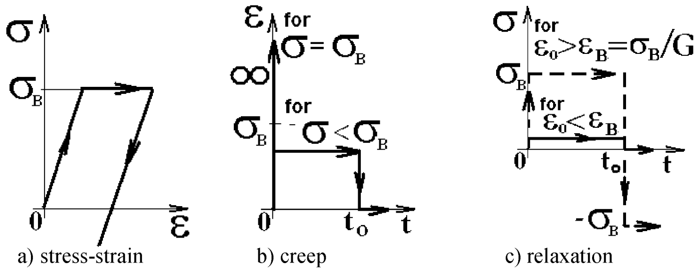


Fig. 8: Basic characteristics of perfect elastic-plastic matter

② **Creep:** Two basic cases should be distinguished here:

- $\sigma < \sigma_B$: In this case the skidding block is locked and only the spring is active; its behaviour is controlled by Hooke's law.

  The matter behaves linear elastic so that the strain of the model under a stress value $\sigma = \sigma_B$ holds $\varepsilon = \sigma/E$, what means it is independent from time. Graphically this situation can be represented by a set of straight lines parallel with the abscissa (time axis t – Fig. 8b).
- for $\sigma = \sigma_B$: Now the skidding block is unlocked so that the element can deform without limitation. For stress values $\sigma = \sigma_B$ the strain increases to infinity, while

for $\sigma_{\mathrm{i}} > \sigma_{\mathrm{B}}$ no static equilibrium can be reached. In the graph $\varepsilon(t)$, this situation is represented by a straight line identical with the ordinate (ε-axis, see Fig. 8b).

③ **Relaxation:** As it holds $\varepsilon_{\mathrm{B}} = \sigma_{\mathrm{B}}/E$, it is necessary to distinguish between the following two cases:

- for $\varepsilon < \varepsilon_{\mathrm{B}}$: In this case $\sigma < \sigma_{\mathrm{B}}$, so that the element deformation is determined by the spring and the situation is represented by a straight line parallel to the abscissa (t-axis – Fig. 8c) – there is no stress relaxation.
- for $\varepsilon \geq \varepsilon_{\mathrm{B}}$: In this case it holds $\sigma = \sigma_{\mathrm{B}}$, because the stress cannot be higher than blocking stress σ_{B} (mostly yield stress σ_{K}) for a perfect plastic matter. Then the 'relaxation curve' is identical with a straight line $\sigma = \sigma_{\mathrm{B}}$ (Fig. 8c); this means there is no stress relaxation.

Aplications: silicon irons.

4.2. Elastic-plastic matter with stiffening

It is characteristic for this matter that it behaves linear elastic during loading until the stress reaches the locking (threshold) value σ_{bB}. After exceeding the value $\sigma = \sigma_{\mathrm{bB}}$, the stress increases linearly with strain, but with another proportionality ratio; moreover several of these threshold values can occur and everyone of them changes the slope of the σ-ε dependence. The lowest threshold value corresponds to the yield stress σ_{K}, the others can be chosen in such a way that the resulting fractional line corresponds to the real non-linear dependence as good as possible. This behaviour can be described by **generalized Saint-Vénant rheological model** (Fig. 9), the structure of which is created by one or more Saint-Vénant models in parallel with a spring. Various models of stiffening can be created by using various numbers of Saint-Vénant models arranged in parallel.

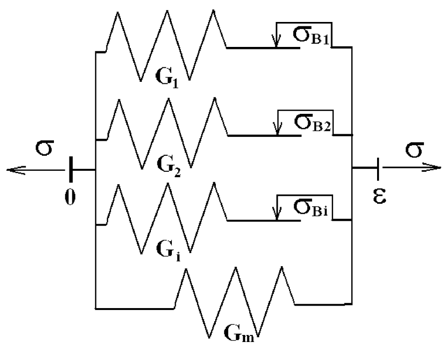


Fig.9: Generalized Saint-Vénant rheological model of a multilinear elastic-platic matter

4.2.1. Elastic-plastic matter with linear stiffening

① **General characteristic:** Linear stiffening can be modelled by Saint-Vénant model with a parallel spring, what represents a special case of generalized Saint-Vénant model for $m = 2$. The σ-ε dependence is in Fig. 10a. The following formulas are valid here:

- for $|\sigma| < \sigma_{\mathrm{bB}}$ it holds: $\varepsilon = \varepsilon_{\mathrm{e}}$, $\sigma = \varepsilon\,(G_1 + G_2)$,
- for $|\sigma| \geq \sigma_{\mathrm{bB}}$ it holds: $\varepsilon = \varepsilon_{\mathrm{e}} + \varepsilon_{\mathrm{p}}$, $\sigma = \sigma_{\mathrm{B}} + \varepsilon\, G_2$.

Here it is important to realize that the threshold stress of skidding block σ_B is not identical with the threshold stress of the material, denoted as σ_bB to distinguish between them (this can equal e.g. yield stress, i.e. $\sigma_\mathrm{bB} = \sigma_\mathrm{K}$). It is in consequence of the parallel arrangement of the model; then the resulting stress is given by summarization of the stresses in both embranchments of the model so that it holds:

$$\sigma_\mathrm{bB} = \sigma_\mathrm{B1} + \sigma_2 = \varepsilon_\mathrm{B}\,(G_1 + G_2) = \frac{\sigma_\mathrm{B}}{G_1}\,(G_1 + G_2) = \sigma_\mathrm{B}\left(1 + \frac{G_2}{G_1}\right) . \tag{f}$$

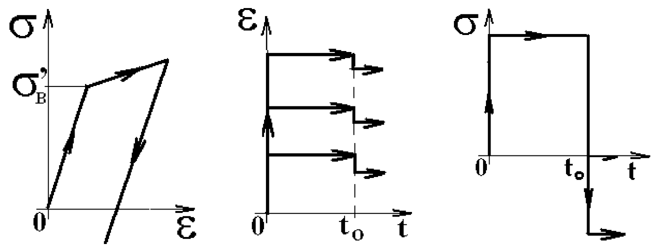


Fig.10: Characteristics of generalized Saint-Vénant model

② **Creep:** No time occurs in constitutive equations, so that the deformation is time-independent. In opposite to the perfect elastic-plastic matter, the relation between stress and strain is mutually unambiguous under monotonnous loading, so that a certain constant strain value corresponds to any stress value σ_0. Therefore no creep occurs, the representation in Fig. 10b is valid for $\sigma_0 > \sigma_\mathrm{bB1}$. For lower stress values, the matter appears to be elastic.

③ **Relaxation:** For the same reasons a unique constant stress value corresponds to any strain value ε_0 under monotonnous loading, no stress relaxation occurs. Fig. 10c is valid for $\varepsilon_0 > \varepsilon_\mathrm{B1}$; for lower strain values, behaviour of the matter is elastic.

4.2.2. Elastic-plastic matter with multilinear stiffening

① **General characteristic:** The real curve of stress-strain characteristic of material stiffening can be replaced by several linear parts; in this way we obtain the **model with multilinear stiffening** (Fig. 11). It is a **generalized Saint-Vénant model** with two or more skidding blocks. Then the relations presented in chap. 4.2.1. can be generalized. If the individual elements are ordered in such a way that their limit strain $\varepsilon_{\mathrm{B}i}$ are enumerated in ascending order ($\varepsilon_\mathrm{B1} < \varepsilon_\mathrm{B2} < \cdots < \varepsilon_{\mathrm{B}i} < \cdots < \varepsilon_{\mathrm{B}m}$), and if the limit stress was just reached in the j^th element (i.e. σ_j lies in the interval $\sigma_j \in \langle \varepsilon_{\mathrm{bB}j}; \varepsilon_{\mathrm{bB}(j+1)} \rangle$, so that the number of unlocked skidding blocks equals j), the stress can be expressed by the following general

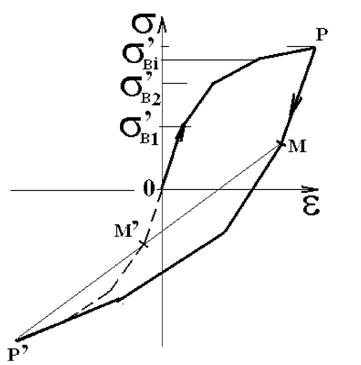


Fig.11: Stress-strain diagram of the generalized Saint-Vénant matter with multilinear stiffening

formula:

$$\sigma = \sum_{i=1}^{j} \sigma_{\mathrm{B}i} + \sum_{i=j+1}^{n} G_i \, \varepsilon \ .$$

The boundary stress values $\sigma'_{\mathrm{B}j}$ of the material are related to the threshold values of the individual skidding blocks $\sigma_{\mathrm{B}i}$ by the following relation:

$$\sigma'_{\mathrm{B}j} = \sum_{i=1}^{j-1} \sigma_{\mathrm{B}i} + \frac{\sigma_{\mathrm{B}j}}{G_j} \sum_{i=j}^{m} G_i \ .$$

Owing to the skidding blocks acting symmetrically (their limit values σ_{B} are the same in tension as well as in compression), the unloading curve differs from the loading one. If the skidding block is unlocked by action of the tensional stress $\sigma_{\mathrm{B}i}$, then if switches into a locked state after a stress decrease and a stress change by $2\,\sigma_{\mathrm{B}i}$ is necessary to unlock the block in the opposite direction. Therefore the unloading curve is given by a central symmetric transformation of the loading curve, with the extension ratio of 2 and the centre in the terminal point P' of the primary compression loading curve (Fig. 11). This model is able to describe a kinematic stiffening and so it can become a simple approximation of Bauschinger effect as well.

② **Creep:** There is no time variable in the constitutive equations, so that the deformation is independent of time. Similar to the linear stiffening, the relation between stress and strain is mutually unambiguous under monotonnous loading so that a certain constant strain value ε corresponds to any stress value σ_0; no creep occurs.

③ **Relaxation:** For the same reasons a unique constant stress value σ corresponds to any strain value ε_0 under monotonnous loading, what means that no stress relaxation occurs.

Aplications: metals and alloys under temperatures lower than a quater of their absolute melting point (°K).

5. Viskoplastic matters

As viscoplastic we denote matters showing permanent (irreversible) deformation under load (like plastic matters) with its value increasing in time. This phenomenon is denoted as **flow** (no equilibrium state occures). Thus a viscoplastic matter shows only viscous behaviour, without any elastic component. Therefore its rheological model does not incorporate any spring (element No. 1 or No. 2 – Table 1 in Part I) but it consists only of a non-linear viscous element (element No. 4 – Norton's element – liquid non-linear damper).

In contrast to viscous liquids, the flow occurs at viscoplastic matters only above a certain threshold stress value σ_{B}, so that the matter shows features of solid state for lower stress values. The dependence of stresses on the strain rate uses to be non-linear above the threshold load so that it can be described by the Norton's rheological model. This behaviour can be modelled as rigid-viscoplastic or elastic-viscoplastic, depending on the significance of the deformation corresponding to stress values below the threshold stress σ_{B}.

5.1. Perfect rigid-viscoplastic matter

① **General characteristics:** This is the simplest rheological model of a viscoplastic matter, represented by parallel arrangement of a non-linear viscous damper and skidding block

(Norton's element No. 4 and element No. 5 in Table 1, Part I, respectively, see Fig. 12). It is analogical to the perfect rigid-plastic model presented in Part I. The behaviour of Norton's element is described by the formula

$$\sigma = \lambda\,\dot{\varepsilon}^{\frac{1}{N}}\ .$$

This relation is derived from the Norton's law $\varepsilon_{\mathrm{p}} = (\sigma/\lambda)^N$, where ε_{p} is plastic strain. At a perfect rigid-viscoplastic matter, stress is then given by the following formula:

$$\sigma = \sigma_{\mathrm{B}} + \lambda \left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right)^{\frac{1}{N}}\ . \qquad \text{(g)}$$

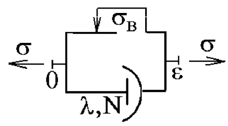


Fig. 12: Rheological model of a perfect rigid-viscoplastic matter

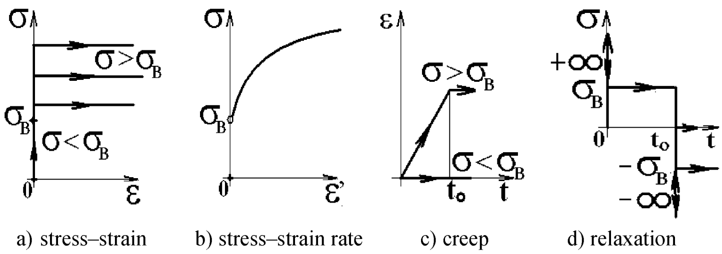


Fig. 13: Basic characteristics of a perfect rigid-viscoplastic matter

- In the standard tension test, the dependence of stress σ on strain ε shows two different regions (Fig. 13a):
  - for $\sigma < \sigma_{\mathrm{B}}$ the skidding block (element No. 5) is locked so that $\varepsilon = 0$,
  - for $\sigma \geq \sigma_{\mathrm{B}}$ the skidding block (element No. 5) is unlocked so that $\varepsilon \neq 0$ – the straight lines representing σ stress are parallel to the ε axis and their position depends on the stress value, or on the strain rate value according to eq. (g).
- In the standard tension test, it holds for the dependence of stress σ on strain rate $\dot{\varepsilon}$ (Fig. 13b):
  - according to the previous relation, stress of value $\sigma = \sigma_{\mathrm{B}}$ occurs even under zero strain rate $\dot{\varepsilon}$,
  - under a non-zero strain rate $\dot{\varepsilon}$ the dependence of stress on strain rate is monotonnously increasing.

② **Creep:** The strain change in time during creep test can be determined for two different regions:

- for $\sigma_0 < \sigma_{\mathrm{B}}$ the stress-strain dependence shows zero strain so that it remains zero also in creep test (Fig. 13c),
- for $\sigma = \sigma_0 \geq \sigma_{\mathrm{B}}$ the equation (g) is used as follows: the derivative $\mathrm{d}\varepsilon/\mathrm{d}t$ is expressed and then integrated:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \left(\frac{\sigma_0 - \sigma_{\mathrm{B}}}{\lambda}\right)^N \quad \rightarrow \quad \varepsilon = \int \left(\frac{\sigma_0 - \sigma_{\mathrm{B}}}{\lambda}\right)^N \mathrm{d}t \quad \Rightarrow \quad \varepsilon = \left(\frac{\sigma_0 - \sigma_{\mathrm{B}}}{\lambda}\right)^N t + c\ .$$

  The initial condition is the same as at the viscous liquid, i.e.: for $t = 0$ it holds $\varepsilon = 0$ so that $c = 0$ in the previous equation. Therefore one can conclude that the deformation is linear dependent on time for $\sigma_0 \geq \sigma_{\mathrm{B}}$ (unlimited creep, see Fig. 13c). After unloading in the time t_0, the strain remains constant.

③ **Relaxation:** The time dependence of stress under relaxation test can be also derived from eq. (g). If we express the derivative $\mathrm{d}\varepsilon/\mathrm{d}t$ from this equation, it must equal zero because relaxation is tested under constant strain-driven loading ($\varepsilon_0 = \text{const.}$). So we obtain $\sigma = \sigma_{\mathrm{B}}$. The following conclusions can then be drawn:

- A stepwise change in strain occurs at the time $t = 0$, so that the stress σ tends to infinity because also strain rate $\dot{\varepsilon} \to \infty$. This fact cannot be changed by the parallel skidding block.
- In the time interval $t \in (0, t_0)$, the strain rate equals zero and therefore the stress $\sigma = \sigma_{\mathrm{B}}$.
- In the time $t = t_0$ it holds $\sigma \to -\infty$, but the stress relaxes immediately to the value $\sigma = -\sigma_{\mathrm{B}}$.
- For time $t > t_0$ the stress holds the constant value $\sigma = -\sigma_{\mathrm{B}}$. The relaxation curve is presented in Fig. 13d.

Note: If the threshold stress $\sigma_{\mathrm{B}} = 0$, we obtain a model of **perfect viscoplastic matter**. This model shows creep under an arbitrary low stress value (no equilibrium state exists), so that it corresponds to a **non-linear viscous liquid**.

Aplications: approximation of behaviour of metals and alloys under temperatures higher than 1/3 of their absolute melting point.

6. Elastic-viscoplastic matters

These matters comprehend all the three basic reological properties, namely elasticity, viscosity and plasticity and their combinations. Elasticity is usually modelled as linear, viscoplasticity as non-linear. With respect to their behaviour after having reached the blocking stress (yield stress), perfect models (without stiffening) or models with (linear or multi-linear) stiffening are used.

6.1. Perfect elastic-viscoplastic matter

① **General characteristics:** Contrary to the perfect rigid-viscoplastic matter, the perfect elastic-viscoplastic matter shows a (linear) elastic deformation in addition to the plastic one. The behaviour of this matter can be described by **Bingham-Norton reological model** (see Fig. 14). It represents a system created by a linear spring (element No. 1) in series with two parallely connected elements, namely a non-linear viscous damper (Norton's element No. 4) and skidding block (element No. 5).

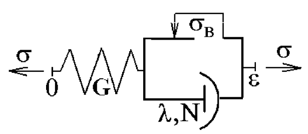


Fig.14: Rheological model of a perfect elastic-viscoplastic matter

This model can be also described as a connection of a spring with the rigid-viscoplastic rheological model, presented in Fig. 12. Function of the skidding block lies in blocking of the non-linear viscous damper under stresses $|\sigma| < \sigma_{\mathrm{B}}$, so that the strain of the Bingham-Norton rheological element is determined only by the properties of the spring in this range of stresses.

- For the stress-strain dependence the following relations hold (see Fig. 15a):
  - for $|\sigma| < \sigma_{\mathrm{B}}$ results in: $\varepsilon = \varepsilon_{\mathrm{e}} = \sigma/G$ – the stress-strain dependence is linear,
  - for $|\sigma| \geq \sigma_{\mathrm{B}}$ results in: $\varepsilon = \varepsilon_{\mathrm{e}} + \varepsilon_{\mathrm{p}}$; strain in a general time t consists of an elastic ε_{e} a plastic ε_{p} components.

- For the stress-strain rate $\dot{\varepsilon}$ dependence the following relations hold (see Fig. 15b):
  - for $|\sigma| < \sigma_{\mathrm{B}}$ results in: $\dot{\varepsilon} = 0$ and $\sigma = \sigma_{\mathrm{B}}$ – stress equals the blocking stress σ_{B} at zero strain rate $\dot{\varepsilon}$,
  - for $|\sigma| \geq \sigma_{\mathrm{B}}$ results in: $\sigma = \sigma_{\mathrm{B}} + \lambda\,\dot{\varepsilon}_{\mathrm{p}}^{1/N}$ – the curve in Fig. 14b is identical with the same curve for perfect rigid-viscoplastic material (Fig. 13b), but the stress achieves the value corresponding to this curve only after the time of value $t > t_{\mathrm{M}} = \sigma_{\mathrm{B}}/(G\,\dot{\varepsilon})$, at any strain rate during tension test, while the stress lies in the interval $\sigma \in (0, \sigma_{\mathrm{B}})$ for the time $t \in (0, t_{\mathrm{M}})$ and the deformation is only elastic in this interval.

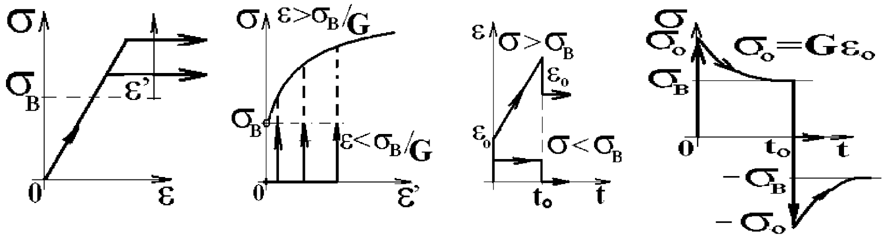


Fig.15: Basic characteristics of elastic-viscoplastic matter

② **Creep:** As there is a skidding block with a threshold stress value in the rheological model, it is necessary to distinguish between the following two cases:

- for $|\sigma_0| < \sigma_{\mathrm{B}}$: In this case the skidding block is locked and the behaviour is driven by the spring only, so that it corresponds to the Hooke's law. The matter behaves linear elastic, the strain is independent of time, given by the relation $\varepsilon = \sigma_0/G$ for any stress $\sigma_0 < \sigma_{\mathrm{B}}$ at any time. This behaviour is represented graphically by a straight line parallel to the time t axis (Fig. 15c), no creep occurs. The stress returns to zero at the time t_0.
- for $|\sigma_0| \geq \sigma_{\mathrm{B}}$: The skidding block is unlocked now, so that the Norton's element can move and model the non-linear viscosity. The creep behaviour is driven by the serial arrangement of the spring and the non-linear viscous damper. The resulting strain is given by addition of the strains of these two elements:

$$\varepsilon(t) = \varepsilon_{\mathrm{e}} + \varepsilon_{\mathrm{p}} = \frac{\sigma_0}{G} + \int \left(\frac{\sigma_0 - \sigma_{\mathrm{B}}}{\lambda}\right)^N \mathrm{d}t$$

$$\text{after integration:} \qquad \varepsilon(t) = \frac{\sigma_0}{G} + \left(\frac{\sigma_0 - \sigma_{\mathrm{B}}}{\lambda}\right)^N t\ .$$

The analysis of this relation results in the following statements:

- For $t = 0$, strain equals $\varepsilon_0 = \sigma_0/G$.
- For $t > 0$, time dependence of strain is linear (Fig. 15c), thus unlimited creep occurs.
- At the time $t = t_0$, i.e. after stress-driven unloading, the strain decreases by the value of ε_0.
- At times $t > t_0$, the strain value keeps a constant value.

③ **Relaxation:** Also here two of the basic cases should be distinguished:

- for $|\varepsilon_0| < \sigma_{\mathrm{B}}/G$: In this case the stress holds $\sigma < \sigma_{\mathrm{B}}$, the skidding block disables plastic deformation so that the deformation is given by the spring only and it is purely elastic. The stress $\sigma = G\,\varepsilon_0$ is constant in time and the situation is represented graphically by a straight line parallel with the t axis (Fig. 15d). No stress relaxation occurs.

– for $|\varepsilon_0| > \sigma_\mathrm{B}/G$: It holds $|\sigma| > \sigma_\mathrm{B}$ in this case. The time dependence of stress will be solved similary to the previous case. The relaxation of stress σ starts from a certain value $\sigma = \sigma_0 > \sigma_\mathrm{K}$ for $t = 0$, and the stress decreases exponencially with time. After having reached the value of $\sigma = \sigma_\mathrm{K}$, the stress keeps this constant value and becomes thus independent of time – it is represented graphically by a straight line parallel to the time axis (Fig. 15d). The exponential dependence of stress decrease can be derived from eq. (g) as follows:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma}{G}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\int\left(\frac{\sigma-\sigma_\mathrm{B}}{\lambda}\right)^N\right] = 0 \qquad \rightarrow$$

$$\rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma}{G}\right) + \frac{(\sigma-\sigma_\mathrm{B})^N}{\lambda^N} = 0 \qquad \rightarrow \qquad \frac{\mathrm{d}\sigma}{(\sigma-\sigma_\mathrm{B})^N} = G\,\frac{\mathrm{d}t}{\lambda^N}\ .$$

After integration and substitution of the boundary condition in the shape $t = 0 \Rightarrow \sigma = G\,\varepsilon_0 > \sigma_\mathrm{B}$, one obtains:

$$\sigma = \sigma_\mathrm{B} + \frac{G\,\varepsilon_0 - \sigma_\mathrm{B}}{\left[1 + \dfrac{(N-1)\,G}{\lambda^N}\,(G\,\varepsilon_0-\sigma_\mathrm{B})^{N-1}\,t\right]^{\frac{1}{N-1}}}\ .$$

6.2. Elastic-viscoplastic matter with stiffening

① **General characteristics:** This matter represents the most comprehensive and complex of the constitutive models reviewed here; above the threshold value (yield stress), the stress depends not only on the plastic strain rate but on the magnitude of this strain (or on another quantity defining the material stiffening). A linear stiffening is only analyzed below.

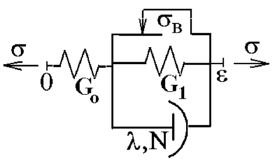


Fig.16: Rheological model of an elastic-viscoplastic matter with linear stiffening

The behaviour of this matter can be described using the rheological model presented in Fig. 16. Its structure corresponds to the rheological model of perfect elastic-viscoplastic matter (Fig. 14) with a spring of stiffness G_1 added in parallel to the skidding block. This spring induces a stress increase in tension test even above the value of threshold (blocking) stress σ_B, because the stress is function of the magnitude as well as of the rate of deformation in this model.

A. In a general case it holds:

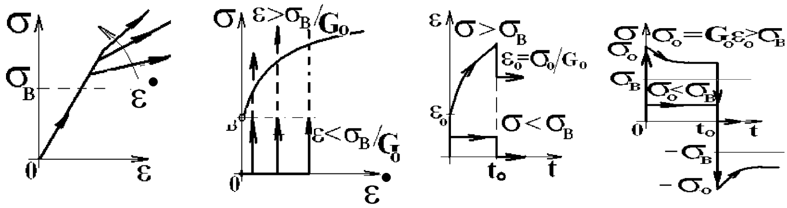
– for $|\sigma| < \sigma_\mathrm{B}$ we obtain $\varepsilon = \varepsilon_\mathrm{e} = \sigma/G_0$; the stress-strain dependence is linear,

– for $|\sigma| \geq \sigma_\mathrm{B}$ we obtain $\varepsilon = \varepsilon_\mathrm{e} + \varepsilon_\mathrm{p}$; at a general time t, strain consists of an elastic ε_e and plastic ε_p components. It holds for stresses: $\sigma = G_0\,\varepsilon_\mathrm{e} = f(\varepsilon_\mathrm{p}, \dot\varepsilon_\mathrm{p})$ while the dependence $f(\varepsilon_\mathrm{p}, \dot\varepsilon_\mathrm{p})$ can be of a general character.

B. For the case with linear stiffening it holds:

– for $|\sigma| < \sigma_\mathrm{B}$ it holds $\varepsilon = \varepsilon_\mathrm{e} = \sigma/G_0$; the stress-strain dependence is linear, similar to the general case,

– for $|\sigma| \geq \sigma_\mathrm{B}$ it holds $\varepsilon = \varepsilon_\mathrm{e} + \varepsilon_\mathrm{p}$; $\sigma = G_0\,\varepsilon_\mathrm{e} = \sigma_\mathrm{B} + G_1\,\varepsilon_\mathrm{p} + \lambda\,\dot{\varepsilon}_\mathrm{p}^{1/N}$; the dependence of stress on the plastic strain is also linear, while the slope of the straight line (in graphical representation) depends on the strain rate. It results from the properties of Norton's element (No. 4) and skidding block (No. 5) that the transition point between the elastic and plastic regions corresponds to various stress values different from the threshold stress σ_B.



a) stress–strain b) stress–strain rate c) creep d) relaxation

Fig.17: Basic characteristics of elastic-viscoplastic matter

② **Creep:** As there is a skidding block in the rheological model, creep occurs only for stress values $|\sigma_0| > \sigma_\mathrm{B}$ and it holds:

– For $t = 0$ the stress is given by elastic response $\varepsilon_0 = \sigma_0/G_0$.

– For $t > 0$ the time dependence of strain is non-linear (Fig. 17c), with the strain magnitude limited by G_1 spring – limited creep occurs.

– In time $t = t_0$, i.e. after removal of the stress-driven load, the strain decreases by the value of ε_0

– In time $t > t_0$ the strain value remains constant, if the stress magnitude σ_0 has not exceeded double σ_B.

③ **Relaxation:** It is also necessary to distinguish between the following two cases:

– for $|\varepsilon_0| < \sigma_\mathrm{B}/G_0$: stress $\sigma = G_0\,\varepsilon_0$ keeps constant – no stress relaxation occurs (Fig. 17d).

– for $|\varepsilon_0| > \sigma_\mathrm{B}/G_0$: in this case it holds $|\sigma| > \sigma_\mathrm{B}$. The dependence $\sigma(\varepsilon)$ can be solved similary to the par. 6.1 and the resulting stress relaxation curve (Fig. 17d) differs from the previous one (Fig. 15d) especially in the fact, that the stress does not tend to the value of σ_B with time going to infinity, but to a higher value given by summation of stresses σ_B acting in the skidding block and of the stress acting in the spring G_1.

Aplications: metals and alloys under medium and high temperatures, wood under high load.

7. Conclusion

The overview of constitutive models presented in the two presented papers tries to be comprehensive in description of constitutive relations of the matters in question; it resigns from attempts to be comprehensive in possible types of matters, of course. It aims mostly at materials used in technology (homogeneous in the technical sense), i.e. matters in solid state; only little attention is paid to fluids and their constitutive models. We have also not dealt with any directional dependence of material properties, all the models presented above are isotropic; any of the materials in question, however, can be anisotropic in some features, and then creation and exploitation of more complex constitutive models could be necessary.

In the case of multi-component materials, such as e.g. composite or porous materials, another feature can become substantial. If a characteristic dimension of the process in question (e.g. length of a stress wave) is on the same order with the characteristic dimensions of material components (e.g. fibre thickness at a fibre composite or particle size at a particle composite), then the classical continuum theory fails in giving realistic results; microcontinuum theories can be used then for a more accurate description of these materials. For example, **Cosserat linear elastic isotropic continuum** incorporates, in addition to the point displacement, also a local rotation of the point, and, in addition to the stress (a force per unit area), also a couple stress (torque per unit area). More information on these materials and theories can be found elsewhere.

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