DIFFERENT REFORMULATIONS OF STOCHASTIC OPTIMIZATION OF THE TRANSVERSE VIBRATION

Eva Žampachová, Pavel Popela*

The applicability of stochastic programming models and methods to PDE constrained stochastic optimization problem is discussed. The problem concerning the transverse vibration of a string is chosen. Therefore, the corresponding mathematical model involves a PDE-type constraint and an uncertain parameter related to the external load. A computational scheme for this type of problems is proposed, including discretization methods for random elements (scenario based two-stage stochastic programming) and the PDE constraint (finite difference method). Several deterministic reformulations are presented and compared using numerical and graphical results.

Keywords: *stochastic programming, PDE, finite difference method*

1. Introduction

Technical processes are very often governed by partial differential equations (PDEs) (e.g. [6]). In engineering practice, we usually need to control and optimize such processes. Consequently, we are facing PDE constrained optimization problems. Strictly speaking, we obtain optimal control problems. Theory of optimal control problems is very well developed for constraints in the form of ordinary differential equations (see [3]). However we are dealing with PDE constrained problems, and therefore, we have to challenge several difficulties. Usually, we have to approximate PDE's solution by discretization. Then, we can approximate our initial PDE constrained optimization problem by mathematical programs (e.g. [9]). Solution methods of these deterministic problems are relatively well developed.

But the real world cannot be approximated by deterministic approach every time, and therefore, we need to include stochastic behaviour of some elements of the considered problem. Hence, we obtain a stochastic optimization problem. Because we are interested in stochastic programming we would like to use its methods for solving such problems.

2. Stochastic optimization problem

As an illustrative example we consider the initial-boundary problem with hyperbolic equation describing the transverse vibration of a string

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2} + h(x,t) \ , \quad x \in \langle 0, l\rangle \ , \quad t \in \langle 0, T\rangle \ , \tag{1}$$

where l is the string length, $a^2 = \sigma/\mu$, σ is the tension in the string [Pa], μ is the mass of the string per unit length [kg m^{-1}], $v(x,t)$ is the displacement [m], $h(x,t)$ is the load [N kg^{-1}].

* Ing. Mgr. E. Žampachová, Ph.D., RNDr. P. Popela, Ph.D., Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology

The boundary conditions are

$$v(0,t) = 0 \ , \qquad v(l,t) = 0 \tag{2}$$

and the initial conditions are

$$v(x,0) = \varphi(x) \ , \qquad \frac{\partial v}{\partial t}(x,0) = \psi(x) \ . \tag{3}$$

We are interested in nondeterministic problems which are very frequent in practice and that's why we assume stochastic load $h(\xi, x, t)$, where $\xi : \Xi \to \mathbb{R}$ is the random variable, (Ξ, Σ, P) is the probability space and $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. We would like to push the vibrations into the required form therefore our objective will be the following:

$$z = \min_{f_1} \int_0^T \int_0^l \big(v(\xi,x,t) - u(x,t)\big)^2 \,\mathrm{d}x \,\mathrm{d}t \ , \tag{4}$$

where $u(x,t)$ is the required displacement [m] and $f_1(\xi, x, t)$ is the control (decision) variable.

This objective together with modified initial-boundary problem results in the underlying continuous stochastic optimization program:

$$z = \min_{f_1} \int_0^T \int_0^l \big(v(\xi,x,t) - u(x,t)\big)^2 \,\mathrm{d}x \,\mathrm{d}t \ , \tag{5}$$

$$\frac{\partial^2 v}{\partial t^2}(\xi,x,t) = a^2 \, \frac{\partial^2 v}{\partial x^2}(\xi,x,t) + f_1(\xi,x,t) + h(\xi,x,t) \ , \quad x \in \langle 0, l \rangle \ , \quad t \in \langle 0, T \rangle \ , \tag{6}$$

$$v(\xi,0,t) = 0 \ , \quad v(\xi,l,t) = 0 \ , \quad t \in \langle 0, T \rangle \ , \tag{7}$$

$$v(\xi,x,0) = \varphi(x) \ , \quad \frac{\partial v}{\partial t}(\xi,x,0) = \psi(x) \ , \quad x \in \langle 0, l \rangle \ . \tag{8}$$

This program provides only syntactically correct description because it depends on ξ, and we cannot compare optimal solutions obtained for different realizations of the random variable. Therefore, we use deterministic reformulations in order that random elements are correctly interpreted and the underlying program makes sense. In this paper, we will assume several deterministic reformulations. At first, the objective function will be optimized on average – expected objective (EO) reformulation, where the expected value $\mathbb{E}$ is taken with respect to a known probability measure P on (Ξ, Σ). It is clear that expected value does not guarantee that there are no outliers. Therefore, we will introduce min-max (MM) reformulation that guarantees avoiding the large fluctuations of the objective function and is therefore more risk averse. In fact, it minimizes the maximum of fluctuations so it is the most pesimistic approach. In applications, there can be found various requirements for optimization, e.g. to increase reliability of some equipment (especially in engineering). Therefore, our last reformulation will optimize probability – probabilistic objective (PO) reformulation (see e.g. [7]). The constraints given by equalities (12), (13) and (14) will be understood in the almost sure sense in all three cases described above.

In our illustrative problem as well as in many real engineering problems, the stage-related decision structure is more adequate than continuous time like in optimal control. As it is often necessary to make decision before a realization of the corresponding random variables becomes known and after it, two-stage stochastic programming can be used (see [10], [11] for the first corresponding models). It means that control function $f_1(\xi,x,t)$ is replaced by two types of the decision functions $f(x)$ and $g(\xi,x,t)$, where $f(x)$ is the first-stage decision variable (here-and-now approach because it does not depend on a realization of ξ) and $g(\xi,x,t)$ is the second-stage decision variable (wait-and-see approach, e.g. [8]).

Hence, we will solve three continuous stochastic optimization programs with one of the following objective functions:

$$z^{\mathrm{EO}} = \min_{f,\,g(\xi)} \mathbb{E}(F(\xi,f,g(\xi))) = \min_{f,\,g(\xi)} \mathbb{E} \int_0^T\!\!\int_0^l \big(v(\xi,x,t)-u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t \;, \tag{9}$$

$$z^{\mathrm{MM}} = \min_{f,\,g(\xi)} \max_{\xi} F(\xi,f,g(\xi)) = \min_{f,\,g(\xi)} \max_{\xi} \int_0^T\!\!\int_0^l \big(v(\xi,x,t)-u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t \;, \tag{10}$$

$$z^{\mathrm{PO}} = \min_{f,\,g(\xi)} P(F(\xi,f,g(\xi))>b) = \min_{f,\,g(\xi)} P\Big(\int_0^T\!\!\int_0^l \big(v(\xi,x,t)-u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t > b\Big) \;, \tag{11}$$

where $b \in \mathbb{R}$ is a certain upper bound for the optimal objective function value that we do not want to exceed.

The constraints are given by these equalities:

$$\frac{\partial^2 v}{\partial t^2}(\xi,x,t) = a^2\,\frac{\partial^2 v}{\partial x^2}(\xi,x,t) + f(x) + g(\xi,x,t) + h(\xi,x,t) \;, \quad x \in \langle 0,l\rangle \;, \quad t \in \langle 0,T\rangle \;, \quad \text{a.e.} \quad \xi \in \Xi \;, \tag{12}$$

$$v(\xi,0,t) = 0 \;, \quad v(\xi,l,t) = 0 \;, \quad t \in \langle 0,T\rangle \;, \quad \text{a.e.} \quad \xi \in \Xi \;, \tag{13}$$

$$v(\xi,x,0) = \varphi(x) \;, \quad \frac{\partial v}{\partial t}(\xi,x,0) = \psi(x) \;, \quad x \in \langle 0,l\rangle \;, \quad \text{a.e.} \quad \xi \in \Xi \;. \tag{14}$$

3. Approximations in the stochastic program

We are not able to solve the above continuous stochastic optimization problems without proper approximations. At first, we need to deal with the random data therefore we use the scenario based approach (see e.g. [1]). Then, we must approximate continuous solution of PDE by the space and time discretization.

We consider that the random variable ξ has a discrete distribution with a finite number R of possible realizations ξ_s (scenarios) with the uniformly distributed corresponding probabilities $p_s = P(\xi=\xi_s) = \frac{1}{R}$. In this case, $\mathbb{E}(F(\xi,f,g)) = \sum_{s=1}^{R} p_s\, F(\xi_s,f,g)$. Because of the easier numerical solution we introduce so-called nonanticipativity constraints (see [2]). It means that we relax the above programs by replacing the first stage decision function $f(x)$ by possibly different functions $f(\xi_s,x)$ and we add $f(\xi_s,x) = \sum_{k=1}^{R} p_k\, f(\xi_k,x)$, $s=1,\dots,R$. Nonanticipativity ensures that the first-stage decision variable does not depend on the second stage realization of the random variable.

The difference equations are derived from the partial differential equation via the finite differences method (e.g. [4]) with the uniform grid spacing $x_i = i\,d$, $i=0,\dots,N$, $d=l/N$

and $t_j = j\,\tau$, $j = 0,\dots,M$, $\tau = T/M$. For convenience we use the following notation: $v(\xi_s, x_i, t_j) = v_{s,i,j}$, $f(\xi_s, x_i) = f_{s,i}$, $g(\xi_s, x_i, t_j) = g_{s,i,j}$, $h(\xi_s, x_i, t_j) = h_{s,i,j}$, $\varphi(x_i) = \varphi_i$ and $\psi(x_i) = \psi_i$. The central-difference formulas for approximating $\partial^2 v(\xi, x, t)/\partial x^2$ and $\partial^2 v(\xi, x, t)/\partial t^2$ are

$$\frac{\partial^2 v}{\partial x^2}(\xi_s, x_i, t_j) = \frac{v_{s,i+1,j} - 2\,v_{s,i,j} + v_{s,i-1,j}}{d^2} + \mathcal{O}(d^2)\ , \tag{15}$$

$$\frac{\partial^2 v}{\partial t^2}(\xi_s, x_i, t_j) = \frac{v_{s,i,j+1} - 2\,v_{s,i,j} + v_{s,i,j-1}}{\tau^2} + \mathcal{O}(\tau^2)\ . \tag{16}$$

We drop the terms $\mathcal{O}(\tau^2)$ and $\mathcal{O}(d^2)$ and use the approximation $V_{s,i,j} \approx v_{s,i,j}$.

The difference formulas (15) and (16) are substituted into (12) and we get the difference equation of the transverse vibration for $i = 1,\dots,N-1$, $j = 1,\dots,M-1$, $s = 1,\dots,R$:

$$\frac{V_{s,i,j+1} - 2\,V_{s,i,j} + V_{s,i,j-1}}{\tau^2} = a^2\,\frac{V_{s,i+1,j} - 2\,V_{s,i,j} + V_{s,i-1,j}}{d^2} + f_{s,i} + g_{s,i,j} + h_{s,i,j}\ . \tag{17}$$

The substitution $r = a\,\tau/d$ is introduced in (17) and we obtain the relation

$$V_{s,i,j+1} = (2 - 2\,r^2)\,V_{s,i,j} + r^2\,(V_{s,i+1,j} + V_{s,i-1,j}) - V_{s,i,j-1} + \tau^2\,(f_{s,i} + g_{s,i,j} + h_{s,i,j})\ . \tag{18}$$

This method is explicit therefore to guarantee stability in formula (18), it is necessary that $r \le 1$.

The values for $i = 0$ and $i = N$ are given by the boundary conditions (13):

$$V_{s,0,j} = 0\ ,\quad V_{s,N,j} = 0\ ,\quad j = 0,\dots,M\ ,\quad s = 1,\dots,R\ . \tag{19}$$

The values corresponding to $j = 0$ and $j = 1$ must be supplied in order to use (18) to compute the values for $j = 2$. From the initial condition (14) we get

$$V_{s,i,0} = \varphi_i\ ,\quad i = 0,\dots,N\ ,\quad s = 1,\dots,R\ . \tag{20}$$

The values for $j = 1$ are constructed via the Taylor formula of order 2:

$$V_{s,i,1} = V_{s,i,0} + \dot{V}_{s,i,0}\,\tau + \frac{\ddot{V}_{s,i,0}\,\tau^2}{2}\ . \tag{21}$$

Hence, we obtain the following equation:

$$\begin{aligned} V_{s,i,1} = \varphi_i + \psi_i\tau + \frac{r^2}{2}\,(\varphi_{i+1} - 2\,\varphi_i + \varphi_{i-1}) + \frac{\tau^2}{2}\,(f_{s,i} + g_{s,i,0} + h_{s,i,0})\ ,\\ i = 1,\dots,N-1\ ,\quad s = 1,\dots,R\ . \end{aligned} \tag{22}$$

Concerning the objective functions we need to modify the probabilistic objective equivalent into the form suitable for computational purposes. For discrete finite probability distribution of ξ the following mixed-integer program solves (11) with constraints (12), (13) and (14):

$$\min_{z,\,f,\,g(\xi)} \left\{ z \,\middle|\, F(\xi_s, f, g(\xi_s)) \le b + M(1-\delta_s), \sum_{s=1}^{R} p_s(1-\delta_s) = z, \delta_s \in \{0{,}1\}, s = 1,\dots,R \right\}, \tag{23}$$

where $F(\xi_s, f, g(\xi_s)) - b$ is bounded from above by M for $\forall \xi_s \in \Xi$ and f, $g(\xi_s)$ which satisfy (12), (13) and (14).

From numerical results in the section 5, namely from the figure 8, one can see that the optimal value of the first-stage function strongly violates the smoothness which we would like to achieve. Therefore, we will slightly modify corresponding objective function by the additional demand on course of the first-stage function as follows.

$$z^{\mathrm{POmod}} = \min_{f(\xi),\, g(\xi)} P\Big(\int_0^T \!\! \int_0^l \big(v(\xi,x,t) - u(x,t)\big)^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^l \big(f(\xi,x) - w(x)\big)^2 \,\mathrm{d}x) > b \Big) \;, \tag{24}$$

where w is the required form of the first stage variable. In our case we will demand course of w similar to the optimal values of the first-stage functions from EO and MM reformulations.

The objective functions are discretized via the composite Simpson's rule ($a_0 = 1 = a_N$, $a_{2k+1} = 4$, $a_{2k+2} = 2$, $b_0 = 1 = b_M$, $b_{2k+1} = 4$, $b_{2k+2} = 2$ and N, M are even numbers):

$$z^{\mathrm{EO}} = \min \sum_{s=1}^{R} \sum_{j=0}^{M} \sum_{i=0}^{N} p_s \frac{d\,\tau}{9}\, a_i\, b_j\, (V_{s,i,j} - u_{i,j})^2 \;, \tag{25}$$

$$z^{\mathrm{MM}} = \min\Big\{ z \,\Big|\, z \geq \sum_{j=0}^{M} \sum_{i=0}^{N} \frac{d\,\tau}{9}\, a_i\, b_j\, (V_{s,i,j} - u_{i,j})^2,\; s = 1, \ldots, R \Big\} \;, \tag{26}$$

$$\begin{aligned} z^{\mathrm{PO}} = \min\Big\{ & \sum_{s=1}^{R} p_s\, (1 - \delta_s) \,\Big|\, \delta_s \in \{0,1\}, s = 1, \ldots, R, \\ & \sum_{j=0}^{M} \sum_{i=0}^{N} \frac{d\,\tau}{9}\, a_i\, b_j\, (V_{s,i,j} - u_{i,j})^2 \leq b + M(1 - \delta_s) \Big\} \;, \end{aligned} \tag{27}$$

$$\begin{aligned} z^{\mathrm{POmod}} = \min\Big\{ & \sum_{s=1}^{R} p_s\, (1 - \delta_s) \,\Big|\, \delta_s \in \{0,1\},\; \sum_{j=0}^{M} \sum_{i=0}^{N} \frac{d\,\tau}{9}\, a_i\, b_j\, (V_{s,i,j} - u_{i,j})^2 + \\ & + \sum_{i=0}^{N} \frac{d}{3}\, a_i\, (f_{s,i} - w_i)^2 \leq b + M\,(1 - \delta_s),\; s = 1, \ldots, R \Big\} \;. \end{aligned} \tag{28}$$

Finally, we set discretized nonanticipativity constraints

$$f_{s,i} = \sum_{k=1}^{R} p_k\, f_{k,i} \;, \quad i = 0, \ldots, N \;, \quad s = 1, \ldots, R \tag{29}$$

and add the following constraints for the second-stage variable:

$$g_{s,i,0} = 0 \;, \quad i = 0, \ldots, N, \quad s = 1, \ldots, R \;. \tag{30}$$

4. Approximating mathematical programs

After realization of the above approximations, we obtain large deterministic quadratic programs in case of EO and MM reformulation and mixed integer program in case of PO reformulation, which we summarize in matrix formulation. All constraints must be satisfied for $s = 1, \dots, R$.

$$z^{\mathrm{EO}} = \min \sum_{s=1}^{R} \sum_{j=0}^{M} \sum_{i=0}^{N} p_s \, \frac{d\,\tau}{9} \, a_i \, b_j \, (V_{s,i,j} - u_{i,j})^2 \ , \tag{31}$$

$$z^{\mathrm{MM}} = \min z \ , \tag{32}$$

$$z \geq \sum_{j=0}^{M} \sum_{i=0}^{N} \frac{d\,\tau}{9} \, a_i \, b_j \, (V_{s,i,j} - u_{i,j})^2 \ , \tag{33}$$

$$z^{\mathrm{PO}} = \min \sum_{s=1}^{R} p_s \, (1 - \delta_s) \ , \quad \delta_s \in \{0, 1\} \ , \tag{34}$$

$$\sum_{j=0}^{M} \sum_{i=0}^{N} \frac{d\,\tau}{9} \, a_i \, b_j \, (V_{s,i,j} - u_{i,j})^2 \leq b + M\,(1 - \delta_s) \ , \tag{35}$$

$$z^{\mathrm{POmod}} = \min \sum_{s=1}^{R} p_s \, (1 - \delta_s) \ , \quad \delta_s \in \{0, 1\} \ , \tag{36}$$

$$\sum_{j=0}^{M} \sum_{i=0}^{N} \frac{d\,\tau}{9} \, a_i \, b_j \, (V_{s,i,j} - u_{i,j})^2 + \sum_{i=0}^{N} \frac{d}{3} \, a_i \, (f_{s,i} - w_i)^2 \leq b + M\,(1 - \delta_s) \ , \tag{37}$$

$$V_{s,0,j} = 0 \ , \quad V_{s,N,j} = 0 \ , \quad j = 0, \dots, M \ , \tag{38}$$

$$V_{s,i,0} = \varphi_i \ , \quad i = 0, \dots, N \ , \tag{39}$$

$$\mathbb{V}_{s,1} = \Phi + \tau\,\Psi + \frac{1}{2}\,\mathbb{K}_1\,\Phi + \frac{\tau^2}{2}\,\mathbb{F}_{s,0} \ , \tag{40}$$

where $\Phi = (\varphi_1 \dots \varphi_{N-1})^{\mathrm{T}}$, $\Psi = (\psi_1 \dots \psi_{N-1})^{\mathrm{T}}$ and

$$\mathbb{K}_1 = \begin{pmatrix} -2\,r^2 & r^2 & 0 & \cdots & 0 \\ r^2 & -2\,r^2 & r^2 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & r^2 & -2\,r^2 & r^2 \\ 0 & \cdots & 0 & r^2 & -2\,r^2 \end{pmatrix} ,$$

$$\mathbb{V}_{s,j+1} = \mathbb{K}\,\mathbb{V}_{s,j} - \mathbb{V}_{s,j-1} + \tau^2\,\mathbb{F}_{s,j} \ , \quad j = 1, \dots, M-1 \ , \tag{41}$$

where

$$\mathbb{K} = \begin{pmatrix} 2 - 2\,r^2 & r^2 & 0 & \cdots & 0 \\ r^2 & 2 - 2\,r^2 & r^2 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & r^2 & 2 - 2\,r^2 & r^2 \\ 0 & \cdots & 0 & r^2 & 2 - 2\,r^2 \end{pmatrix} ,$$

$$\mathbb{V}_{s,j} = \begin{pmatrix} V_{s,1,j} \\ \vdots \\ V_{s,N-1,j} \end{pmatrix} , \quad \mathbb{F}_{s,j} = \begin{pmatrix} f_{s,1} + g_{s,1,j} + h_{s,1,j} \\ \vdots \\ f_{s,N-1} + g_{s,N-1,j} + h_{s,N-1,j} \end{pmatrix} ,$$

$$f_{s,i} = \sum_{k=1}^{R} p_k \, f_{k,i} \ , \tag{42}$$

$$g_{s,i,0} = 0 \ , \quad i = 0, \dots, N \ . \tag{43}$$

5. Solution

The numerical solution was found using optimization software GAMS and solvers CPLEX which can solve quadratic programming problems and BONMIN which can solve mixed-integer nonlinear programs. Vizualization of the solution for the constants $l = 1\,\mathrm{m}$, $a = 2\,\mathrm{s}^{-1}$,

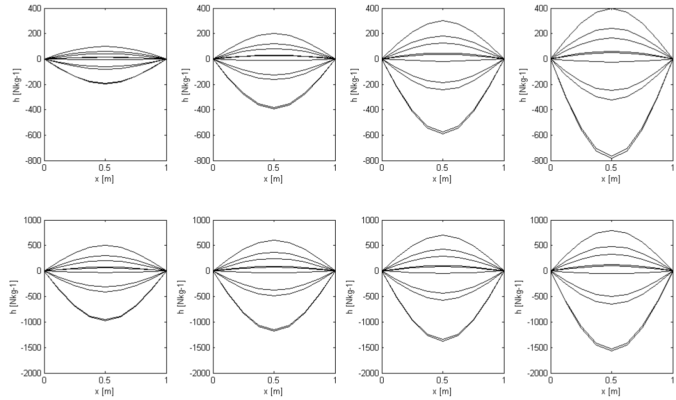

Fig.1: The load $h(\xi,x,t)$

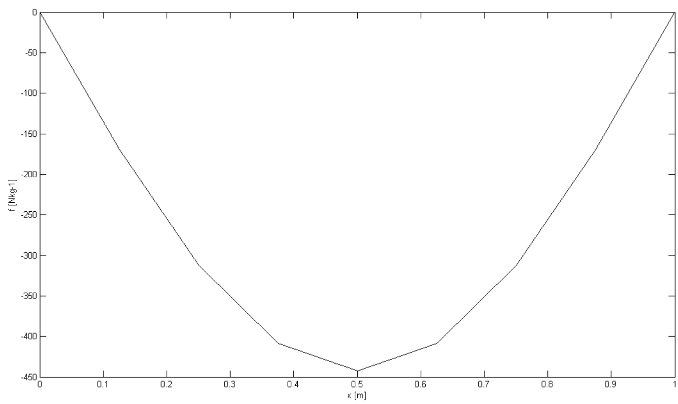

Fig.2: The first stage variable $f^{\mathrm{EO}}(x)$

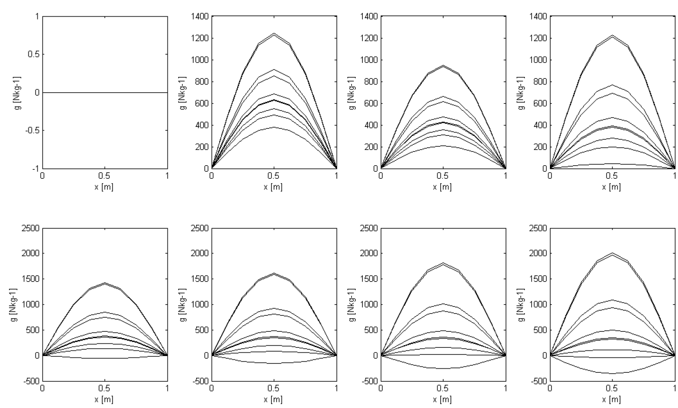

Fig.3: The second stage variable $g^{\mathrm{EO}}(\xi,x,t)$

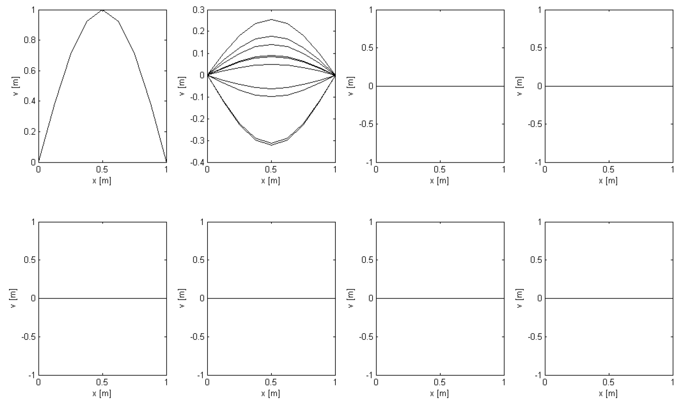

Fig.4: The displacement $v^{\mathrm{EO}}(\xi, x, t)$

$T = 0.5\,\mathrm{s}$, $b = 200$, $N = 8$, $M = 8$, $R = 10$, initial conditions $\varphi(x) = \sin \pi x$, $\psi(x) = 0$, $u(x,t) = 0$ and load $h(\xi, x, t) = -2\,b\,\xi\,\sin 2\pi x(t + \tau)$, $\xi \approx U(-5, 10)$ are in the figures 1–11. For modified objective function in case of PO reformulation, we consider the following form of the required first stage variable: $w(x) = -c \sin \pi x$, where $c = 430$ which is between the values given by EO and MM reformulations.

If we want to know how useful is the application of the stochastic programming approach we have to solve three problems. At first, we solve the stochastic programming problem mentioned above with the objective function (9) and we obtain the following value $z^{\mathrm{EO}} = 0.0119$. Then, the deterministic optimization approach is used when the random variable ξ is replaced by its mean $\mathbf{E}\,\xi$ and we solve the following optimization problem:

$$z^{\mathrm{EV}} = \min F\big(\mathbf{E}\,\xi,\ f,\ g(\mathbf{E}\,\xi)\big)\ . \tag{44}$$

$z^{\mathrm{EV}} = 0.0104$ and the optimal value of the first stage decision variable is $f^{\mathrm{EV}}_{\min}$, where EV means expected value. Finally the problem

$$z^{\mathrm{EEV}} = \min \mathbf{E}(F(\xi, f^{\mathrm{EV}}_{\min}, g(\xi))) \tag{45}$$

is solved and the solution is $z^{\mathrm{EEV}} = 0.0119$.

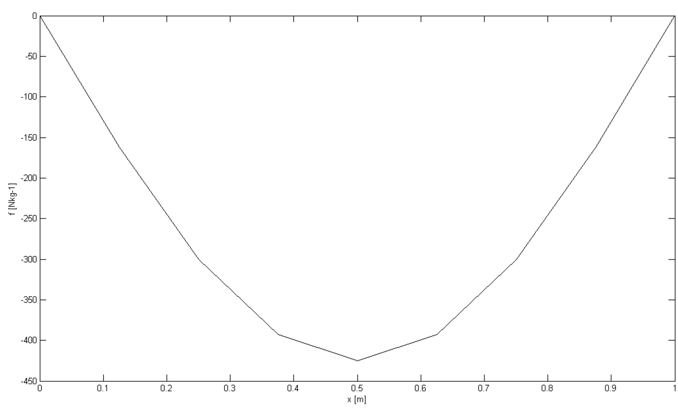

Fig.5: The first stage variable $f^{\mathrm{MM}}(x)$

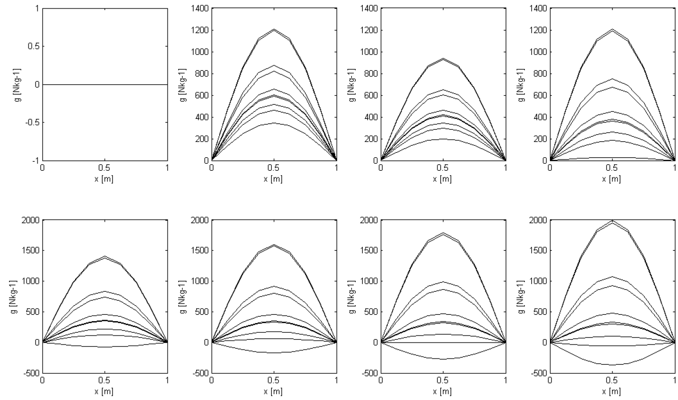

Fig.6: The second stage variable $g^{\mathrm{MM}}(\xi, x, t)$

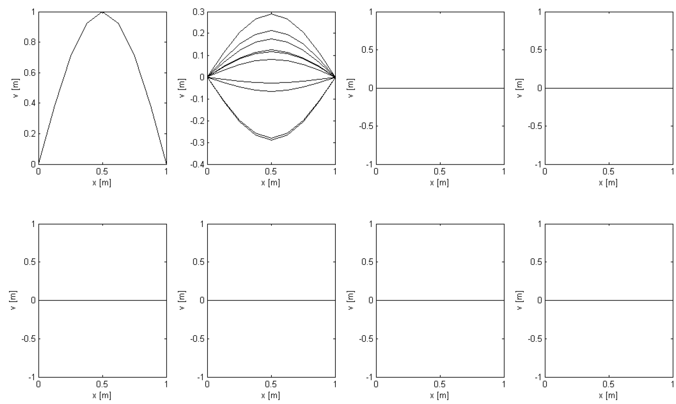

Fig.7: The displacement $v^{\mathrm{MM}}(\xi, x, t)$

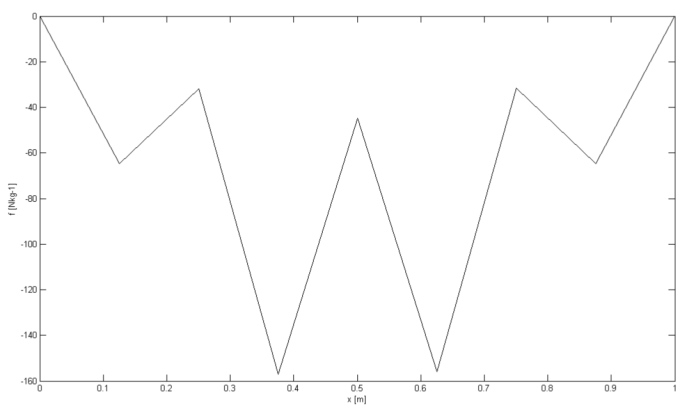

Fig.8: The first stage variable $f^{\mathrm{PO}}(x)$

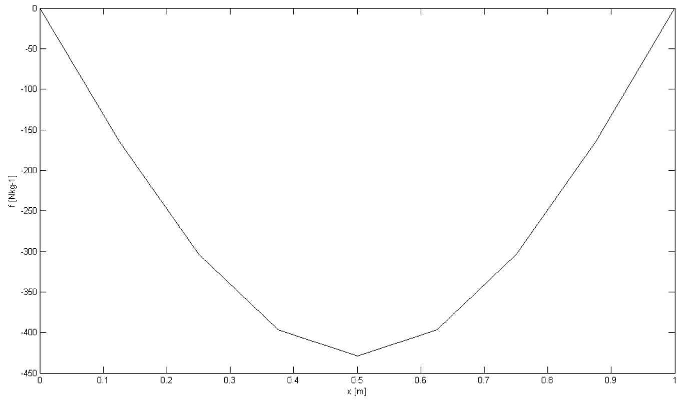

Fig.9: The first stage variable $f^{\mathrm{POmod}}(x)$

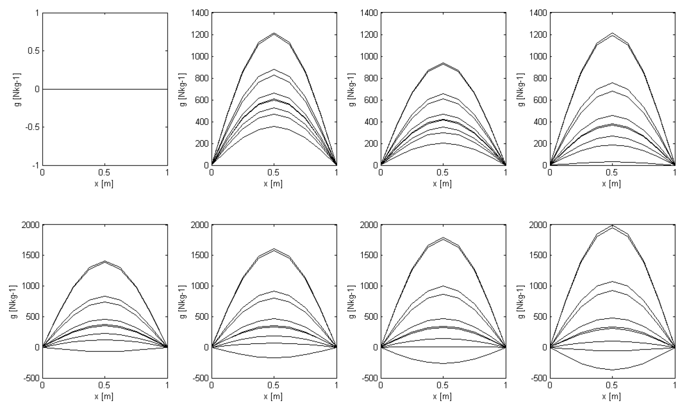

Fig.10: The second stage variable $g^{\mathrm{PO}}(\xi, x, t)$

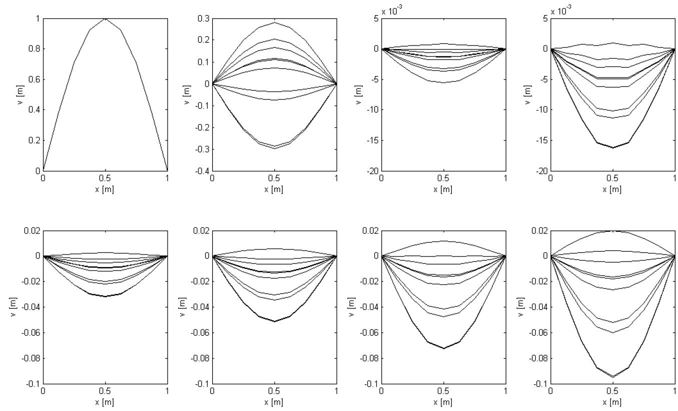

Fig.11: The displacement $v^{\mathrm{PO}}(\xi, x, t)$

Then, the value of stochastic solution $VSS = z^{\mathrm{EEV}} - z^{\mathrm{EO}}$ is the criterion for comparing stochastic and deterministic approach. The larger is this value the better is using stochastic approach. In our case, $VSS = 0$ so the deterministic and stochastic approaches give the same results.

For the second reformulation we obtain optimal value $z^{\mathrm{MM}} = 0.0139$ which is greater than optimal value for EO reformulation because this approach is more pessimistic and gives us the value of the objective function for the worst scenario.

Concerning the probabilistic reformulation we get $z^{\mathrm{PO}} = 0 = z^{\mathrm{POmod}}$. These values are the smallest from considered equivalents therefore it may seem that probabilistic reformulation is the best choice.

But the choice of the reformulation depends on the requirements which arise from practice and reformulation with larger objective value may give better results. In our case, the optimal solutions are very similar for all reformulations but for real engineering problems the differences may be larger. Furthermore, solution quality can be determined via Monte Carlo method as can be found in [12].

6. Conclusions

We proposed the numerical approach for solving PDE constrained stochastic optimization problem based on two-stage scenario based stochastic programming and simple discretization method. Several deterministic reformulations were presented with numerical and graphical results.

In the future, we would like to apply the presented approach to the engineering problems associated with reliability especially in the civil engineering (e.g. [5]).

Acknowledgement

This work was supported by projects from MSMT of the Czech Republic No. 2E08017 Procedures and Methods to Increase Number of Researchers and No. 1M06047 Center for Quality and Reliability of Production, by grant from Grant Agency of the Czech Republic (Czech Science Foundation) Reg. No. 103/08/1658 Advanced optimum design of composed concrete structures and by research plan from MSMT of the Czech Republic No. MSM0021630519 Progressive reliable and durable structures.

References

[1] Birge J., Louveaux F.: Introduction to Stochastic Programming, New York: Springer, 1997

[2] Kall P., Wallace S.W.: Stochastic Programming, New York: Wiley and Sons, 1995

[3] Lee E.B., Markus, L.: Foundations of Optimal Control Theory, Wiley, 1967

[4] Mathews J.H., Fink K.D.: Numerical Methods Using Matlab, New Jersey: Pearson Prentice Hall, 2004

[5] Plšek J., Štěpánek P.:Optimization of design of cross-section in concrete structures, In Proceding of the 4th International Conference Concrete and Concrete Structures, Žilina, Slovakia: University of Žilina, 2005, pp. 325–330

[6] Polyanin A.D.: Handbook of Linear Partial Differential Equations for Engineers and Scientists, Chapman & Hall/CRC, 2001

[7] Popela P.: Numerical Techniques and Available Software, In: Stochastic Modeling in Economics and Finance (authors: Dupačová J., Hurt J., Štěpán J.), Chapter 8 in Part II, Kluwer Academic Publishers, 2002, pp. 206–227

[8] Ruszczyński A., Shapiro A. (ed.): Handbooks in Operations Research and Management Science, vol. 10: Stochastic Programming, Amsterdam: Elsevier, 2003
[9] Tabak D., Kuo B.C.: Optimal Control by Mathematical Programming, Prentice-Hall, 1971
[10] Žampachová E.: Approximation of the continuous stochastic programs via mathematical programming (in Czech), In Proceedings from 16th seminar Modern mathematical methods in engineering, Ostrava, 2007, pp. 334–338
[11] Žampachová E., Popela P.: The selected PDE constrained stochastic programming problem, In Proceedings of the Risk, Quality and Reliability Conference, Ostrava, 2007, pp. 233–237
[12] Žampachová E:. Determining solution quality of the selected stochastic programming problem via Monte Carlo method (in Czech). In Proceedings of conference CQR – REQUEST '08, Brno, 2008, pp. 244–249

Received in editor's office: January 3, 2009
Approved for publishing: December 21, 2009

Note: This paper is an extended version of the contribution presented at the international conference *STOPTIMA 2007* in Brno.