WEIBULL FUZZY PROBABILITY DISTRIBUTION FOR RELIABILITY OF CONCRETE STRUCTURES

Zdeněk Karpíšek*, Petr Štěpánek**, Petr Jurák*

Based on the fuzzy probability distribution and its properties, the paper defines the fuzzy reliability and its characteristics for the double-state probability model of object. Two fuzzy reliability models are described that are based on the Weibull fuzzy distribution. The results can be applied to determining the reliability of real objects in cases where pre-failure times are of a vague numerical type.

Keywords : *fuzzy probability distribution, fuzzy reliability, Weibull fuzzy distribution, concrete structures*

1. Introduction

The design of concrete structures and their mathematical modeling is rather subjective in its nature. Ordered increasingly with respect to the cost of input information acquisition, a comparison shows that deterministic analyses are the cheapest, however, their results are of limited validity. Probabilistic analyses provide the designer with extensive information including the distribution of the sought quantities, however, the input data acquisition is considerably expensive and, in some cases such as the design or calculation of residual lifetime of unique structures, its use is irrelevant due to the lack of knowledge about the input parameters.

Whereas reliability theory only defines the likelihood of outcomes (does an event occur?), fuzzy logic is an excellent means to describe the extent or frequency to which an event occurs also if few samples are available.

Evaluation of the safety level in concrete structures should be carried out considering the stochastic behavior of the main parameters involved, not only under ultimate conditions, but also under serviceability and durability conditions. Particularly in concrete structures, the large variability of mechanical and rheological parameters may give rise to significant deviations from the expected behavior if a deterministic approach is used. On the other hand, it is well known that the probability density function and its parameters cannot be univocally defined. To overcome this problem, the fuzzy probabilistic theory may be used in the processing of stochastic parameters, taking into account their fuzzy nature.

In order to preserve also the less probable data items in addition to the 'completely' true ones, we can describe them using tools of the theory of fuzzy sets. They particularly include the description of each quantitative piece of data in the form of a fuzzy number where the membership function value corresponds to the 'probability' of the quantity measured.

* doc. RNDr. Z. Karpíšek, CSc., Ing. P. Jurák, Ph.D., Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2, 616 69 Brno, Czech Republic

** prof. RNDr. Ing. P. Štěpánek, CSc., Institute of Concrete and Masonry Structures, Faculty of Civil Engineering, Brno University of Technology, Veveří 95, 602 00 Brno, Czech Republic

This approach then makes it possible to model the reliability of an object as a reliability system with different membership degrees, using fuzzy reliability, based on the notion of fuzzy probability [1, 4]. Hence, for example, we can obtain the fuzzy hazard function and the fuzzy mean value of the time to failure. Even though a reverse interpretation does not lead to a 'focusing' of the resulting information, it does make it possible to assess the values of the characteristics analyzed in a maximum range including the degrees of membership of their individual values as measures of their reliability. Our fuzzification of the stochastic model of reliability is based on the following notions [1, 4].

Let $\Omega \neq \emptyset$ be a universal set (*basic space*). A fuzzy set $\underset{\sim}{A} = (\Omega, \mu_{\underset{\sim}{A}})$ with a Borel measurable membership function $\mu_{\underset{\sim}{A}}$ is called *fuzzy event*. Every crisp random event $A \subset \Omega$ is evidently a fuzzy event. We define the relationships $\subset$, $\subseteq$, $=$ and operations $\bar{}$, $\cap$, $\cup$, $\cdot$, $+$ with fuzzy events also for fuzzy sets.

A nonempty set Σ of fuzzy events $\underset{\sim}{A} = (\Omega, \mu_{\underset{\sim}{A}})$ is called a *fuzzy Borel field of fuzzy events* over the universal set Ω, if Σ has the following properties:

1. $A \in \Sigma, \forall \alpha \in [0,1] \Rightarrow \alpha A \in \Sigma$.
2. $\underset{\sim}{A} \in \Sigma \Rightarrow \bar{\underset{\sim}{A}} \in \Sigma$.
3. $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots \in \Sigma \Rightarrow \bigcup_{i=1}^{\infty} \underset{\sim}{A}_i \in \Sigma$.
4. $\underset{\sim}{A}_1, \underset{\sim}{A}_2 \in \Sigma \Rightarrow \underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2 \in \Sigma$.

Let $\Omega = \mathbb{R}^m$ ($\mathbb{R}$ denotes the set of all real numbers) where $m \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of all natural numbers) be a universal set, $\mathit{\Sigma}$ a crisp Borel field of random events over Ω, Π a nonempty set of probability measures P on $(\Omega, \mathit{\Sigma})$, Σ a fuzzy Borel field of fuzzy events on Ω, and $\underset{\sim}{A} = (\Omega, \mu_{\underset{\sim}{A}}) \in \Sigma$ a fuzzy event. Let, further, $\underset{\sim}{P} = (\Pi, \mu_{\underset{\sim}{P}})$ be such a fuzzy bunch on Π that $\exists P \in \Pi$ and $\mu_{\underset{\sim}{P}}(P) = 1$. Then the fuzzy bunch $\underset{\sim}{P}$ is called a *fuzzy probability* on Ω and the *fuzzy probability of a fuzzy event* $\underset{\sim}{A}$ is the fuzzy set $\underset{\sim}{P}(\underset{\sim}{A}) = ([0,1], \mu_{\underset{\sim}{P}(\underset{\sim}{A})})$ where

$$\mu_{\underset{\sim}{P}(\underset{\sim}{A})}(p) = \sup_{\substack{P \\ \int_\Omega \mu_{\underset{\sim}{A}} \,\mathrm{d}P = p}} \mu_{\underset{\sim}{P}}(P)$$

for $\forall p \in [0,1]$. If no measure $P \in \Pi$ exists such that $\int_\Omega \mu_{\underset{\sim}{A}} \,\mathrm{d}P = p$, we put $\mu_{\underset{\sim}{P}(\underset{\sim}{A})} = 0$. The triplet $(\mathbb{R}^m, \Sigma, \underset{\sim}{P})$ is called a *fuzzy probability space* on $\mathbb{R}^m$.

The restriction $\Omega = \mathbb{R}^m$ is given by the application framework and by the possibility of specifying probability measures by distribution functions of random variables. Every crisp probability P is evidently a particular instance of a fuzzy probability $\underset{\sim}{P}$.

Usually, we express the fuzzy probability by means of fuzzy real numbers [4]. The properties of fuzzy real numbers and the extended binary arithmetic operations $\oplus$, $\ominus$, $\odot$, $\oslash$ are described in [2, 3].

2. Fuzzy probability distributions

If probability measures P for a fuzzy probability $\underset{\sim}{P}$ are given by distribution functions $F(x) = P(X < x)$ where X are random variables on $\mathbb{R}^m$, the notion of a random variable can be extended. For simplicity, we put $m = 1$. The theorems in this part describe the fundamental attributes of fuzzy probability distribution. Proofs of these theorems and additional propositions are shown in [6].

Definition 1. Let X be a nonempty set of random variables X on $\mathbb{R}$, Φ a set of their distribution functions and Σ a fuzzy Borel field over $\Omega = \mathbb{R}$. Let $\underset{\sim}{F} = (\Phi, \mu_{\underset{\sim}{F}})$ be a fuzzy bunch of distribution functions F on $\mathbb{R}$ such that $\exists F \in \Phi$ and $\mu_{\underset{\sim}{F}}(F) = 1$ for $\forall x \in \mathbb{R}$. The fuzzy set $\underset{\sim}{X} = (X, \mu_{\underset{\sim}{X}})$ where $\mu_{\underset{\sim}{X}}(X) = \mu_{\underset{\sim}{F}}(F)$, for $\forall X$, is called a *fuzzy random variable* and its *fuzzy distribution function* is the fuzzy bunch $\underset{\sim}{F} = (\Phi, \mu_{\underset{\sim}{F}})$. The pair $(\underset{\sim}{X}, \underset{\sim}{F})$ is called a *fuzzy distribution of probability*.

Remark 1. By the *fuzzy value of a fuzzy random variable* $\underset{\sim}{X}$ we understand an arbitrary fuzzy event from the fuzzy Borel field Σ. Then the fuzzy probability of a fuzzy random variable $\underset{\sim}{X}$ assuming the fuzzy value $\underset{\sim}{A} = (\mathbb{R}, \mu_{\underset{\sim}{A}})$ is the fuzzy probability $\underset{\sim}{P}(\underset{\sim}{A}) = ([0,1], \mu_{\underset{\sim}{P}(\underset{\sim}{A})})$ where

$$\mu_{\underset{\sim}{P}(\underset{\sim}{A})}(p) = \sup_{\substack{F \\ \int_{\mathbb{R}} \mu_{\underset{\sim}{A}}\, \mathrm{d}F = p}} \mu_{\underset{\sim}{F}}(F) \ .$$

In particular, for $\forall x \in \mathbb{R}$, the *value of the fuzzy distribution function* $\underset{\sim}{F}(x) = \underset{\sim}{P}(\underset{\sim}{X} < x)$ is the fuzzy probability $\underset{\sim}{Y} = ([0,1], \mu_{\underset{\sim}{Y}})$ where

$$\mu_{\underset{\sim}{Y}}(y) = \sup_{\substack{F \\ F(x) = y}} \mu_{\underset{\sim}{F}}(F(x)) \ .$$

If all the distribution functions F are absolutely continuous, then the fuzzy random variable $\underset{\sim}{X}$ is continuous and can be specified by the so-called *fuzzy density of probability*, that is, by the fuzzy bunch $\underset{\sim}{f}$ of densities of probability f. In this case, the Lebesgue-Stieltjes integral is equal to the Riemann integral.

Definition 2. Let $\underset{\sim}{P}$, $\underset{\sim}{Q}$ be fuzzy probabilities. We say that $\underset{\sim}{P}$ is *equal to or less than* $\underset{\sim}{Q}$ writing $\underset{\sim}{P} \trianglelefteq \underset{\sim}{Q}$ if $\exists q \in \mathrm{Ker}(\underset{\sim}{Q})$ such that, for $\forall p \in \mathrm{Ker}(\underset{\sim}{P})$, $p \leq q$. By analogy, we define the relations $\underset{\sim}{P} \trianglerighteq \underset{\sim}{Q}$, $\underset{\sim}{P} \triangleleft \underset{\sim}{Q}$ and $\underset{\sim}{P} \triangleright \underset{\sim}{Q}$. For crisp probabilities P, Q the relations $\trianglelefteq$ and $\leq$ they are identical, etc.

Definition 3. Let $\underset{\sim}{x} = (\mathbb{R}, \mu_{\underset{\sim}{x}})$ be a fuzzy number. We say that the value of a fuzzy random variable $\underset{\sim}{X}$ is *less than* a fuzzy number $\underset{\sim}{x}$ writing $\underset{\sim}{X} \lesssim \underset{\sim}{x}$ if its value is a fuzzy event $\underset{\sim}{B} = (\mathbb{R}, \mu_{\underset{\sim}{B}})$ where

$$\mu_{\underset{\sim}{B}}(x) = \begin{cases} 1 - \mu_{\underset{\sim}{x}}(x) \ , & x \in (-\infty, \inf \mathrm{Ker}(\underset{\sim}{x})] \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

We say that the value of a fuzzy random variable $\underset{\sim}{X}$ is *greater than or equal to* a fuzzy number $\underset{\sim}{x}$ writing $\underset{\sim}{X} \gtrsim \underset{\sim}{x}$ if its value is a fuzzy event $\underset{\sim}{B} = (\mathbb{R}, \mu_{\underset{\sim}{B}})$ where

$$\mu_{\underset{\sim}{B}}(x) = \begin{cases} \mu_{\underset{\sim}{x}}(x) \ , & x \in (-\infty, \sup \mathrm{Ker}(\underset{\sim}{x})] \ , \\ 1 & \text{otherwise} \ . \end{cases}$$

By analogy, we define the relations $\underset{\sim}{X} \lessapprox \underset{\sim}{x}$ and $\underset{\sim}{X} \gtrsim \underset{\sim}{x}$.

Theorem 1. If $\underset{\sim}{x} = (\mathbb{R}, \mu_{\underset{\sim}{x}})$, $\underset{\sim}{x}_1 = (\mathbb{R}, \mu_{\underset{\sim}{x}_1})$ and $\underset{\sim}{x}_2 = (\mathbb{R}, \mu_{\underset{\sim}{x}_2})$ are fuzzy numbers, then, for the fuzzy value of the fuzzy distribution function $\underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{P}(\underset{\sim}{X} \lesssim \underset{\sim}{x})$, the following holds:

a) $0 \leq \underset{\sim}{F}(\underset{\sim}{x}) \leq 1$ and $\underset{\sim}{P}(\underset{\sim}{X} \geq \underset{\sim}{x}) = 1 \ominus \underset{\sim}{F}(\underset{\sim}{x})$ for all fuzzy numbers $\underset{\sim}{x}$,

b) $\underset{\sim}{F}(\underset{\sim}{x}_1) \trianglelefteq \underset{\sim}{F}(\underset{\sim}{x}_2)$ for arbitrary fuzzy numbers $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ such that $\inf \mathrm{Ker}(\underset{\sim}{x}_1) < \inf \mathrm{Ker}(\underset{\sim}{x}_2)$ and $\inf \mathrm{Supp}(\underset{\sim}{x}_1) < \inf \mathrm{Supp}(\underset{\sim}{x}_2)$.

Corollary 1. It follows from Theorem 1 that, for the fuzzy value of a fuzzy distribution function $\underset{\sim}{F}$ and real numbers x, x_1 and x_2, the following holds:

a) $0 \leq \underset{\sim}{F}(x) \leq 1$ and $\underset{\sim}{P}(\underset{\sim}{X} \underset{\sim}{\geq} x) = \underset{\sim}{P}(\underset{\sim}{X} \geq x) = 1 \ominus \underset{\sim}{F}(x)$ for all x,

b) $\underset{\sim}{F}(x_1) \trianglelefteq \underset{\sim}{F}(x_2)$ for all $x_1 < x_2$.

Theorem 2. For any two continuous fuzzy numbers $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ such that $\inf \operatorname{Ker}(\underset{\sim}{x}_1) < \inf \operatorname{Supp}(\underset{\sim}{x}_2)$, we have

$$\underset{\sim}{P}(\underset{\sim}{x}_1 \underset{\sim}{\leq} \underset{\sim}{X} \underset{\sim}{\leq} \underset{\sim}{x}_2) = \underset{\sim}{P}(\underset{\sim}{B}_2 - \underset{\sim}{B}_1) \ ,$$

and $\mu_{\underset{\sim}{B}_2 - \underset{\sim}{B}_1}(x) = \mu_{\underset{\sim}{B}_2}(x) - \mu_{\underset{\sim}{B}_1}(x)$ for $\forall x \in \mathbb{R}$ where the fuzzy event $\underset{\sim}{B}_1$ corresponds to the relation $\underset{\sim}{X} \underset{\sim}{\leq} \underset{\sim}{x}_1$ and the fuzzy event $\underset{\sim}{B}_2$ corresponds to the relation $\underset{\sim}{X} \underset{\sim}{\leq} \underset{\sim}{x}_2$.

Using the extension principle, we define the fuzzy numerical characteristics of the fuzzy random variable $\underset{\sim}{X}$.

Definition 4. Let $\underset{\sim}{F}$ be the fuzzy distribution function of a fuzzy random variable $\underset{\sim}{X}$, $E(X)$ the means of the random variables X from the fuzzy bunch $\underset{\sim}{X}$ and F their distribution functions. The fuzzy set $\underset{\sim}{E}(\underset{\sim}{X}) = (\mathbb{R}, \mu_{\underset{\sim}{E}(\underset{\sim}{X})})$ where

$$\mu_{\underset{\sim}{E}(\underset{\sim}{X})} = \sup_{\substack{F \\ E(X)=x}} \mu_{\underset{\sim}{F}}(F)$$

is called the *fuzzy mean of the fuzzy random variable* $\underset{\sim}{X}$. If a random variable X has no mean or no random variable X exists such that $E(X) = x$, we put $\mu_{\underset{\sim}{E}(\underset{\sim}{X})} = 0$.

In a similar way, also other moment fuzzy characteristics are defined of the fuzzy random variable X and also of a fuzzy random vector where $m > 1$ using the fuzzy mean of $\underset{\sim}{g}(\underset{\sim}{X})$ where $g(x)$ is a Lebesgue measurable function defined on $\mathbb{R}^m$.

Theorem 3. If a fuzzy random variable $\underset{\sim}{X}$ has the fuzzy mean $\underset{\sim}{E}(\underset{\sim}{X})$, then $\underset{\sim}{E}(a \oplus b\,\underset{\sim}{X}) = a \oplus b\,\underset{\sim}{E}(\underset{\sim}{X})$ for any two real numbers a, b. Particularly, for the fuzzy number $\underset{\sim}{E}(\underset{\sim}{X})$, $\underset{\sim}{E}(a \oplus b\,\underset{\sim}{X})$ is also a fuzzy number.

Definition 1 admits the following fuzzification of a crisp distribution function using a fuzzy parameter.

The *first fuzzy probability distribution model* is based on a parametric system of distribution functions of the class of probability distribution $F(x, s)$. Replacing the parameter $s \in \mathbb{R}^k$ where $k \in \mathbb{N}$ by a fuzzy parameter $\underset{\sim}{S} = (\mathbb{R}^k, \mu_{\underset{\sim}{S}})$ with $\operatorname{Ker}(\underset{\sim}{S}) \neq \emptyset$ yields a fuzzy distribution function with the fuzzy parameter $\underset{\sim}{F}(x, \underset{\sim}{S})$ putting $\mu_{\underset{\sim}{F}}(F) = \mu_{\underset{\sim}{S}}(s)$ for $\forall x \in \mathbb{R}$ and $\forall s \in \mathbb{R}^k$. Thus, the fuzzy random variable $\underset{\sim}{X}$ is defined in the sense of Definition 1. Then the fuzzy probability of the fuzzy random variable $\underset{\sim}{X}$ assuming the fuzzy value $\underset{\sim}{A} = (\mathbb{R}, \mu_{\underset{\sim}{A}})$ is the fuzzy probability $\underset{\sim}{P}(\underset{\sim}{A}) = ([0,1], \mu_{\underset{\sim}{P}(\underset{\sim}{A})})$ where

$$\mu_{\underset{\sim}{P}(\underset{\sim}{A})}(p) = \sup_{\substack{s \\ \int_{\mathbb{R}} \mu_{\underset{\sim}{A}} \, \mathrm{d}F = p}} \mu_{\underset{\sim}{S}}(s)$$

and the value of the fuzzy distribution function $\underset{\sim}{F}(x, \underset{\sim}{S}) = \underset{\sim}{P}(\underset{\sim}{X} < x)$ is the fuzzy probability $\underset{\sim}{Y} = ([0,1], \mu_{\underset{\sim}{Y}})$ where

$$\mu_{\underset{\sim}{Y}}(y) = \sup_{\substack{s \\ F(x,\, s)=y}} \mu_{\underset{\sim}{S}}(s) \ .$$

Similarly, we obtain a fuzzy mean $\underset{\sim}{E}(\underset{\sim}{X}) = (\mathbb{R}, \mu_{\underset{\sim}{E}(\underset{\sim}{X})})$ where

$$\mu_{\underset{\sim}{E}(\underset{\sim}{X})}(x) = \sup_{\substack{s \\ E(X)=x}} \mu_{\underset{\sim}{S}}(s) \ .$$

A continuous fuzzy random variable $\underset{\sim}{X}$ may also be defined by fuzzifying the probability density $f(x,s)$ using the fuzzy probability density with the fuzzy parameter $\underset{\sim}{f}(x,\underset{\sim}{S})$. In this case, we put $\mu_{\underset{\sim}{f}}(f)=\mu_{\underset{\sim}{S}}(s)$ for $\forall x\in\mathbb{R}$ and $\forall s\in\mathbb{R}^k$.

The *second fuzzy probability distribution model* is again based on a parametric system of distribution functions, but only on its subset related to the parameter. This is a fuzzy probability distribution of the type $\underset{\sim}{F}(x,\underset{\sim}{S})=F(x)\oplus\underset{\sim}{S}\,\eta(x)$ where the term $\underset{\sim}{S}\,\eta(x)$ indicates the 'fuzziness' of the main probability distribution value $F(x)$.

Theorem 4. Let $\underset{\sim}{F}(x,\underset{\sim}{S})=F(x)\oplus\underset{\sim}{S}\,\eta(x)$ be a fuzzy distribution function where $F(x)$ is the crisp distribution function of the random variable X, $F(x)$ and $\eta(x)\geq 0$ are continuous functions for $\forall x\in\mathbb{R}$. Let the fuzzy parameter $\underset{\sim}{S}$ be a continuous fuzzy number with the main value $s=0$ and $\mu_{\underset{\sim}{S}}(s)=0$ for $s\notin[-1,1]$. Then

$$\eta(x)\leq 1-R(x)=F(x)\qquad\text{for}\qquad x\in(-\infty,x_{\mathrm{M}}]\ ,$$
$$\eta(x)\leq R(x)\qquad\text{for}\qquad x\in[x_{\mathrm{M}},+\infty)\ ,$$
$$\lim_{x\to-\infty}\eta(x)=\lim_{x\to+\infty}\eta(x)=0$$

where x_{M} signs the median of the random variable X with the reliability function $R(x)=$ $=1-F(x)$.

Remark 2. If, for the function $\eta(x)$ from Theorem 4,

$$R(x_2)-R(x_1)\leq\eta(x_2)-\eta(x_1)\leq F(x_2)-F(x_1)$$

for $\forall x_1<x_2$, then all the functions $F(x)+s\,\eta(x)$ are evidently non-decreasing for $\forall s\in[-1,1]$.

Corollary 2. The fuzzy value of the fuzzy distribution function $\underset{\sim}{F}(x,\underset{\sim}{S})=F(x)\oplus\underset{\sim}{S}\,\eta(x)$ from Theorem 4 is a continuous fuzzy number for $\forall x\in\mathbb{R}$, and thus, using α-cuts, we obtain

$$\underset{\sim}{F}(x,\underset{\sim}{S})=\bigcup_{\alpha\in[0,\,1]}\alpha[F(x)+s_{1\alpha}\,\eta(x),F(x)+s_{2\alpha}\,\eta(x)]$$

where $s_{1\alpha}$ is the minimum and $s_{2\alpha}$ the maximum root of the equation $\mu_{\underset{\sim}{S}}(s)=\alpha$. The function $F(x)$ is the main value of $\underset{\sim}{F}(x,\underset{\sim}{S})$ and

$$\mu_{\underset{\sim}{F}(x)}(y)=\begin{cases}\mu_{\underset{\sim}{S}}\left(\dfrac{y-F(x)}{\eta(x)}\right)\ , & y\in[F(t)-\eta(x),F(t)+\eta(x)]\ ,\\ 0 & \text{otherwise}\ .\end{cases}$$

for $\forall x\in\mathbb{R}$. For $x\to-\infty$, or $x\to+\infty$, we have $\mu_{\underset{\sim}{F}}(0)=1$, or $\mu_{\underset{\sim}{F}}(1)=1$ respectively and $\mu_{\underset{\sim}{F}}(y)=0$ otherwise.

Theorem 5. Let a fuzzy distribution function $\underset{\sim}{F}(x,\underset{\sim}{S})=F(x)\oplus\underset{\sim}{S}\,\eta(x)$ have the properties from Theorem 4 and a crisp random variable X with the distribution function $F(x)$ have a mean $E(x)$. Then the fuzzy mean of the fuzzy random variable $\underset{\sim}{X}$ is the continuous fuzzy number

$$\underset{\sim}{E}(\underset{\sim}{X})=E(X)\oplus\underset{\sim}{S}\,\psi\ ,$$

where $\psi=\int_{-\infty}^{+\infty}\eta(x)\,\mathrm{d}x$. Particularly,

$$\mu_{\underset{\sim}{E}(\underset{\sim}{X})}=\begin{cases}\mu_{\underset{\sim}{S}}\left(\dfrac{x-E(X)}{\psi}\right)\ , & y\in[E(X)-\psi,E(X)+\psi]\ ,\\ 0 & \text{otherwise}\ .\end{cases}$$

3. Fuzzy reliability models with Weibull fuzzy distribution

We assume that the object (element or system) under investigation is either in a failure-free or in a failure state. The failure-free state time is a random variable T, which assumes values $t \in [0, +\infty)$. Next we assume that only the transition from a failure-free state to a failure one is possible. The crisp reliability function of object is $R(t) = (P(T \geq t) = 1 - F(t)$ where $F(t)$ is the distribution function of T, and $F(t) = 0$ for $\forall t \in (-\infty, 0]$. The fuzzy probability model of reliability presumes that the time of such a transition is a fuzzy random variable $\underset{\sim}{T}$, which describes the vagueness of the transition time t and the uncertainty of the probability distribution. We define the fuzzy reliability by means of the fuzzy distribution function of fuzzy random variable $\underset{\sim}{T}$ (see Section 2).

Definition 5. The *fuzzy reliability (fuzzy lifetime, fuzzy survival) function* is the fuzzy probability $\underset{\sim}{R}(t) = \underset{\sim}{P}(\underset{\sim}{T} \underset{\sim}{\geq} t)$ for $\forall t \in [0, +\infty)$.

Remark 3. It follows from Corollary 1 that $\underset{\sim}{R}(t) = \underset{\sim}{P}(\underset{\sim}{T} \geq t) = 1 \ominus \underset{\sim}{F}(t)$, $\underset{\sim}{F}(t) = 1 \ominus \underset{\sim}{R}(t)$ for $\forall t \in (-\infty, 0]$. In addition $\underset{\sim}{R}(0) = 1$, and $\underset{\sim}{R}(+\infty) = 0$ is evident.

Definition 6. Let $\underset{\sim}{F}(t)$ be a fuzzy distribution function, where all the distribution functions $F(t)$ are absolutely continuous into $[0, +\infty)$, and $f(t) = \mathrm{d}F(t)/\mathrm{d}t$ the densities of probability. The *fuzzy hazard function (fuzzy failure rate)* is the fuzzy function $\underset{\sim}{\lambda}(t) = ([0, +\infty), \mu_{\underset{\sim}{\lambda}})$ where for $t \in [0, +\infty)$

$$\mu_{\underset{\sim}{\lambda}}(\lambda) = \sup_{\substack{F \\ \frac{f}{1-F} = \lambda}} \mu_{\underset{\sim}{F}}(F) \ .$$

Remark 4. We cannot write $\underset{\sim}{\lambda} = \underset{\sim}{f} \oslash \underset{\sim}{R}$ or $\underset{\sim}{\lambda} = \underset{\sim}{f} \odot (\underset{\sim}{R}^{-1})$ as the fuzzy functions $\underset{\sim}{f}$ and $\underset{\sim}{R}$ are dependent.

Definition 7. The *fuzzy mean time to failure* is the fuzzy mean $\underset{\sim}{E}(\underset{\sim}{T})$ of fuzzy random variable $\underset{\sim}{T}$.

Remark 5. If $\underset{\sim}{R}(t) = 1 \ominus \underset{\sim}{F}(t)$ is a fuzzy reliability function, where the all distribution functions $F(t)$ are absolute continuous into $[0, +\infty)$, then the fuzzy mean time to failure

$$\underset{\sim}{E}(\underset{\sim}{T}) = \int_0^{+\infty} \underset{\sim}{R}(t)\, \mathrm{d}t = ([0, +\infty), \mu_{\underset{\sim}{E}(\underset{\sim}{T})})$$

where

$$\mu_{\underset{\sim}{E}(\underset{\sim}{T})}(t^*) = \sup_{\substack{F \\ \int_0^{+\infty} R(t)\, \mathrm{d}t = t^*}} \mu_{\underset{\sim}{F}}(F) \ ,$$

since $\int_0^{+\infty} t \, \mathrm{d}F = \int_0^{+\infty} R \, \mathrm{d}t$ and $\mu_{\underset{\sim}{R}}(R) = \mu_{\underset{\sim}{F}}(F)$ for $R = 1 - F$.

A crisp random variable T with the *two-parameter Weibull probability distribution* $W(b, \delta)$ where $b > 0$ is the *shape parameter*, $\delta > 0$ is the *scale parameter*, and $t \in [0, \infty)$ has these functional and numerical characteristics:

$$\text{distribution function} \qquad F(t) = 1 - \exp\left[-\left(\frac{t}{\delta}\right)^b\right] \ ,$$

$$\text{reliability function} \qquad R(t) = 1 - F(t) = \exp\left[-\left(\frac{t}{\delta}\right)^b\right] \ ,$$

$$\text{hazard function} \qquad \lambda(t) = \frac{b}{\delta}\left(\frac{t}{\delta}\right)^{b-1} ,$$

$$\text{mean (expected value)} \qquad E(T) = \delta\,\Gamma\!\left(\frac{1}{b}+1\right)$$

$$\text{where Gamma function} \qquad \Gamma(z) = \int\limits_0^\infty y^{z-1}\,\exp(-y)\,\mathrm{d}y ,$$

$$\text{and the } P\text{-percentile} \qquad t_P = \delta[-\ln(1-P)]^{\frac{1}{b}} \quad \text{for} \quad P \in [0,1) .$$

Fuzzy reliability model A. This model results from the first model of fuzzy probability distribution (see Section 2). We assume that the values of a fuzzy random variable $\underset{\sim}{T}$ are the fuzzy numbers $\underset{\sim}{t} = ([0,\infty), \mu_{\underset{\sim}{t}})$ and $\underset{\sim}{t} = \underset{\sim}{\kappa}\,t$ where t is the observed value of a crisp random variable T and $\underset{\sim}{\kappa}$ is a so-called *vagueness coefficient*. The vagueness coefficient is a real triangular fuzzy number $\underset{\sim}{\kappa} = ([0,\infty), \mu_{\underset{\sim}{\kappa}})$ with the main value $\kappa = 1$ and membership function

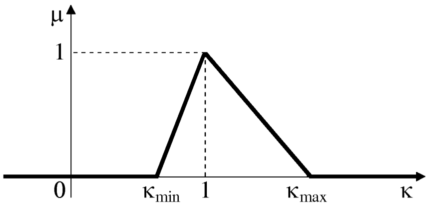

Fig.1

$$\mu_{\underset{\sim}{\kappa}}(\kappa) = \begin{cases} \dfrac{\kappa-\kappa_{\min}}{1-\kappa_{\min}} , & \kappa \in [\kappa_{\min}, 1] , \\ \dfrac{\kappa-\kappa_{\max}}{1-\kappa_{\max}} , & \kappa \in [1, \kappa_{\max}] , \\ 0 & \text{otherwise} \end{cases}$$

where $0 < \kappa_{\min} \le 1 \le \kappa_{\max}$, and boundary values $\kappa_{\min}$, $\kappa_{\max}$ they are given by an expert's estimate. Fig. 1 shows graph of $\mu_{\underset{\sim}{\kappa}}(\kappa)$.

If a random variable T has a crisp Weibull probability distribution $W(b,\delta)$ then the corresponding fuzzy random variable $\underset{\sim}{T}$ with the Weibull fuzzy probability distribution $\underset{\sim}{W}(b,\delta)$ has the following fuzzy characteristics. For $\forall t \in [0,\infty)$ the fuzzy distribution function $\underset{\sim}{F}(t) = 1 - \exp\{-[t/(\underset{\sim}{\kappa}\,\delta)]^b\}$, so that for $\forall\alpha \in [0,1]$ the α-cuts of fuzzy distribution function

$$F_\alpha(t) = [F_{1\alpha}(t), F_{2\alpha}(t)] =$$
$$= \left[1 - \exp\left[-\left(\frac{t}{\delta[(1-\kappa_{\max})\,\alpha + \kappa_{\max}]}\right)^b\right], 1 - \exp\left[-\left(\frac{t}{\delta[(1-\kappa_{\min})\,\alpha + \kappa_{\min}]}\right)^b\right]\right] .$$

For $\forall t \in [0,\infty)$ the fuzzy reliability function $\underset{\sim}{R}(t) = \exp\{-[t/(\underset{\sim}{\kappa}\,\delta)]^b\}$, so that for $\forall\alpha \in [0,1]$ the α-cuts of fuzzy reliability function

$$R_\alpha(t) = [R_{1\alpha}(t), R_{2\alpha}(t)] =$$
$$= \left[\exp\left[-\left(\frac{t}{\delta[(1-\kappa_{\min})\,\alpha + \kappa_{\min}]}\right)^b\right], \exp\left[-\left(\frac{t}{\delta[(1-\kappa_{\max})\,\alpha + \kappa_{\max}]}\right)^b\right]\right] .$$

For $\forall t \in [0,\infty)$ the fuzzy hazard function $\underset{\sim}{\lambda}(t) = b\,t^{b-1}/(\underset{\sim}{\kappa}\,\delta)^b$, so that for $\forall\alpha \in [0,1]$ the α-cuts of fuzzy hazard function

$$\lambda_\alpha(t) = [\lambda_{1\alpha}(t), \lambda_{2\alpha}(t)] = \left[\frac{b\,t^{b-1}}{(\delta[(1-\kappa_{\max})\,\alpha + \kappa_{\max}])^b}, \frac{b\,t^{b-1}}{(\delta[(1-\kappa_{\min})\,\alpha + \kappa_{\min}])^b}\right] .$$

The fuzzy mean of fuzzy random variable $\underset{\sim}{T}$ is triangular fuzzy number $\underset{\sim}{E}(\underset{\sim}{T}) = \underset{\sim}{\kappa}\,\delta\,\Gamma(1/b+1)$ where

$$\mu_{\underset{\sim}{E}(\underset{\sim}{T})}(t) = \begin{cases} \dfrac{t - \kappa_{\min}\,\delta\,\Gamma\big(\frac{1}{b}+1\big)}{(1-\kappa_{\min})\,\delta\,\Gamma\big(\frac{1}{b}+1\big)}\ , & t \in \big[\kappa_{\min}\,\delta\,\Gamma\big(\frac{1}{b}+1\big)\,,\delta\,\Gamma\big(\frac{1}{b}+1\big)\big]\ , \\ \dfrac{t - \kappa_{\max}\,\delta\,\Gamma\big(\frac{1}{b}+1\big)}{(1-\kappa_{\max})\,\delta\,\Gamma\big(\frac{1}{b}+1\big)}\ , & t \in \big[\delta\,\Gamma\big(\frac{1}{b}+1\big)\,,\kappa_{\max}\,\delta\,\Gamma\big(\frac{1}{b}+1\big)\big]\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

The fuzzy P-percentile is $\underset{\sim}{t}_P = \underset{\sim}{\kappa}\,\delta\,[-\ln(1-P)]^{\frac{1}{b}}$, for $\forall P \in [0,1)$, where

$$\mu_{\underset{\sim}{t}_P}(t) = \begin{cases} \dfrac{t - \kappa_{\min}\,\delta\,[-\ln(1-P)]^{\frac{1}{b}}}{(1-\kappa_{\min})\,\delta\,[-\ln(1-P)]^{\frac{1}{b}}}\ , & t \in \Big[\kappa_{\min}\,\delta\,[-\ln(1-P)]^{\frac{1}{b}}, \delta\,[-\ln(1-P)]^{\frac{1}{b}}\Big]\ , \\ \dfrac{t - \kappa_{\max}\,\delta\,[-\ln(1-P)]^{\frac{1}{b}}}{(1-\kappa_{\max})\,\delta\,[-\ln(1-P)]^{\frac{1}{b}}}\ , & t \in \Big[\delta\,[-\ln(1-P)]^{\frac{1}{b}}, \kappa_{\max}\,\delta\,[-\ln(1-P)]^{\frac{1}{b}}\Big]\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Fuzzy reliability model B. This model results from the second model of fuzzy probability distribution (see Section 2). Let a random variable T have again the crisp Weibull probability distribution $W(b,\delta)$, but with the corresponding fuzzy random variable $\underset{\sim}{T}$ having a fuzzy Weibull probability distribution $\underset{\sim}{W}(b,\delta)$ with the fuzzy distribution function

$$\underset{\sim}{F}(t,\S) = \begin{cases} 1 - \exp\left[-\left(\dfrac{t}{b}\right)^{b}\right] \oplus \underset{\sim}{S}\,\eta(t)\ , & t \in [0,\infty)\ , \\ 0 & \text{otherwise} \end{cases}$$

where the function

$$\eta(t) = \begin{cases} \exp(-r\,t^b) - \exp[-(r+q)\,t^b]\ , & t \in [0,\infty)\ , \\ 0 & \text{otherwise} \end{cases}$$

contains real constants r, q. It can be shown that, for $r \in [\delta^{-b},\infty)$ and $q \in [0,\delta^{-b})$, $\eta(t)$ has the properties shown in the proposition of Theorem 4 and Remark 2 so that $\underset{\sim}{F}(t,\underset{\sim}{S})$ is a fuzzy distribution function for the continuous real fuzzy number $\underset{\sim}{S} = ([-1,1],\mu_{\underset{\sim}{S}})$, $\mu_{\underset{\sim}{S}}(0) = 1$. Fig. 2 shows the graph of $\eta(t)$ where the number $t_{\mathrm{M}} = \delta\,\ln 2/b$ is the median of random variable T. The fuzzy distribution function $\underset{\sim}{F}(t,\underset{\sim}{S})$ has the main value $1-\exp[-(t/\delta)^b]$ and

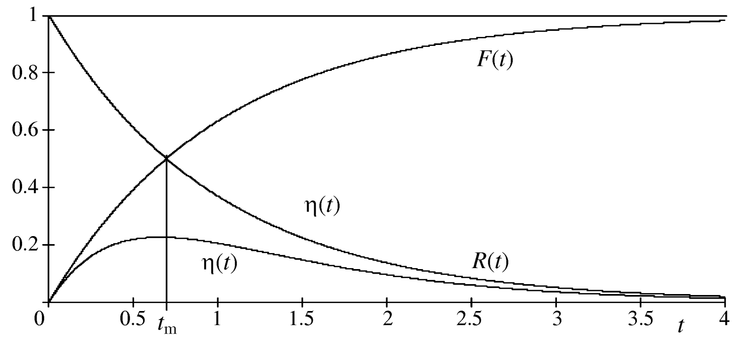

Fig.2

$$\mu_{\underset{\sim}{F}(t,\underset{\sim}{S})} = \begin{cases} \mu_{\underset{\sim}{S}}\left(\dfrac{y-1+\exp\left[-\left(\frac{t}{\delta}\right)^b\right]}{\eta(t)}\right) , & y \in \left[1-\exp\left[-\left(\frac{t}{\delta}\right)^b\right]-\eta(t),\right. \\ & \qquad \left. 1-\exp\left[-\left(\frac{t}{\delta}\right)^b\right]+\eta(t)\right] , \\ 0 & \text{otherwise} \end{cases}$$

for $\forall t \in (0,\infty)$. For $\forall t \in (-\infty, 0]$ we have $\mu_{\underset{\sim}{F}(t,\underset{\sim}{S})}(0) = 1$, and $\mu_{\underset{\sim}{F}(t,\underset{\sim}{S})}(y) = 0$ holds for $y \neq 0$. For $\forall \alpha \in [0,1]$ the α-cuts of the fuzzy distribution function $\underset{\sim}{F}(t,\underset{\sim}{S})$

$$F_\alpha(t,\underset{\sim}{S}) = \left[1-\exp\left[-\left(\frac{t}{\delta}\right)^b\right] + s_{1\alpha}\,\eta(t), 1-\exp\left[-\left(\frac{t}{\delta}\right)^b\right] + s_{2\alpha}\,\eta(t)\right]$$

where $[s_{1\alpha}, s_{2\alpha}]$ are the α-cuts of the fuzzy parameter $\underset{\sim}{S}$. By analogy, we acquire the membership function and α-cuts of the fuzzy reliability function $\underset{\sim}{R}(t)$. The α-cuts of fuzzy hazard function $\underset{\sim}{\lambda}(t)$ can be numerical calculated by means of interval arithmetic [2, 3, 5]. Since $E(T) = \delta\,\Gamma(1/b+1)$ and

$$\psi = \int_{-\infty}^{+\infty} \eta(t)\,\mathrm{d}t = \left(r^{-\frac{1}{b}} - (r+q)^{-\frac{1}{b}}\right)\Gamma\left(\frac{1}{b}+1\right) ,$$

we get the fuzzy mean

$$\underset{\sim}{E}(\underset{\sim}{T}) = \delta\,\Gamma\left(\frac{1}{b}+1\right) \oplus \underset{\sim}{S}\left(r^{-\frac{1}{b}} - (r+q)^{-\frac{1}{b}}\right)\Gamma\left(\frac{1}{b}+1\right) ,$$

where

$$\mu_{\underset{\sim}{E}(\underset{\sim}{T})}(t) = \begin{cases} \mu_{\underset{\sim}{S}}\left(\dfrac{t-\delta\,\Gamma\left(\frac{1}{b}+1\right)}{\psi}\right) , & t \in \left[\delta\,\Gamma\left(\frac{1}{b}+1\right)-\psi, \delta\,\Gamma\left(\frac{1}{b}+1\right)+\psi\right] , \\ 0 & \text{otherwise} . \end{cases}$$

For $\forall P \in [0,1)$ and $\forall t \in [0,\infty)$ we get the α-cuts of fuzzy P-percentile from the nonlinear equations

$$\exp\left[-\left(\frac{t}{\delta}\right)^b\right] - s_{1\alpha}\,\eta(t) = 1-P , \qquad \exp\left[-\left(\frac{t}{\delta}\right)^b\right] - s_{2\alpha}\,\eta(t) = 1-P .$$

In the above-mentioned models A and B, the selection of the membership function of the vagueness coefficient $\underset{\sim}{\kappa}$ or of the fuzzy parameter $\underset{\sim}{S}$ is of a subjective nature. For applications, an expert's estimate of this function is used as a base, its form being chosen to be as simple as possible and respecting the expert's evaluations of the accuracy degrees of the values observed. In the model B constants r, q can furthermore be selected for the norm of $\eta(t)$ to be minimal or maximal [8].

4. Conclusion

The fuzzy-set approach described above allows for a logical and systematic analysis of uncertainties. The models of fuzzy reliability with Weibull fuzzy distributions could be used for fuzzy reliability determining of a concrete structure cross section or of a concrete member. Uncertain parameters (like reinforcement area with influence of corrosion, change of concrete strength due to degradation/carbonation/sulphation in time) can be expressed as fuzzy sets. It is possible to process the fuzzy uncertainties in reliability analysis of a concrete member. Fuzzy uncertainty could be incorporated in the estimated probability of failure. Then the optimization models designed for reliable design problems [8] can be modified by included fuzzy parameters and the a posteriori verification of results can also be generalized in the similar way [9]. The approach allows an assessment of the likelihood that a particular concrete cross-section (or concrete member studied) will have a higher failure probability than the failure probability of the deterministic designed cross section or member [7].

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