

OPTIMISATION OF BUILDING STRUCTURES II (RBSO)

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Reliability-Based Structural Optimisation (RBSO) incorporates probabilistic structural reliability analysis into structural optimisation. A sample definition of an RBSO problem and its solution are presented for the optimisation of an RC cross-section, which is subjected to combinations of normal force and bending moments. The presented RBSO algorithm utilizes the LHS (Latin Hypercube Sampling) approximate simulation method for reliability computations. Numerical results for the particular data set are presented.

Keywords: optimization, building, structures, RBSO

1. Introduction

The principal task of RBSO is the calculation of reliability (failure probability), which is computationally demanding (due to the inherent multidimensional integrals), requires advanced knowledge, computer skills and entails more work for the designer. This is the cost of attaining a higher quality of structural design based on RBSO (in comparison with DBSO). Furthermore, the optimisation of real problems requires repeated computations involving updated design values, which, especially in the case of probability computations, implies a huge amount of calculation effort and time; therefore, a compromise between accuracy and computational claims must be accepted. It is advantageous to consider a minimal number of uncertainty sources, i.e. to identify those sources which most of all affect structural reliability, and possibly to replace the aggregate impacts of less significant sources. A number of approximate methods for failure probability assessment exist, e.g. FORM (First Order Reliability Method), SORM (Second Order Reliability Method) and Monte Carlo type methods (simulative).

The majority of existing studies attempt to use standard nonlinear optimisation algorithms to solve reliability-based optimal structural design problems. In the following list of studies probability quantities were involved as constraints or in target functions. These efforts include e.g. Murotsu and Shao [18], Kim and Kwak [19] regarding shape optimization; Mahadevan [20] concerning the design of frames; Liu and Moses [21] on the design of trusses; Lin and Frangopol [22] on the design of reinforced concrete structures. Maintenance planning for deteriorating structures was dealt with by Mori and Ellingwood [23] and Frangopol [24]. No attempts were made in these works to show that the resulting design problems satisfy the necessary requirements for the use of the standard nonlinear optimization algorithms which were employed. These requirements are that all the functions in the problem are continuously differentiable (smooth) and can be evaluated exactly in finite computational time. In the case of failure probability, which is generally defined as a multi-

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dimensional integral over an implicitly defined region, possibly discontinuous, these conditions are apparently not fulfilled. While one cannot prove that the exact failure probability is differentiable with respect to the design variables (Polak et al. [16]), one can easily show that approximations of the failure probability as obtained by first-order or second-order reliability methods (FORM, SORM) or Monte Carlo simulation are not differentiable. Under such conditions, standard nonlinear optimization algorithms may fail and may not converge to a solution of the problem.

In the FORM, an approximation of the failure probability is itself given by an optimisation problem, which is an unpleasant complication, therefore designers Madsen and Hansen [25], and later Kuschel and Rackwitz [26] replaced this inner optimisation problem with its necessary first-order optimality conditions and solved the resulting reliability-based optimal design problem by employing standard nonlinear algorithms. The transcribed problem requires second-order derivatives even for the solution of the first-order reliability approximating problem, which may be costly to compute. Furthermore, according to Luo et al. [27] this transcription may result in an optimization problem with a problematic constraint set. Researchers Moses [28] and Smilowitz and Madanat [29] have used Markov models to describe the evolution of system performance over time, leading to design problems of linear programming. Augusti et al. [30] considered a probabilistic model for a highway network, which led to a discrete nonlinear optimisation problem solvable by dynamic programming techniques. These formulations, though convenient from a computational standpoint, impose severe restrictions on the probabilistic models that can be used to describe structural behaviour.

Kirjner-Neto et al. [31] showed the equivalence between a special case of RBSO involving multiple component failure probabilities defined by affine limit-state functions and a semi-infinite optimisation problem; the presented outer approximation algorithm converges to stationary points. Kiureghian and Polak [32], Polak et al. [16] proved a similar equivalence for other cases of RBSO. To account for non-affine limit-state functions, they parameterised the semi-infinite optimization problem and solved a sequence of such semi-infinite problems for a range of parameter values. The parameter values were determined by separate calculations of the failure (by arbitrary method). These parameters serve as a FORM correction of such semi-infinite problems. Royset et al. [33, 34] contains an application of this technique to a series of structural systems.

Gasser and Schueller [35], Liaw and DeVries [36] used the ‘response surface’ method to approximate the failure probability via a smooth function defined in the space of design variables. This approach is numerically robust since the response surface is smooth and, unlike the real failure probability function, this substitute function is easily differentiated. However, the overall efficiency of the method strongly depends on the accuracy of the response surface and the computational cost of establishing it, which tends to be high for problems with many design variables.

There have also been attempts to apply gradient-free algorithms to solve optimal design problems with uncertainty. Genetic algorithms were used by e.g. Itoh and Liu [37], Nakamura et al. [38], Cheng and Ang [39], Thampan and Krishnamoorthy [40]. These algorithms are applicable to most reliability-based optimal design problems, including those that contain non-differentiable functions. However, they are known to have slow convergence, and the computational effort required to achieve a solution can be extremely high. On the boundary

between stochastic and deterministic optimization, there is some important literature on robust or worst-case scenario design, see e.g. Gu et al. [41, 42]. However, no probabilistic characterisations of uncertainties are employed in these approaches.

2. General formulation of RBSO

The general formulation of RBSO is analogical to DBSO formulation; see the preceding paper Optimisation of building structures I, differing in the occurrence of probabilistic quantities, i.e.

$$\vec{x}_{\text{opt}} = \arg \left\{ \underset{\vec{x}}{\text{bal}} \vec{f}(\vec{x}, \vec{p}_f(\vec{x})) \right\}, \quad (1)$$

where all restrictive conditions in the form of equalities

$$\vec{h}(\vec{x}_{\text{opt}}, \vec{p}_f(\vec{x}_{\text{opt}})) = 0, \quad (2)$$

and in the form of inequalities

$$\vec{g}(\vec{x}_{\text{opt}}, \vec{p}_f(\vec{x}_{\text{opt}})) \leq 0, \quad (3)$$

$$p_{f,j}(\vec{x}_{\text{opt}}) \leq p_{0f,j}, \quad (4)$$

are satisfied. Symbols in (1)–(4) mean

- bal ... symbolic operator calling for mutual criteria balance,
- $\vec{f}(\vec{x}, \vec{p}_f(\vec{x}))$... vector of target functions with possible inclusion of probability quantities $\vec{p}_f(\vec{x})$,
- $\vec{h}(\vec{x}, \vec{p}_f(\vec{x}))$... vector of restrictive conditions in the form of equalities with the possible inclusion of $\vec{p}_f(\vec{x})$,
- $\vec{g}(\vec{x}, \vec{p}_f(\vec{x}))$... vector of restrictive conditions in the form of inequalities with the possible inclusion of $\vec{p}_f(\vec{x})$,
- $p_{f,j}(\vec{x}_{\text{opt}})$... j -th member of the vector $\vec{p}_f(\vec{x})$ expressing the j -th probability of an inadmissible structural condition (e.g. failure),
- $p_{0f,j}$... upper bound of the probability quantity $\vec{p}_{f,j}(\vec{x})$.

3. Formulation of a DBSO RC cross-section task

The purpose of the task is quite identical with the RC cross-section optimisation formulated in the preceding paper Optimisation of building structures I (DBSO), section 4. This time, the cross-sectional bearing capacity against the interaction of normal force and bending moments will be evaluated in a probabilistic manner, thus it is an RBSO problem which must be solved. It is unnecessary to repeat the whole formulation in detail; it will suffice to note the differences between this and the previous one.

The objective criteria remain unchanged and all design variables $\vec{\mathbf{x}} = \{\vec{\mathbf{r}}^T, \phi^T, f_c, f_s\}^T$ are taken as continuous. The cross-section is stressed by a set of loading effects $\vec{\mathbf{L}}\vec{E}_j = (N, M_y, M_z)_j^T$, $1 \leq j \leq n_L$; they are described by random vectors with known joint probability distributions. The dependencies of vector components are for simplicity specified by second order cross moments, i.e. by correlation matrices. The design of the cross-section must bear every loading effect $\vec{\mathbf{L}}\vec{E}_j$ safely. Cross-section reliability is expressed by the probability of failure $p_{f,j}$, which depends on \vec{E}_j as well as on cross-section geometry, material

parameters and their stochasticity. The principal formulation difference thus lies in the enforcement of cross-section reliability (constraint), which is based on a direct probability approach (unlike DBSO, which utilises a partial factors method).

3.1. Bounds and constraints

Primary constraints are based on the Ultimate Limit State (ULS) of the cross-section exposed to normal force and bending moments. The primary set of restrictive conditions is based on the ULS (Ultimate Limit State) capacity of a cross-section loaded by normal force and bending moments during short-term stress, as described in [6]. The aim is to determine the failure probability p_f value. Let us consider the general ULS reliability condition with random quantities:

$$E(\xi) \leq R(\xi) , \quad (5)$$

where $E(\xi)$ is the unfavourable effect of actions and $R(\xi)$ is the cross-section resistance, and the formal quantity ξ symbolizes the randomness of the dependent quantity. If the effect and resistance are expressed as vectors $\bar{\mathbf{E}}(\xi)$ and $\bar{\mathbf{R}}(\xi)$ instead of scalars $E(\xi)$ and $R(\xi)$, respectively, it is possible to obtain expressions analogous with (5) by introducing new scalar quantities K_R, Z :

$$K_R \bar{\mathbf{E}}(\xi) = \bar{\mathbf{R}}(\xi) , \quad (6)$$

$$Z = |\bar{\mathbf{R}}(\xi)| - |\bar{\mathbf{E}}(\xi)| = (K_R - 1) |\bar{\mathbf{E}}(\xi)| , \quad (7)$$

$$Z \geq 0 , \quad (8)$$

where the same orientation of vectors $\bar{\mathbf{E}}(\xi)$ and $\bar{\mathbf{R}}(\xi)$ is assumed; K_R is called the ULS coefficient of capacity ($K_R > 0$) and Z is called the safety margin. Both of these scalars are random variables, because they depend on $\bar{\mathbf{E}}(\xi)$ and $\bar{\mathbf{R}}(\xi)$. Therefore, condition (6) is not yet correct and it is necessary to reformulate the derived constraint (8) conveniently:

$$p_f = P(Z < 0) , \quad (9)$$

$$p_f \leq p_f^{\max} , \quad (10)$$

where p_f is the failure probability restricted by the defined upper constraint p_f^{\max} . Thus the quartet of restrictive conditions (6), (7), (9) and (10) is obtained.

For the j -th loading case $\bar{\mathbf{L}}_j(\xi)$, corresponding to the effect of actions $\bar{\mathbf{E}}(\xi)$, the following formulas are derived from equations of equilibrium of cross-sections in the ULS:

$$N_j(\xi) = N_j^R(\sigma_{M(y,z)}(\varepsilon_j(y,z)); \bar{\mathbf{D}}(\xi, \bar{\mathbf{x}})) , \quad (11a)$$

$$K_{R,j} M_{y,j}(\xi) = M_{y,j}^R(\sigma_{M(y,z)}(\varepsilon_j(y,z)); \bar{\mathbf{D}}(\xi, \bar{\mathbf{x}})) , \quad (11b)$$

$$K_{R,j} M_{z,j}(\xi) = M_{z,j}^R(\sigma_{M(y,z)}(\varepsilon_j(y,z)); \bar{\mathbf{D}}(\xi, \bar{\mathbf{x}})) , \quad (11c)$$

$$\varepsilon_j(y,z) = \varepsilon_c^j + K_y^j y + K_z^j z , \quad (12)$$

$$\Delta \varepsilon_j^{\min} = \min \left\{ \begin{array}{l} \min_{\Omega_c}(\varepsilon_j(y,z) - \varepsilon_{c,\min}) \\ \min_q(\varepsilon_j(y_q, z_q) - \varepsilon_{st,\min}) \\ \min_{\Omega_c}(\varepsilon_{c,\max} - \varepsilon_j(y,z)) \\ \min_q(\varepsilon_{st,\max} - \varepsilon_j(y_q, z_q)) \end{array} \right\} , \quad (13)$$

$$\Delta \varepsilon_j^{\min} = 0 , \quad (14)$$

$$Z = (K_{R,j} - 1) \sqrt{M_{y,j}(\xi)^2 + M_{z,j}(\xi)^2} , \tag{15}$$

$$p_{f,j} = P(Z < 0) , \tag{16}$$

$$p_{f,j} \leq p_{f,j}^{\max} , \tag{17}$$

for each $j \in \{1 \dots n_L\}$, where the symbols used have the following meanings :

$N_j(\xi), M_{y,j}(\xi), M_{z,j}(\xi) \dots$ random components of loading effect $\bar{\mathbf{L}}\bar{E}_j(\xi)$,

$N_j^R, M_{y,j}^R, M_{z,j}^R \dots$ random components of internal forces of the cross-section, which are determined as integral functions,

$\bar{\mathbf{D}}(\xi, \bar{\mathbf{x}}) \dots$ a vector of all considered cross-section parameters; they may be random and dependent on the design variables $\bar{\mathbf{x}}$, explained in Section 3.1,

$\varepsilon_{c, \min}, \varepsilon_{c, \max} \dots$ boundaries of admissible strain for concrete,

$\varepsilon_{st, \min}, \varepsilon_{st, \max} \dots$ boundaries of admissible strain for steel,

$q \dots$ the index of the reinforcing bar, $q \in \{1 \dots N_\phi\}$,

$\Delta\varepsilon_j^{\min} \dots$ the strain margin of the cross-section, generally a continuous non-smooth function; if equal to zero, the cross-section is in the ULS.

The system of equations (11) represents equilibrium conditions between inner forces, $\bar{\mathbf{L}}_j(\xi)$ in terms of the external load and $\bar{\mathbf{R}}(\xi)$ in terms of cross-section response. This response appears as a change in the deformation state characterized by parameters of deformation $\varepsilon_c^j, K_y^j, K_z^j$; formula (12) preserves cross-sectional planarity after its deformation. Including $K_{R,j}$, there are four independent quantities that occur in (11) and (12). The definiteness of this system is assured by the additional limit state condition (14) (zero strain margin), hence the system of equations (11)–(14), which contains stochastic items and characterizes the joint probability distribution of the random vector $(\varepsilon_c, K_y, K_z, K_R, Z)_j^T$. Thus, the random variable Z is statistically computable and formula (16) is meaningful. With respect to $j \in \{1 \dots n_L\}$, restrictive conditions (11)–(17) are linked just by the design variables $\bar{\mathbf{x}}$, therefore (11)–(17) can be formally replaced by a single condition

$$p_{f,j}(\bar{\mathbf{x}}) \leq p_{f,j}^{\max} . \tag{18}$$

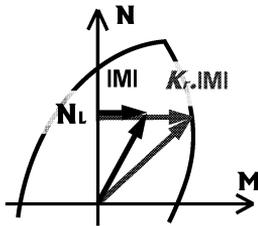


Fig.1: Illustration of a cross-section in the ULS resulting from system (11), where M stands for effect

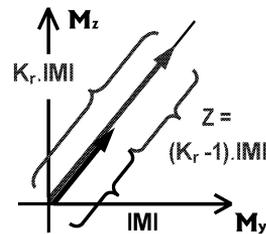


Fig.2: Illustration of formula (15) and the background of safety margin Z , where M stands for effect

4. Comments and specifications regarding RBSO

4.1. Stochastic model of a cross-section for $p_{f,j}(\mathbf{x})$ calculation

Vector $\bar{\mathbf{d}}$ denotes a set of cross-section parameters, e.g. the height and width of a rectangular cross-section, reinforcement cover and diameters, material strengths and so on; the

vector of design variables $\bar{\mathbf{x}}$ is a subset of $\bar{\mathbf{d}}$; this fact is in (2) expressed by dependence $\bar{\mathbf{d}}(\bar{\mathbf{x}})$. Cross-section parameters $\bar{\mathbf{d}}$ are considered to be mean values of corresponding random quantities $\bar{\mathbf{D}}(\xi)$, which are a source of uncertainties in $R(\xi)$. Letting $\bar{\mathbf{H}}_j(\xi)$ be a vector of all relevant random quantities which affect cross-section reliability, the equation then can be written as

$$\bar{\mathbf{H}}_j(\xi) = (\bar{\mathbf{L}}_j(\xi)^T, \bar{\mathbf{D}}(\xi)^T)^T. \quad (19)$$

$\bar{\mathbf{H}}_j(\xi)$ may depend on the design variables. Let us assume that each element of $\bar{\mathbf{H}}_j(\xi)$ depends on just a single corresponding design variable.

If both quantities (random variable and corresponding design variable) are marked by the same letter and the previous assumption stands, it can be succinctly written as

$$X^*(\xi) \sim F_{X^*}(x^*; x), \quad (20)$$

where $X^*(\xi)$ is a random variable with a distribution function, $F_{X^*}(x^*; x)$, and a mean value, x . On the other hand, one design variable can be a parameter for more random quantities, e.g. real strengths of reinforcement material are mutually independent, but usually belong to the same material class. If there is a more parametric distribution, e.g. normal or lognormal, where also a standard deviation occurs, dependence can be modified, for example, by the formula

$$\sigma(X^*(\xi)) = \gamma_{X^*} x, \quad (21)$$

where σ is the standard deviation and γ_{X^*} the coefficient of variation.

4.2. $p_f(\mathbf{x})$ calculation obstacles

The solution of the above-mentioned optimisation models requires the application of nonlinear optimisation algorithms. A couple of assumptions must be held:

- a) the existence of continuous derivatives at least of the first order with respect to all design variables,
- b) the computability of function values and their first order derivatives in the whole domain.

Calculation of reliability generally involves the necessity of computing a multi-dimensional integral over an implicitly defined region, so condition a) is practically never satisfied, see for example [3]. Approximate methods such as the FORM (First Order Reliability Method), SORM (Second Order Reliability Method) and Monte Carlo-like methods allow the bypassing of this problem by replacing the exact value $p_f(\bar{\mathbf{x}})$ by its estimation $\hat{p}_f(\bar{\mathbf{x}})$, which is much easier to calculate. Despite the impossibility of proving that $p_f(\bar{\mathbf{x}})$ is continuously differentiable, it can be shown that the estimations based on the above-mentioned approximate methods are not continuously differentiable (see [16]). This aspect results in malfunctions in nonlinear algorithms. Therefore, an approximate computational algorithm that meets both required conditions was proposed. It is based on LHS and makes several demands regarding random quantities:

- 1) Uncertainties are specified by continuous random quantities.
- 2) Their probability distributions are known in the form of distribution functions.
- 3) Distribution function parameters can depend on design variables.
- 4) This functional dependence is first order continuously differentiable.
- 5) Stochastic dependence among random quantities is given by a correlation matrix, independent of design variables.

5. Problem solution

Both the DBSO and RBSO problems were processed in the GAMS (General Algebraic Modelling System, see [7]), which is a high-level tool for mathematical programming. The main advantages of GAMS are the possibility of defining the problem formally using mathematical notation without a prescriptive definition of the solution method by choosing a solver, and the ability to implement any external programs or DLLs (Dynamic Link Libraries) in the form of a ‘black box’. DLLs are utilized for the calculation of cross-section responses by DBSO and for the calculation of cross-section reliability $p_f(\bar{\mathbf{x}})$ by RBSO.

A standard nonlinear GRG algorithm (General Reduced Gradient, detailed for example in [2]) is used for problem calculation; specifically, the CONOPT solver proved useful (see [4]). The optimisation progress being iterative, each iteration improves the objective function value, and therefore the initial solution matches the upper limits of the design variables of the stiffest cross-section and proceeds ‘from the top downwards’. The principles of the proposed approximate method for the calculation of $p_f(\bar{\mathbf{x}})$ are described in the following sections.

For the sake of transparency, let $N_L = 1$ and the number of design variables be 1, with random vector $\bar{\mathbf{H}}_1(\xi)$ thus being denotable as $\bar{\mathbf{H}}(\xi)$ and the vector $\bar{\mathbf{x}}$ as a scalar x .

5.1. Estimation of functional value $p_f(\mathbf{x})$

Let $\bar{\mathbf{H}}(\xi)$ be defined in accordance with demands 1) to 5) in Section 4.2 and contain N_H random quantities, i.e.

$$\bar{\mathbf{H}}(\xi) = \{H_1(\xi), \dots, H_i(\xi), \dots, H_{N_H}(\xi)\}^T, \tag{22}$$

where $i \in \{1 \dots N_H\}$, $H_i(\xi) \sim F_i(h_i; x)$. We are interested in the probability distribution of the random quantity Z , which is driven by the following relationship

$$\mathbf{u}(Z, K_R, \varepsilon_c, K_Y, K_Z; \bar{\mathbf{H}}(\xi)) = \mathbf{0}, \tag{23}$$

where $\mathbf{u}(\cdot) = \mathbf{0}$ is a system (11)–(15). The approximate solution will be achieved with the help of the LHS method, see [1]. Following the LHS method, it is necessary to choose the number of simulations N_S , divide the definition domains of $H_i(\xi)$ into N_S subintervals of the same probability $1/N_S$, and assign the k -th subinterval, $k \in \{1 \dots N_S\}$, a representative value h_i^k (Fig. 3). Therefore, a set of N_S deterministic values is obtained for all N_H random quantities.

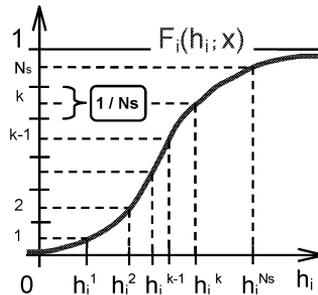


Fig.3: Principle of LHS sample generation according to a given cumulative distribution function $F_i(h_i; x)$

Each of the sets is submitted to permutation, and the representative values of random quantities in the k -th position after permutation belong to the k -th simulation $\bar{\mathbf{s}}_k$. It is convenient to work with a permutation table, which contains in each of its N_H columns a permutation of numbers $1 \dots N_S$ (the i -th column contains N_S indexes of representative values of the i -th random quantity $H_i(\xi)$), and each of the lines represents one simulation. However, the permutations cannot be carried out arbitrarily, because they control statistical dependence among simulated quantities, which should be in best accordance with the given correlation matrix. It is advantageous to base the comparison between achieved and required correlation on Spearman's coefficients of consecutive correlation, which are not dependent on representative values but only on the permutation table, the reason for which is explained in the following section. Now the computation of simulations

$$\mathbf{u}(z_k, k_{R,k}, \varepsilon_c, k_{y,k}, k_{z,k}; \bar{\mathbf{s}}_k) = \mathbf{0} , \quad (24)$$

must be performed specially for all of $k \in \{1 \dots N_S\}$. The set of z_k values is statistically computable and it is possible to obtain, for example, an estimation of mean value $\hat{\mu}(Z)$ standard deviation $\hat{\sigma}(Z)$ and estimates of higher order moments, and possibly to estimate the distribution law via an empirical cumulative distribution function, $\hat{F}_Z(z)$. If the probability law of Z is known (or estimated) except for some parameters, we can use point estimations instead and easily calculate \hat{p}_f . The calculation becomes more simplified if Gauss's normal distribution is assumed, and the following holds true:

$$\beta = \frac{\mu(Z(\xi))}{\sigma(Z(\xi))} \approx \frac{\hat{\mu}(Z(\xi))}{\hat{\sigma}(Z(\xi))} , \quad (25)$$

$$p_f = \Phi(-\beta) , \quad (26)$$

where $\hat{\mu}(\cdot)$ and $\hat{\sigma}(\cdot)$ are point estimations of mean value and standard deviation, β is Cornell's index of reliability, and $\Phi(\cdot)$ is the normal distribution function.

5.2. Estimation of function value $\partial p_f(x)/\partial x$

A smooth change in design variable x causes, in accordance with assumption 4) in Section 4.2, a smooth change in the shape of the distribution function $F_i(h_i; x)$, which results in a smooth change in representative values h_i^k for all of $k \in \{1 \dots N_S\}$. Furthermore, the correlation matrix is independent of design variables $\bar{\mathbf{x}}$, hence the corresponding Spearman's coefficients and permutation table remain constant and the members of simulation $\bar{\mathbf{s}}_k$ remain unchanged as well. As the functions in subsequent calculations (11)–(15), (25) and (26) are also smooth or sufficiently smooth, the estimation of failure probability \hat{p}_f is continuously differentiable with respect to x , and the condition a) in Section 4.2 is satisfied.

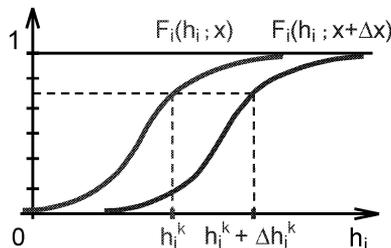


Fig.4: A smooth change in cumulative distribution function parameter(s) results in a smooth change in sample representative values

6. Numerical example

Let a simply supported beam be uniformly loaded all along its span l by its own weight g and imposed load q ; the beam is of constant RC rectangular cross-section.

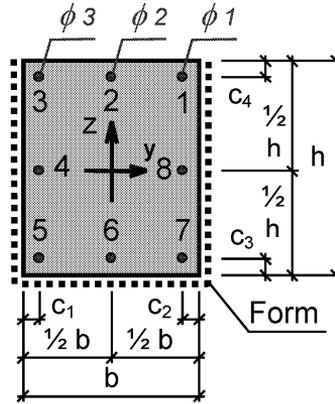


Fig.5: Example cross-section

The objective is to design all unknown cross-section parameters with the help of the proposed RBSO and DBSO algorithms and to compare the acquired results. Some of the parameters are given: strength classes of materials (f_c and f_s), number of reinforcement bars n_ϕ , and their positions in the cross-section (covers c_1 to c_4 , see Fig. 5). The cross-section is made of concrete C25/30 and steel B500(R) and reinforced with $N_\phi = 8$ reinforcement bars, of which the diameter intervals are $\phi_{1,3,5,7} \in \langle 10, 22 \rangle$ mm, $\phi_{2,4,6,8} \in \langle 0, 22 \rangle$ mm and cover $c_{1...4} = 40$ mm. Reinforcement bar diameters ϕ and cross-section dimensions b and h (width and height, respectively) are left for optimisation. The calculations are carried out in 5 variants of weighted sums $\alpha_p, \alpha_{co}, \alpha_{so}$, see Table 5, and 2 variants of cross-section dimensions, which are either a) optimised $b, h \in \langle 200, 600 \rangle$ mm or b) given (fixed) $b = 300$ mm, $h = 500$ mm.

6.1. Loading case specifications

Number of loading cases, $n_L = 1$. A simply supported beam is strained the most in the middle of the span by simple bending, so

$$\bar{\mathbf{L}}_1(\xi) = (0, M_y(\xi), 0)^T, \tag{27}$$

$$M_y(\xi) = -\frac{1}{8} [g(\xi) - q(\xi)] l^2, \tag{28}$$

where l is the effective span (8 m), $g(\xi)$ and $q(\xi)$ are random variables with mean values g and q , respectively, obtained from

$$g = g_k = b h \gamma_{cncr}, \tag{29}$$

$$q = q_k, \tag{30}$$

where b and h are the width and height of the cross-section (design variables), respectively, γ_{cncr} is the reinforced concrete unit weight (25 kN/m^3) and q_k is the given characteristic value of imposed load (14 kN/m).

6.2. Stochastic model parameters

Vector $\overline{\mathbf{H}}_1(\xi)$ is transparently shown in Table 1; the values were adopted from [11].

$\overline{\mathbf{H}}_1(\xi)$	Random variable	Mean value	Unit	γ or σ	
				Type	Value
$\overline{\mathbf{L}}_1(\xi)$	$g(\xi)$	g	kN/m	γ	0.213
	$q(\xi)$	9.333	kN/m	γ	0.304
$\overline{\mathbf{D}}_1(\xi)$	$b(\xi), h(\xi)$	b, h	mm	γ	0.02
	$f_c(\xi)$	25	MPa	γ	0.12
	$f_{s,1}(\xi) \dots f_{s,8}(\xi)$	490	MPa	γ	0.06
	$c_1(\xi) \dots c_4(\xi)$	40	mm	σ	5
	$\phi_1(\xi) \dots \phi_8(\xi)$	$\phi_1 \dots \phi_8$	mm	γ	0.05

Tab.1: Random quantities and their parameters

Each of the random quantities is assumed to be normally distributed without mutual correlations. The mean value of $q(\xi)$ is determined in such a way that value q_k is its 95% fractile. Multipliers γ_g and γ_q are relevant partial factors for permanent and variable loads in accordance with [5], $\gamma_g = 1.35$ and $\gamma_q = 1.50$.

6.3. Objective function parameters

All parameters needed in the objective function are contained in Tables 2–5, i.e. unit emission amounts (Table 2), unit costs (Table 3), reference values (Table 4) and summation weights (Table 5).

Amount	CO ₂	SO ₂
Concr kg/m ³	321	1.11
Steel kg/t	767	3.67

Tab.2: Unit emissions of cross-section construction materials (data from [8])

Concrete	Steel	Form
EUR/m ³	EUR/kg	EUR/m ²
83.29	0.675	14.22

Tab.3: Unit costs of construction materials and form

Cost	CO ₂	SO ₂
EUR	kg	kg
44.31	63.3	0.239

Tab.4: Reference values; derived from a $300 \times 500 \text{ mm}^2$ rectangular cross-section reinforced by 8 bars of 22 mm diameter (maximal allowable reinforcement)

Weight	1	2	3	4	5
Cost	1.00	0.750	0.500	0.250	0
CO ₂	0	0.125	0.250	0.375	0.500
SO ₂	0	0.125	0.250	0.375	0.500

Tab.5: Summation weights in objective function; five variants are considered

6.4. Other parameters and specifications

ULS strain limits for steel are $\varepsilon_{\min}^{\text{St}} = -3.5 \times 10^{-3}$, $\varepsilon_{\max}^{\text{St}} = 10 \times 10^{-3}$ and those for concrete are $\varepsilon_{\min}^{\text{C}} = -3.5 \times 10^{-3}$, $\varepsilon_{\max}^{\text{C}} = \infty$. The required cross-section reliability is given by the minimal Cornell safety index $\beta_{\min} = 3.8$, which corresponds to the maximal failure probability $p_f^{\max} \approx 7.2 \times 10^{-5}$. The number of simulations for p_f estimation is set to $N_S = 100$. $Z(\xi)$ is assumed to be normally distributed, therefore (25) and (26) are applicable. The proportion of b to h is constrained by a lower bound of 0.4. The cross-section is kept vertically symmetrical, so additional constraints must be set up for bar diameters:

$$\phi_1 = \phi_3, \tag{31a}$$

$$\phi_4 = \phi_8, \tag{31b}$$

$$\phi_5 = \phi_7. \tag{31c}$$

DBSO load design values g_d and q_d are determined in accordance with code [1], i.e.

$$g_d = \gamma_g g_k, \tag{32a}$$

$$q_d = \gamma_q q_k. \tag{32b}$$

6.5. Results

The numerical results are listed in Tables 6 and 7 and graphically represented by the accompanying Figures 6 to 11. The variant with fixed cross-section dimensions produced the same results for DBSO without regard to summation weight changes, thus all five cases are written in a single column, similarly as with RBSO.

Weight Dimensions	Weights 1-5	Weights 1	Weights 2	Weights 3	Weights 4	Weights 5
	Fixed	Optimised	Optimised	Optimised	Optimised	Optimised
$\phi_1 = \phi_3$	10.0	10.0	10.0	10.0	10.0	10.0
$\phi_5 = \phi_7$	21.5	22.0	22.0	22.0	22.0	22.0
ϕ_2 [mm]	0.0	0.0	0.0	0.0	0.0	0.0
$\phi_4 = \phi_8$	0.0	0.0	6.1	6.3	6.3	0.0
ϕ_6	20.9	22.0	22.0	22.0	22.0	22.0
Width b [mm]	300	200	200	200	200	200
Height h [mm]	500	454	446	446	445	454
Steel Area [mm ²]	1222	1297	1356	1361	1366	1297
Concrete Area [mm ²]	150000	90752	89201	89105	88991	90752
Form Length [mm]	1300	1108	1092	1091	1090	1108
Cost of Steel [EUR]	6.48	6.88	7.19	7.21	7.24	6.88
Cost of Concrete [EUR]	12.49	7.56	7.43	7.42	7.41	7.56
Cost of Form [EUR]	18.49	15.75	15.53	15.51	15.50	15.75
Total Cost [EUR]	37.46	30.19	30.15	30.15	30.15	30.19
Amount of CO ₂ [kg]	55.510	36.943	36.800	36.795	36.788	36.943
Amount of SO ₂ [kg]	0.202	0.13811	0.13809	0.13810	0.13812	0.13811
Target function [-]	0.846	0.681	0.655	0.63	0.605	0.581
Reliability Index [-]	3.8	3.8	3.8	3.8	3.8	3.8

Tab.6: Results of DBSO; all 'b, h = const' variants always led to the same results as shown in first column; reliability indexes for DBSO were calculated additionally after termination of the optimisation process

Weight Dimensions	Weights 1-5 Fixed	Weights 1 Optimised	Weights 2 Optimised	Weights 3 Optimised	Weights 4 Optimised	Weights 5 Optimised
$\phi_1 = \phi_3$	10.0	17.1	16.5	22.0	21.9	22.0
$\phi_5 = \phi_7$	22.0	22.0	22.0	22.0	22.0	22.0
ϕ_2 [mm]	0.0	14.8	22.0	15.8	20.6	21.5
$\phi_4 = \phi_8$	7.1	17.3	19.0	20.4	21.7	22.0
ϕ_6	22.0	22.0	22.0	22.0	22.0	22.0
Width b [mm]	300	254	244	235	228	227
Height h [mm]	500	424	406	392	381	378
Steel Area [mm ²]	1376	2244	2516	2753	2967	3024
Concrete Area [mm ²]	150000	107693	99038	92326	86878	85507
Form Length [mm]	1300	1102	1056	1020	989	982
Cost of Steel [EUR]	7.29	11.90	13.34	14.60	15.73	16.04
Cost of Concrete [EUR]	12.49	8.97	8.25	7.69	7.24	7.12
Cost of Form [EUR]	18.49	15.66	15.02	14.50	14.07	13.96
Total Cost [EUR]	38.27	36.53	36.61	36.79	37.04	37.11
Amount of CO ₂ [kg]	56.433	48.081	46.937	46.213	45.752	45.567
Amount of SO ₂ [kg]	0.206	0.18419	0.18241	0.18180	0.18191	0.18204
Target function [-]	0.877	0.825	0.808	0.788	0.766	0.742
Reliability Index [-]	4.38	4.51	4.48	4.47	4.43	4.42

Tab.7: Results of RBSO; all ' $b, h = \text{const}$ ' variants always led to the same results shown in first column; the minimal required reliability index was 3.8 (main restrictive condition of the optimisation)

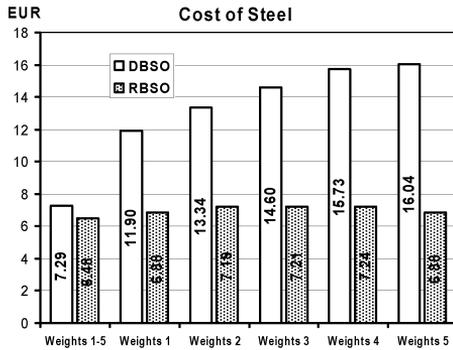


Fig.6: Costs of reinforcement : DBSO and RBSO

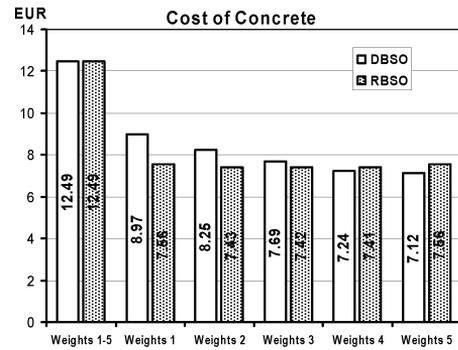


Fig.7: Costs of concrete : DBSO and RBSO

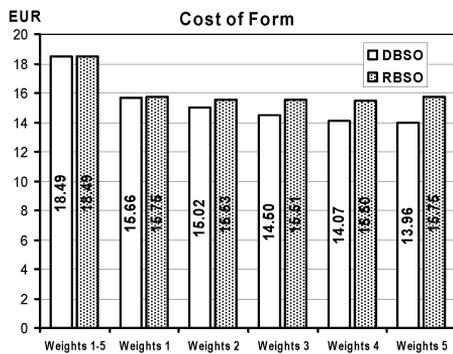


Fig.8: Form costs : DBSO and RBSO

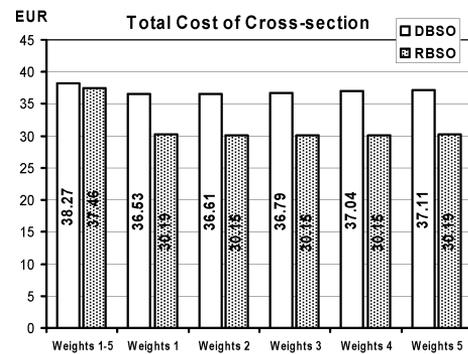


Fig.9: Total cross-section costs : DBSO and RBSO

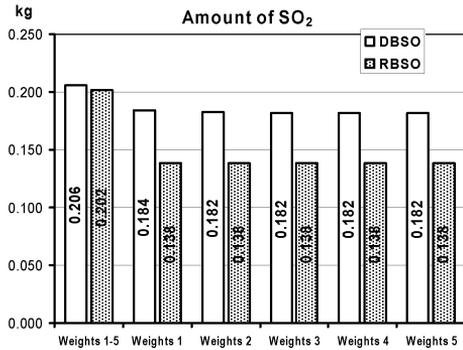


Fig.10: Amounts of emitted SO₂ :
DBSO and RBSO

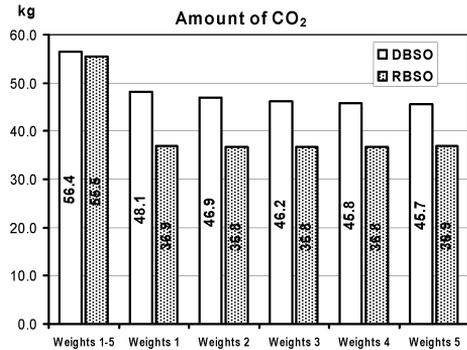


Fig.11: Amounts of emitted CO₂ :
DBSO and RBSO

The amount of steel computed by DBSO is always greater than that obtained by RBSO, and the amount of concrete is almost equal in all cases. Modifications of summation weights caused only small changes in the CO₂ and SO₂ emission results. There are substantial differences in the reliability indexes (on average 17% for ‘optimised dimensions’ cases) and total costs (on average 22% for ‘optimised dimensions’ cases).

- 1) Materially-optimised structures seem to be environmentally-friendly without there being any explicitly prescribed goal of taking into account ecological aspects. This is because material savings mean less pollution.
- 2) RBSO is more flexible in taking into account the varying contributions to the failure probability (according to the particular case) created by constituent random elements. Partial factors (used by corresponding DBSO problems) are fixed constants without regard to a particular case and are assigned in a more or less transparent way; therefore RBSO is a safer method than DBSO.
- 3) With the help of RBSO, it is possible to achieve a structural design with a well-proportioned level of failure probability (with respect to various failure modes), therefore the structure is economically and environmentally advantageous.

7. Summary

The RBSO approach for RC cross-section optimisation is presented and supported with an illustrative numerical example to compare the proposed approaches. Models were solved in GAMS (see [7]), which allows items such as arbitrary cross-section shape and structural materials such as FRP reinforcement (Fibre Reinforced Polymer) to be flexibly dealt with. The GAMS/CONOPT (see [4]) implementation of the GRG algorithm is used, as all design variables are considered to be continuous. The cross-section reliability of RBSO is based on failure probability p_f , which is estimated via the LHS method with an acceptable number of simulations N_S , and which controls the design accuracy. Stochastic correlation between random variables may be simply included. The results of the numerical examples document the applicability of the proposed RBSO approach.

8. Conclusions

From the related theory of stochastic programming, it follows that in most cases that involve random parameters, the stochastic approach is more advantageous than the de-

terministic approach based on the use of expectations. As the proposed RBSO approach represents one possible stochastic programming model, it is robust enough to be applicable for RBSO optimisation of whole structures (in pre-chosen critical cross-sections), not just in RC cross-sections, and the principles remain unchanged. The most important result of the presented theory is the fact that the authors have shown how to deal with the principal difficulties that must be solved when the stochastic programming approach is utilised in the optimum structural design area (design and material-related constraints, multi-criteria objectives, complex dependency structures, reliability requirements, etc.). The main reference (see [17]) shows that these difficulties have not allowed researchers to successfully apply the stochastic programming approach to optimum structural design in a straightforward manner as is typical in other application areas. An indirect approach combining specific techniques and motivated by the application area has proved to be more suitable and equivalent in terms of the stochastic programming interpretation of the final model.

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