FINITE ELEMENT FOR ANALYSIS OF BEAMS STRENGTHENED BY BONDED COMPOSITE STRIPS

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This paper presents a finite element for the analysis of beams strengthened by composite strips. The element is based on a mixed formulation of the mechanical model of the strengthened beam, the adhesive and the composite strip, working simultaneously. This solution allows us to study the influence of the adhesive layer on the behaviour of the strengthened beam.

The proposed element is verified with solutions of other analytical models or different finite element models and schemes.

Keywords: *FRP strengthening, finite element formulation*

1. Introduction

The increase of beams' bending capacity is considered to be the most effective application of FRP in the building industry. Many laboratory tests deal with failure modes of strengthened beams. They show that the strengthened system, which includes the beam, the adhesive and the FRP strip, has a different behaviour than other types of strengthening such as reinforced concrete jacketing, prestressing of the structural elements, steel jacketing or addition of external steel elements.

For design purposes, the strengthened system is assumed to fail when the FRP strip breaks. In many cases, the failure mode of the system is the FRP strip debonding. This dangerous brittle failure cannot be neglected.

A lot of research work focuses on the analysis of this failure. The laboratory tests and the analytical models related to the behaviour of the bonded joints and the separation of the adherents are studied in details in [7, 13, 21]. The effect of debonding of the composite strip due to bending of strengthened beams is also studied very carefully. In addition to the experimental tests in [3, 12], the separation of the FRP from the beam is analyzed by analytical methods presented in [2, 4, 5, 8, 9, 11, 14, 16, 17, 20–22, 24–26] and by finite element models applied in [1, 3, 5, 14, 19, 23].

The proposed finite element is based on the analytical model presented in [5]. This model is constrained by the assumptions for the adhesive layer and the composite strip, suggested in [26] and [22]. The model assumes that the FRP strip has only axial stiffness and the strengthened beam is modelled according to the Timoshenko hypothesis. The bonding effect between the two adherents is expressed by the work of the shear stresses in the adhesive layer. This assumption implies constant shear stress in the adhesive. The differential element of the strengthened system is shown in Fig. 1.

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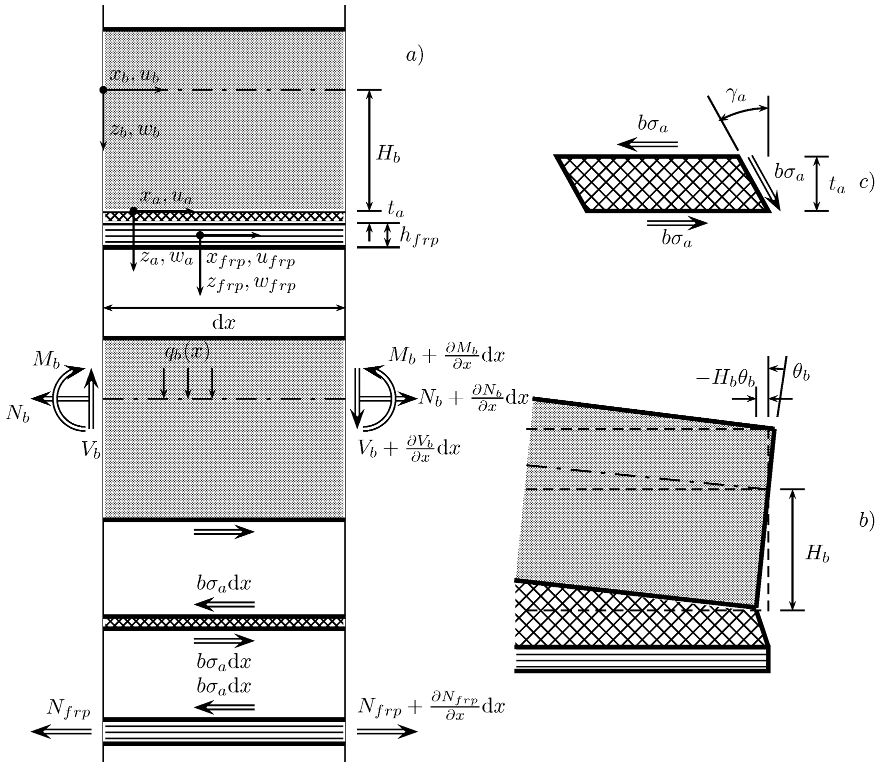

Fig.1: Differential element (a) and kinematics of deformations (b) and (c)

The equilibrium equations for the strengthened system are

$$
\begin{aligned}
E_\mathrm{b}\, A_\mathrm{b}\, u_{\mathrm{ob,xx}} + b\,\sigma_\mathrm{a} &= 0\ , \\
A_{11}\, u_{\mathrm{ofrp,xx}} - b\,\sigma_\mathrm{a} &= 0\ , \\
k_\mathrm{b}\, G_\mathrm{b}\, A_\mathrm{b}\,(w_{\mathrm{b,xx}} - \theta_{\mathrm{b,x}}) + q_\mathrm{b} &= 0\ , \\
E_\mathrm{b}\, I_\mathrm{b}\, \theta_{\mathrm{b,xx}} + k_\mathrm{b}\, G_\mathrm{b}\, A_\mathrm{b}\,(w_{\mathrm{b,x}} - \theta_\mathrm{b}) - b\, H_\mathrm{b}\,\sigma_\mathrm{a} &= 0\ , \\
u_\mathrm{ob} - u_\mathrm{ofrp} - H_\mathrm{b}\,\theta_\mathrm{b} + \frac{t_\mathrm{a}}{G_\mathrm{a}}\,\sigma_\mathrm{a} &= 0\ ,
\end{aligned}
\qquad (1)
$$

where $(.)_{,\mathrm{x}}$ is differentiation of $(.)$ with respect to x; E_b is modulus of elasticity of the beam's material; G_b is shear modulus of the beam's material; A_b and I_b are the area and the moment of inertia of the beam's cross-section; H_b is depth of the beam; k_b is shear correction factor for the beam's cross-section; b and t_a are the width and thickness of the FRP strip; A_{11} is membrane stiffness of the laminate [18]; G_a is shear module of the adhesive; n_b, q_b and m_b are the axially and transversally distributed loads and moments along the beam's axis; u_ofrp, u_ob , w_b and θ_b are the axial and transversal displacements and rotations of the FRP strip and the strengthened beam; σ_a is shear stress in the adhesive layer.

2. Element formulation

The general-purpose computer programs are the common tools for finite element analysis of beams strengthened with bonded fiber reinforced polymer (FRP) strips. Usually, the beam and the FRP strip are modelled with frame elements. Spring elements represent the adhesive layer. Another option is presented by plane stress elements for the whole strengthened system. A third possibility is to apply brick (solid) elements for the beam and the adhesive and shell or plate elements for the FRP strip. The significant differences between thicknesses of the components of the strengthened system leads to difficulties with the finite element meshing. Some problems call for including specific elements, which represent the non-linear behaviour of the materials. This additionally complicates the solution of the problem.

This study focuses on the formulation, implementation and validation of a proposed finite element, which represents the behavior of the strengthened beam, the adhesive and the FRP strip, working simultaneously.

The formulation of the presented finite element is based on the idea of the Hellinger-Reissner mixed functional [6]. The approximated stress master field is presented by the shear force in the strengthened beam. The generalized displacements of the strengthened beam and the FRP strip are the displacements master fields used for the finite element formulation.

The next steps for the derivation of the stiffness matrix are based on the mechanics of materials approach. The assumed shear force field is used for the solution of the equation which is related to the shear stresses in the adhesive layer. The remaining equilibrium equations are solved with appropriate boundary conditions.

The last equation of (1) gives the expression for the shear stress in the adhesive layer

$$\sigma_{\rm a} = \frac{G_{\rm a}}{t_{\rm a}} \left(u_{\rm ofrp} - u_{\rm ob} + H_{\rm b}\,\theta_b\right) . \tag{2}$$

The first-order shear-deformation beam theory (Timoshenko hypothesis) gives the relationships between the internal forces and the displacements

$$\begin{aligned} N_{\rm b} &= E_{\rm b}\,A_{\rm b}\,u_{\rm ob,x} , \qquad & M_{\rm b} &= -E_{\rm b}\,I_{\rm b}\,\theta_{\rm b,x} , \\ V_{\rm b} &= k_{\rm b}\,G_{\rm b}\,A_{\rm b}\,(w_{\rm b,x} - \theta_{\rm b}) , \qquad & N_{\rm frp} &= A_{11}\,u_{\rm ofrp,x} . \end{aligned} \tag{3}$$

The equilibrium equations for the differential element of the strengthened system (Fig. 1) show the relation between internal forces and shear stresses

$$N_{\rm b,x} = -b\,\sigma_{\rm a} , \qquad N_{\rm frp} = b\,\sigma_{\rm a} , \qquad V_{\rm b,x} = -q_{\rm b} , \qquad M_{\rm b,x} = V_{\rm b} - b\,H_{\rm b}\,\sigma_{\rm a} . \tag{4}$$

Differentiation of (2) and the combination with (3) and (4) leads to

$$\sigma_{\rm a,xx} - \alpha^2\,\sigma_{\rm a} + \frac{H_{\rm b}\,G_{\rm a}}{E_{\rm b}\,I_{\rm b}\,t_{\rm a}}\,V_{\rm b} = 0 , \tag{5}$$

where

$$\alpha^2 = \frac{G_{\rm a}\,b}{t_{\rm a}} \left(\frac{1}{A_{11}} + \frac{1}{E_{\rm b}\,A_{\rm b}} + \frac{H_{\rm b}^2}{E_{\rm b}\,I_{\rm b}}\right) . \tag{6}$$

The general solution of the differential equation (5) can be written as

$$\sigma_{\rm a} = C_1\,{\rm e}^{(\alpha x)} + C_2\,{\rm e}^{(-\alpha x)} + \sigma_{\rm a}^{\rm p} . \tag{7}$$

The particular solution $\sigma_{\rm a}^{\rm p}$ depends on the function of shear force in the strengthened beam, which is the non-homogeneous part of the equation (5). In order to derive the elements of the stiffness matrix, the generalized displacements of the end nodes of the strengthened system are assigned unit values. The solution of the equations (1) shows that the shear force V_b is a constant function along the beam's length. The constants of integration C_1 and C_2 are arbitrary and can be derived from the boundary conditions for the adhesive layer. They may also depend on loading.

In order to apply the Timoshenko beam theory we can introduce

$$\Phi = \frac{12\, E_\mathrm{b}\, I_\mathrm{b}}{k_\mathrm{b}\, G_\mathrm{b}\, A_\mathrm{b}\, L^2} \;, \tag{8}$$

a dimensionless measure of the flexural-to-shear stiffness ratio [15]. The equilibrium equations (1) can be re-written as

$$\begin{aligned}
E_\mathrm{b}\, A_\mathrm{b}\, u_\mathrm{ob,xx} + b\, \sigma_\mathrm{a} &= 0 \;,\\
A_{11}\, u_\mathrm{ofrp,xx} - b\, \sigma_\mathrm{a} &= 0 \;,\\
12\, \frac{E_\mathrm{b}\, I_\mathrm{b}}{\Phi\, L^2}\, (w_\mathrm{b,xx} - \theta_\mathrm{b,x}) + q_\mathrm{b} &= 0 \;,\\
E_\mathrm{b}\, I_\mathrm{b}\, \theta_\mathrm{b,xx} + 12\, \frac{E_\mathrm{b}\, I_\mathrm{b}}{\Phi\, L^2}\, (w_\mathrm{b,x} - \theta_\mathrm{b}) - b\, H_\mathrm{b}\, \sigma_\mathrm{a} &= 0 \;,\\
u_\mathrm{ob} - u_\mathrm{ofrp} - H_\mathrm{b}\, \theta_\mathrm{b} + \frac{t_\mathrm{a}}{G_\mathrm{a}}\, \sigma_\mathrm{a} &= 0 \;.
\end{aligned} \tag{9}$$

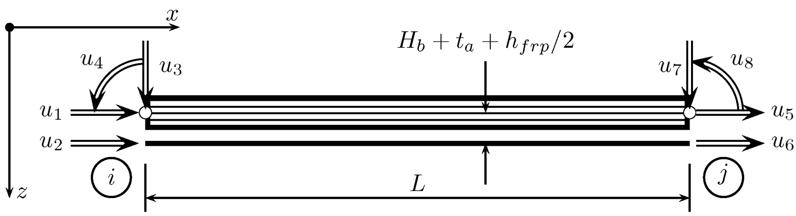

Fig.2: Generalized displacements for the proposed finite element

The nodal parameters selected for the displacements' degrees of freedom are (Fig. 2):

- An axial displacement of the left (right) end node of the strengthened element – u_1 (u_5);
- An axial displacement of the left (right) end node of the strengthening strip – u_2 (u_6);
- A transversal displacement of the left (right) end node of the strengthened element – u_3 (u_7);
- A rotation of the left (right) end node of the strengthened element – u_4 (u_8).

Stiffness elements are derived from internal forces with the corresponding generalized displacements $u_i = 1$

$$\begin{gathered}
k_{1i} = -N_\mathrm{b}(0) \;, \quad k_{2i} = -N_\mathrm{frp}(0) \;, \quad k_{3i} = -M_\mathrm{b,x}(0) - b\, H_\mathrm{b}\, \sigma_\mathrm{a}(0) \;, \quad k_{4i} = -M_\mathrm{b}(0) \;,\\
k_{5i} = N_\mathrm{b}(L) \;, \quad k_{6i} = N_\mathrm{frp}(L) \;, \quad k_{7i} = M_\mathrm{b,x}(L) + b\, H_\mathrm{b}\, \sigma_\mathrm{a}(L) \;, \quad k_{8i} = M_\mathrm{b}(L) \;.
\end{gathered} \tag{10}$$

2.1. Unit axial displacement u_1 of the left end node of the strengthened frame element

In case of axial displacement $u_1 = 1$ the function of displacement can be set as $u_\mathrm{ob}(0) = 1$ and the shear force function can be assumed to be $V_\mathrm{b}(x) = 0$. The necessary boundary conditions for the frame element and the FRP strip used to determine the integration constants in the general solution are

$$\begin{aligned} &u_{\mathrm{b}}(0) = 1\ , \qquad u_{\mathrm{frp}}(0) = 0\ , \qquad w_{\mathrm{b}}(0) = 0\ , \qquad \theta_{\mathrm{b}}(0) = 0\ , \\ &u_{\mathrm{b}}(L) = 0, \qquad u_{\mathrm{frp}}(L) = 0\ , \qquad w_{\mathrm{b}}(L) = 0\ , \qquad \theta_{\mathrm{b}}(L) = 0\ . \end{aligned} \tag{11}$$

The substitution of the boundary conditions (11) in (2) leads to the boundary conditions for the shear stresses in the adhesive layer

$$\sigma_{\mathrm{a}}(0) = -\frac{G_{\mathrm{a}}}{t_{\mathrm{a}}}\ , \qquad \sigma_{\mathrm{a}}(L) = 0\ . \tag{12}$$

The boundary conditions (12) will be used to complete the general solution for the shear stresses, which are necessary to solve the remaining differential equations.

The general solution of eqn. (5) is

$$\sigma_{\mathrm{a}}(x) = \frac{\mathrm{e}^{-x\alpha}\,(\mathrm{e}^{2x\alpha} - \mathrm{e}^{2L\alpha})\,G_{\mathrm{a}}}{(\mathrm{e}^{2L\alpha} - 1)\,t_{\mathrm{a}}}\ . \tag{13}$$

Now the expression for shear stress (13) can be substituted in the first four equations of (9). These equations, together with the boundary conditions (11), lead to the analytical solutions for the generalized displacements. Relationships (3) give the functions for the internal forces.

The stiffness coefficients are derived from (10) and are

$$\begin{aligned}
k_{11} &= \frac{A_{\mathrm{b}}\,(-1+\mathrm{e}^{2L\alpha})\,E_{\mathrm{b}}\,t_{\mathrm{a}}\,\alpha^2 + b\,G_{\mathrm{a}}\,[1 + L\,\alpha + \mathrm{e}^{2L\alpha}(-1+L\,\alpha)]}{(-1+\mathrm{e}^{2L\alpha})\,L\,t_{\mathrm{a}}\,\alpha^2}\ , \\
k_{21} &= -\frac{b\,G_{\mathrm{a}}\,[1 + L\,\alpha + \mathrm{e}^{2L\alpha}\,(-1+L\,\alpha)]}{(-1+\mathrm{e}^{2L\alpha})\,L\,t_{\mathrm{a}}\,\alpha^2}\ , \\
k_{31} &= \frac{6\,b\,G_{\mathrm{a}}\,H_{\mathrm{b}}\,[2 + L\,\alpha + \mathrm{e}^{L\alpha}\,(-2+L\,\alpha)]}{(1+\mathrm{e}^{L\alpha})\,L^3\,t_{\mathrm{a}}\,\alpha^3\,(1+\Phi)}\ , \\
k_{41} &= \frac{1}{(-1+\mathrm{e}^{2L\alpha})\,L^2\,t_{\mathrm{a}}\,\alpha^3\,(1+\Phi)}\Big(b\,G_{\mathrm{a}}\,H_{\mathrm{b}}\,\big\{6 - 12\,\mathrm{e}^{L\alpha} + L^2\,\alpha^2\,(1+\Phi) + \\
&\qquad + L\,\alpha\,(4+\Phi) + \mathrm{e}^{2L\alpha}\,[6 + L^2\,\alpha^2\,(1+\Phi) - L\,\alpha\,(4+\Phi)]\big\}\Big)\ , \\
k_{51} &= \frac{-A_{\mathrm{b}}\,(-1+\mathrm{e}^{2L\alpha})\,E_{\mathrm{b}}\,t_{\mathrm{a}}\,\alpha^2 + b\,G_{\mathrm{a}}\,(-1+\mathrm{e}^{2L\alpha} - 2\,\mathrm{e}^{L\alpha}\,L\,\alpha)}{(-1+\mathrm{e}^{2L\alpha})\,L\,t_{\mathrm{a}}\,\alpha^2}\ , \\
k_{61} &= \frac{b\,G_{\mathrm{a}}\,(1 - \mathrm{e}^{2L\alpha} + 2\,\mathrm{e}^{L\alpha}\,L\,\alpha)}{(-1+\mathrm{e}^{2L\alpha})\,L\,t_{\mathrm{a}}\,\alpha^2}\ , \\
k_{71} &= -\frac{6\,b\,G_{\mathrm{a}}\,H_{\mathrm{b}}\,(2 + L\,\alpha + \mathrm{e}^{L\alpha}\,(-2+L\,\alpha))}{(1+\mathrm{e}^{L\alpha})\,L^3\,t_{\mathrm{a}}\,\alpha^3\,(1+\Phi)}\ , \\
k_{81} &= \frac{1}{(-1+\mathrm{e}^{2L\alpha})\,L^2\,t_{\mathrm{a}}\,\alpha^3(1+\Phi)}\Big(b\,G_{\mathrm{a}}\,H_{\mathrm{b}}\,\big\{6 + \mathrm{e}^{2L\alpha}\,[6 + L\,\alpha\,(-2+\Phi)] - \\
&\qquad - L\,\alpha\,(-2+\Phi) - 2\,\mathrm{e}^{L\alpha}\,[6 + L^2\,\alpha^2\,(1+\Phi)]\big\}\Big)\ .
\end{aligned} \tag{14}$$

2.2. Unit axial displacement u_2 of the left end node of the strengthening strip

Similar to the previous case $u_2 = 1$, $u_{\mathrm{ofrp}}(0) = 1$ and $V_{\mathrm{b}}(x) = 0$. The boundary conditions for the frame element and the FRP strip are

$$\begin{aligned} &u_{\mathrm{b}}(0) = 0\,, \quad &&u_{\mathrm{frp}}(0) = 1\,, \quad &&w_{\mathrm{b}}(0) = 0\,, \quad &&\theta_{\mathrm{b}}(0) = 0\,, \\ &u_{\mathrm{b}}(L) = 0\,, &&u_{\mathrm{frp}}(L) = 0\,, &&w_{\mathrm{b}}(L) = 0\,, &&\theta_{\mathrm{b}}(L) = 0\,. \end{aligned} \tag{15}$$

The boundary conditions for the shear stresses in the adhesive layer are

$$\sigma_{\mathrm{a}}(0) = \frac{G_{\mathrm{a}}}{t_{\mathrm{a}}}\,, \qquad \sigma_{\mathrm{a}}(L) = 0\,. \tag{16}$$

The general solution for the shear stress is

$$\sigma_{\mathrm{a}}(x) = -\frac{\mathrm{e}^{-x\alpha}\,(\mathrm{e}^{2x\alpha} - \mathrm{e}^{2L\alpha})\,G_{\mathrm{a}}}{(\mathrm{e}^{2L\alpha} - 1)\,t_{\mathrm{a}}}\,. \tag{17}$$

This expression for shear stress (17) and the boundary conditions (15) can be used to solve the first four equations of (9).

The stiffness coefficients are derived from (10) and are

$$\begin{aligned}
k_{22} &= \frac{A_{11}\,(-1 + \mathrm{e}^{2L\alpha})\,t_{\mathrm{a}}\,\alpha^2 + b\,G_{\mathrm{a}}[1 + L\,\alpha + \mathrm{e}^{2L\alpha}\,(-1 + L\,\alpha)]}{(-1 + \mathrm{e}^{2L\alpha})\,L\,t_{\mathrm{a}}\,\alpha^2}\,, \\
k_{32} &= -\frac{6\,b\,G_{\mathrm{a}}\,H_{\mathrm{b}}\,[2 + L\,\alpha + \mathrm{e}^{L\alpha}\,(-2 + L\,\alpha)]}{(1 + \mathrm{e}^{L\alpha})\,L^3\,t_{\mathrm{a}}\,\alpha^3\,(1 + \Phi)}, \\
k_{42} &= -\frac{1}{(-1 + \mathrm{e}^{2L\alpha})\,L^2\,t_{\mathrm{a}}\,\alpha^3(1 + \Phi)}\Big(b\,G_{\mathrm{a}}\,H_{\mathrm{b}}\,\big\{6 - 12\,\mathrm{e}^{L\alpha} + L^2\,\alpha^2\,(1 + \Phi) + \\
&\qquad + L\,\alpha\,(4 + \Phi) + \mathrm{e}^{2L\alpha}\,[6 + L^2\,\alpha^2\,(1 + \Phi) - L\,\alpha\,(4 + \Phi)]\big\}\Big)\,, \\
k_{52} &= \frac{b\,G_{\mathrm{a}}\,[1 - \mathrm{e}^{2L\alpha} + 2\,\mathrm{e}^{L\alpha}\,L\,\alpha]}{(-1 + \mathrm{e}^{2L\alpha})\,L\,t_{\mathrm{a}}\,\alpha^2}\,, \\
k_{62} &= \frac{-A_{11}\,(-1 + \mathrm{e}^{2L\alpha})\,t_{\mathrm{a}}\,\alpha^2 + b\,G_{\mathrm{a}}\,(-1 + \mathrm{e}^{2L\alpha} - 2\,\mathrm{e}^{L\alpha}\,L\,\alpha)}{(-1 + \mathrm{e}^{2L\alpha})\,L\,t_{\mathrm{a}}\,\alpha^2}\,, \\
k_{72} &= \frac{6\,b\,G_{\mathrm{a}}\,H_{\mathrm{b}}\,[2 + L\,\alpha + \mathrm{e}^{L\alpha}\,(-2 + L\,\alpha)]}{(1 + \mathrm{e}^{L\alpha})\,L^3\,t_{\mathrm{a}}\,\alpha^3\,(1 + \Phi)}, \\
k_{82} &= -\frac{1}{(-1 + \mathrm{e}^{2L\alpha})\,L^2\,t_{\mathrm{a}}\,\alpha^3\,(1 + \Phi)}\Big(b\,G_{\mathrm{a}}\,H_{\mathrm{b}}\,\big\{6 + \mathrm{e}^{2L\alpha}\,[6 + L\,\alpha\,(-2 + \Phi)] - \\
&\qquad - L\,\alpha\,(-2 + \Phi) - 2\,\mathrm{e}^{L\alpha}\,[6 + L^2\,\alpha^2\,(1 + \Phi)]\big\}\Big)\,.
\end{aligned} \tag{18}$$

2.3. Unit transversal displacement u_3 of the left end node of the frame element

Let the transversal displacement of the left node of the frame element be $u_3 = 1$, i.e. $w_{\mathrm{b}}(0) = 1$. For this case with accuracy of 0.1 % the shear force function can be assumed to be

$$V_{\mathrm{b}}(x) = -\frac{12\,E_{\mathrm{b}}\,I_{\mathrm{b}}}{L^3\,(1 + \Phi)}\,. \tag{19}$$

The boundary conditions for the frame element and the FRP strip are

$$\begin{aligned} u_{\mathrm{b}}(0) &= 0 \,, & u_{\mathrm{frp}}(0) &= 0 \,, & w_{\mathrm{b}}(0) &= 1 \,, & \theta_{\mathrm{b}}(0) &= 0 \,, \\ u_{\mathrm{b}}(L) &= 0 \,, & u_{\mathrm{frp}}(L) &= 0 \,, & w_{\mathrm{b}}(L) &= 0 \,, & \theta_{\mathrm{b}}(L) &= 0 \,. \end{aligned} \tag{20}$$

The boundary conditions for the shear stresses in the adhesive layer are

$$\sigma_{\mathrm{a}}(0) = 0 \,, \qquad \sigma_{\mathrm{a}}(L) = 0 \,. \tag{21}$$

The general solution for the shear stress can be written as

$$\sigma_{\mathrm{a}}(x) = \frac{12\,\mathrm{e}^{-x\alpha}\,(\mathrm{e}^{x\alpha}-1)(\mathrm{e}^{x\alpha}-\mathrm{e}^{L\alpha})\,G_{\mathrm{a}}\,H_{\mathrm{b}}}{(1+\mathrm{e}^{L\alpha})\,L^3\,t_{\mathrm{a}}\,\alpha^2\,(1+\Phi)} \,. \tag{22}$$

This expression for the shear stress (22) and the boundary conditions (20) can be used to solve the first three equations of (9).

The stiffness coefficients are

$$\begin{aligned}
k_{33} &= \frac{12}{(1+\mathrm{e}^{L\alpha})\,L^6\,t_{\mathrm{a}}\,\alpha^5\,(1+\Phi)^2} \Big\{ b\,G_{\mathrm{a}}\,H_{\mathrm{b}}^2 \left[-24 - 12\,L\,\alpha + L^3\,\alpha^3 + \right. \\
&\qquad \left. + \mathrm{e}^{L\alpha}(24 - 12\,L\,\alpha + L^3\,\alpha^3)\right] + (1+\mathrm{e}^{L\alpha})\,E_{\mathrm{b}}\,I_{\mathrm{b}}\,L^3\,t_{\mathrm{a}}\,\alpha^5\,(1+\Phi) \Big\} \,, \\
k_{43} &= -\frac{6}{(1+\mathrm{e}^{L\alpha})\,L^5\,t_{\mathrm{a}}\,\alpha^5\,(1+\Phi)^2} \Big((1+\mathrm{e}^{L\alpha})\,E_{\mathrm{b}}\,I_{\mathrm{b}}\,L^3\,t_{\mathrm{a}}\,\alpha^5\,(1+\Phi) - \\
&\qquad - b\,G_{\mathrm{a}}\,H_{\mathrm{b}}^2 \left\{ 24 + 12\,L\,\alpha + L^3\,\alpha^3\,\Phi + 2\,L^2\,\alpha^2\,(1+\Phi) + \right. \\
&\qquad \left. + \mathrm{e}^{L\alpha}\left[-24 + 12\,L\,\alpha + L^3\,\alpha^3\,\Phi - 2\,L^2\,\alpha^2\,(1+\Phi)\right] \right\} \Big) \,, \\
k_{53} &= \frac{6\,b\,G_{\mathrm{a}}\,H_{\mathrm{b}}\,[2 + L\,\alpha + \mathrm{e}^{L\alpha}\,(-2 + L\,\alpha)]}{(1+\mathrm{e}^{L\alpha})\,L^3\,t_{\mathrm{a}}\,\alpha^3\,(1+\Phi)} \,, \\
k_{63} &= -\frac{6\,b\,G_{\mathrm{a}}\,H_{\mathrm{b}}\,[2 + L\,\alpha + \mathrm{e}^{L\alpha}\,(-2 + L\,\alpha)]}{(1+\mathrm{e}^{L\alpha})\,L^3\,t_{\mathrm{a}}\,\alpha^3\,(1+\Phi)} \,, \\
k_{73} &= -\frac{12}{(1+\mathrm{e}^{L\alpha})\,L^6\,t_{\mathrm{a}}\,\alpha^5\,(1+\Phi)^2} \Big\{ b\,G_{\mathrm{a}}\,H_{\mathrm{b}}^2 \left[-24 - 12\,L\,\alpha + L^3\,\alpha^3 + \right. \\
&\qquad \left. + \mathrm{e}^{L\alpha}\,(24 - 12\,L\,\alpha + L^3\,\alpha^3)\right] + (1+\mathrm{e}^{L\alpha})\,E_{\mathrm{b}}\,I_{\mathrm{b}}\,L^3\,t_{\mathrm{a}}\,\alpha^5\,(1+\Phi) \Big\} \,, \\
k_{83} &= -\frac{6}{(1+\mathrm{e}^{L\alpha})\,L^5\,t_{\mathrm{a}}\,\alpha^5\,(1+\Phi)^2} \Big((1+\mathrm{e}^{L\alpha})\,E_{\mathrm{b}}\,I_{\mathrm{b}}\,L^3\,t_{\mathrm{a}}\,\alpha^5\,(1+\Phi) - \\
&\qquad - b\,G_{\mathrm{a}}\,H_{\mathrm{b}}^2 \left\{ 24 + 12\,L\,\alpha + L^3\,\alpha^3\,\Phi + 2\,L^2\,\alpha^2\,(1+\Phi) + \right. \\
&\qquad \left. + \mathrm{e}^{L\alpha}\left[-24 + 12\,L\,\alpha + L^3\,\alpha^3\,\Phi - 2\,L^2\,\alpha^2\,(1+\Phi)\right] \right\} \Big) \,.
\end{aligned} \tag{23}$$

2.4. Unit rotation u_4 of the left end node of the frame element

The rotation of the left node of the frame element $u_4 = 1$ implies $\theta_{\mathrm{b}}(0) = -1$ and the function of shear force is

$$V_{\mathrm{b}}(x) = 6\,\frac{E_{\mathrm{b}}\,I_{\mathrm{b}}}{L^2\,(1+\Phi)} \,. \tag{24}$$

The boundary conditions for the frame element and the FRP strip are

$$\begin{aligned} &u_\mathrm{b}(0)=0\,, \qquad u_\mathrm{frp}(0)=0\,, \qquad w_\mathrm{b}(0)=0\,, \qquad \theta_\mathrm{b}(0)=-1\,,\\ &u_\mathrm{b}(L)=0\,, \qquad u_\mathrm{frp}(L)=0\,, \qquad w_\mathrm{b}(L)=0\,, \qquad \theta_\mathrm{b}(L)=0\,. \end{aligned} \tag{25}$$

The boundary conditions for the shear stresses in the adhesive layer are obtained as follows

$$\sigma_\mathrm{a}(0)=-\frac{G_\mathrm{a}\,H_\mathrm{b}}{t_\mathrm{a}}\,, \qquad \sigma_\mathrm{a}(L)=0\,. \tag{26}$$

The equations (26) give the integration constants and the general solution for the shear stress is

$$\begin{aligned} \sigma_\mathrm{a}(x)=&-\frac{1}{\left(-1+\mathrm{e}^{2L\alpha}\right)L^2\,t_\mathrm{a}\,\alpha^2\,(1+\Phi)}\,\mathrm{e}^{-x\alpha}\left(-\mathrm{e}^{L\alpha}+\mathrm{e}^{x\alpha}\right)G_\mathrm{a}\,H_\mathrm{b}\left(6-6\,\mathrm{e}^{L\alpha}-\right.\\ &\left.-\,6\,\mathrm{e}^{x\alpha}+6\,\mathrm{e}^{L\alpha+x\alpha}-\mathrm{e}^{L\alpha}\,L^2\,\alpha^2-\mathrm{e}^{x\alpha}\,L^2\,\alpha^2-\mathrm{e}^{L\alpha}\,L^2\,\alpha^2\,\Phi-\mathrm{e}^{x\alpha}\,L^2\,\alpha^2\,\Phi\right)\,. \end{aligned} \tag{27}$$

The expression for shear stress (27) and the boundary conditions (25) can be used to solve the four equations of (9).

In this case the stiffness coefficients are

$$\begin{aligned} k_{44}=&\frac{1}{\left(-1+\mathrm{e}^{2L\alpha}\right)L^5\,t_\mathrm{a}\,\alpha^5\,(1+\Phi)^2}\Big(\left(-1+\mathrm{e}^{2L\alpha}\right)E_\mathrm{b}\,I_\mathrm{b}\,L^4\,t_\mathrm{a}\,\alpha^5\,(1+\Phi)\,(4+\Phi)-\\ &+b\,G_\mathrm{a}\,H_\mathrm{b}^2\,L\left\{72+36\,L\,\alpha+12\,L^2\,\alpha^2\,(1+\Phi)+L^4\,\alpha^4\,(1+\Phi)^2+\right.\\ &+L^3\,\alpha^3\,(4+8\,\Phi+\Phi^2)-24\,\mathrm{e}^{L\alpha}\,[6+L^2\,\alpha^2\,(1+\Phi)]+\\ &+\mathrm{e}^{2L\alpha}\,[72-36\,L\,\alpha+12\,L^2\,\alpha^2\,(1+\Phi)+L^4\,\alpha^4\,(1+\Phi)^2-\\ &\left.-\,L^3\,\alpha^3\,(4+8\,\Phi+\Phi^2)]\right\}\Big)\,,\\ k_{54}=&\frac{1}{\left(-1+\mathrm{e}^{2L\alpha}\right)L^2\,t_\mathrm{a}\,\alpha^3\,(1+\Phi)}\,b\,G_\mathrm{a}\,H_\mathrm{b}\Big\{6+\mathrm{e}^{2L\alpha}\,[6+L\,\alpha\,(-2+\Phi)]-\\ &-L\,\alpha\,(-2+\Phi)-2\,\mathrm{e}^{L\alpha}\,[6+L^2\,\alpha^2\,(1+\Phi)]\Big\}\,,\\ k_{64}=&-\frac{1}{\left(-1+\mathrm{e}^{2L\alpha}\right)L^2\,t_\mathrm{a}\,\alpha^3\,(1+\Phi)}\,b\,G_\mathrm{a}\,H_\mathrm{b}\Big\{6+\mathrm{e}^{2L\alpha}\,[6+L\,\alpha\,(-2+\Phi)]-\\ &-L\,\alpha\,(-2+\Phi)-2\,\mathrm{e}^{L\alpha}\,[6+L^2\,\alpha^2\,(1+\Phi)]\Big\}\,,\\ k_{74}=&\frac{6}{\left(1+\mathrm{e}^{L\alpha}\right)L^5\,t_\mathrm{a}\,\alpha^5\,(1+\Phi)^2}\Big(\left(1+\mathrm{e}^{L\alpha}\right)E_\mathrm{b}\,I_\mathrm{b}\,L^3\,t_\mathrm{a}\,\alpha^5\,(1+\Phi)-\\ &-b\,G_\mathrm{a}\,H_\mathrm{b}^2\left\{24+12\,L\,\alpha+L^3\,\alpha^3\,\Phi+2\,L^2\,\alpha^2\,(1+\Phi)+\right.\\ &\left.+\,\mathrm{e}^{L\alpha}[-24+12\,L\,\alpha+L^3\,\alpha^3\,\Phi-2\,L^2\,\alpha^2\,(1+\Phi)]\right\}\Big)\,,\\ k_{84}=&\frac{1}{\left(-1+\mathrm{e}^{2L\alpha}\right)L^4\,t_\mathrm{a}\,\alpha^5\,(1+\Phi)^2}\Big(-\left(-1+\mathrm{e}^{2L\alpha}\right)E_\mathrm{b}\,I_\mathrm{b}\,L^3\,t_\mathrm{a}\,\alpha^5\,(-2-\Phi+\Phi^2)+\\ &+b\,G_\mathrm{a}\,H_\mathrm{b}^2\left\{72+36\,L\,\alpha+12\,L^2\,\alpha^2\,(1+\Phi)+L^3\,\alpha^3\,(2+4\,\Phi-\Phi^2)-\right.\\ &-2\,\mathrm{e}^{L\alpha}\,[72+12\,L^2\,\alpha^2\,(1+\Phi)+L^4\,\alpha^4\,(1+\Phi)^2]+\\ &\left.+\,\mathrm{e}^{2L\alpha}\,[72-36\,L\,\alpha+12\,L^2\,\alpha^2\,(1+\Phi)+L^3\,\alpha^3\,(-2-4\,\Phi+\Phi^2)]\right\}\Big)\,. \end{aligned} \tag{28}$$

Due to the symmetry of the proposed finite element, the generalized displacements of the right end node will yield similar stiffness coefficients

$$
\begin{aligned}
&k_{55} = k_{11}\ , \quad && k_{65} = k_{21}\ , \quad && k_{75} = -k_{31}\ , \quad && k_{85} = k_{41}\ , \\
& && k_{66} = k_{22}\ , \quad && k_{76} = -k_{32}\ , \quad && k_{86} = k_{42}\ , \\
& && && k_{77} = k_{33}\ , \quad && k_{87} = -k_{43}\ , \\
& && && && k_{88} = k_{44}\ .
\end{aligned}
\tag{29}
$$

The final form of the stiffness matrix is

$$
\mathbf{k} = \begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\
 & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\
 & & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\
 & & & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\
 & & & & k_{55} & k_{56} & k_{57} & k_{58} \\
 & \text{symm.} & & & & k_{66} & k_{67} & k_{68} \\
 & & & & & & k_{77} & k_{78} \\
 & & & & & & & k_{88}
\end{bmatrix} .
\tag{30}
$$

The stiffness matrix can be easily transformed, according to the Euler-Bernoulli theory, by setting the coefficient $\Phi = 0$.

The elements of the generalized nodal forces caused by a uniform transversal load are

$$
\begin{aligned}
f_1 &= -\frac{b\,G_\mathrm{a}\,H_\mathrm{b}\,q_\mathrm{b}\,[-12 - 6\,L\,\alpha - L^2\,\alpha^2 + \mathrm{e}^{L\alpha}\,(12 - 6\,L\,\alpha + L^2\,\alpha^2)]}{12\,(-1 + \mathrm{e}^{L\alpha})\,E_\mathrm{a}\,I_\mathrm{b}\,t_\mathrm{a}\,\alpha^4}\ , \\
f_2 &= \frac{b\,G_\mathrm{a}\,H_\mathrm{b}\,q_\mathrm{b}\,[-12 - 6\,L\,\alpha - L^2\,\alpha^2 + \mathrm{e}^{L\alpha}\,(12 - 6\,L\,\alpha + L^2\,\alpha^2)]}{12\,(-1 + \mathrm{e}^{L\alpha})\,E_\mathrm{b}\,I_\mathrm{b}\,t_\mathrm{a}\,\alpha^4}\ , \\
f_3 &= -\frac{L\,q_\mathrm{b}}{2}\ , \\
f_4 &= \frac{1}{12\,(-1 + \mathrm{e}^{L\alpha})\,E_\mathrm{b}\,I_\mathrm{b}\,t_\mathrm{a}\,\alpha^4}\,q_\mathrm{b}\,\Big\{(-1 + \mathrm{e}^{L\alpha})\,E_\mathrm{b}\,I_\mathrm{b}\,L^2\,t_\mathrm{a}\,\alpha^4 + \\
&\qquad + b\,G_\mathrm{a}\,H_\mathrm{b}^2\,[12 + 6\,L\,\alpha + L^2\,\alpha^2 - \mathrm{e}^{L\alpha}\,(12 - 6\,L\,\alpha + L^2\,\alpha^2)]\Big\}\ , \\
f_5 &= \frac{b\,G_\mathrm{a}\,H_\mathrm{b}\,q_\mathrm{b}\,[-12 - 6\,L\,\alpha - L^2\,\alpha^2 + \mathrm{e}^{L\alpha}\,(12 - 6\,L\,\alpha + L^2\,\alpha^2)]}{12\,(-1 + \mathrm{e}^{L\alpha})\,E_\mathrm{b}\,I_\mathrm{b}\,t_\mathrm{a}\,\alpha^4}\ , \\
f_6 &= -\frac{b\,G_\mathrm{a}\,H_\mathrm{b}\,q_\mathrm{b}\,[-12 - 6\,L\,\alpha - L^2\,\alpha^2 + \mathrm{e}^{L\alpha}\,(12 - 6\,L\,\alpha + L^2\,\alpha^2)]}{12\,(-1 + \mathrm{e}^{L\alpha})\,E_\mathrm{b}\,I_\mathrm{b}\,t_\mathrm{a}\,\alpha^4}\ , \\
f_7 &= -\frac{L\,q_\mathrm{b}}{2}\ , \\
f_8 &= \frac{1}{12\,(-1 + \mathrm{e}^{L\alpha})\,E_\mathrm{b}\,I_\mathrm{b}\,t_\mathrm{a}\,\alpha^4}\,q_\mathrm{b}\,\Big\{-(-1 + \mathrm{e}^{L\alpha})\,E_\mathrm{b}\,I_\mathrm{b}\,L^2\,t_\mathrm{a}\,\alpha^4 + \\
&\qquad + b\,G_\mathrm{a}\,H_\mathrm{b}^2\,[-12 - 6\,L\,\alpha - L^2\,\alpha^2 + \mathrm{e}^{L\alpha}\,(12 - 6\,L\,\alpha + L^2\,\alpha^2)]\Big\}\ .
\end{aligned}
\tag{31}
$$

The load vector can be expressed as

$$
\mathbf{f} = [\,f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6 \quad f_7 \quad f_8\,]^\mathrm{T}\ .
\tag{32}
$$

3. Numerical examples and verification

The numerical test, given below, shows the applicability of the proposed finite element. A single-span simply-supported beam with uniformly distributed loading is analyzed in all examples (Fig. 3). An FRP strip strengthens the full span of the beam.

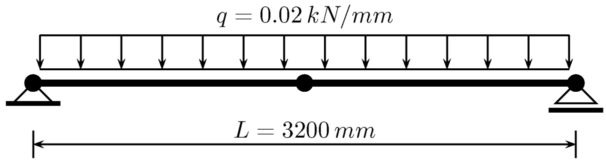

Fig.3: Test example – simply supported beam

The results of the solution with the proposed element are compared to those obtained by using different modelling schemes or a finite element mesh for the strengthened beam, the adhesive and the FRP strip. The geometry of the cross-section of the beam and the properties of the used materials are as follows: cross-section of the strengthened beam – rectangular with dimensions $B = 200$ mm and $H = 400$ mm; thickness of the adhesive layer – $t_\mathrm{a} = 2$ mm; FRP strip with dimensions $b = 200$ mm and $h_\mathrm{frp} = 4$ mm; module of elasticity of the beam's material – $E_\mathrm{b} = 30$ GPa; module of elasticity of the FRP strip's material – $E_{11} = 200$ GPa; shear module of the adhesive's material – $G_\mathrm{a} = 3$ GPa.

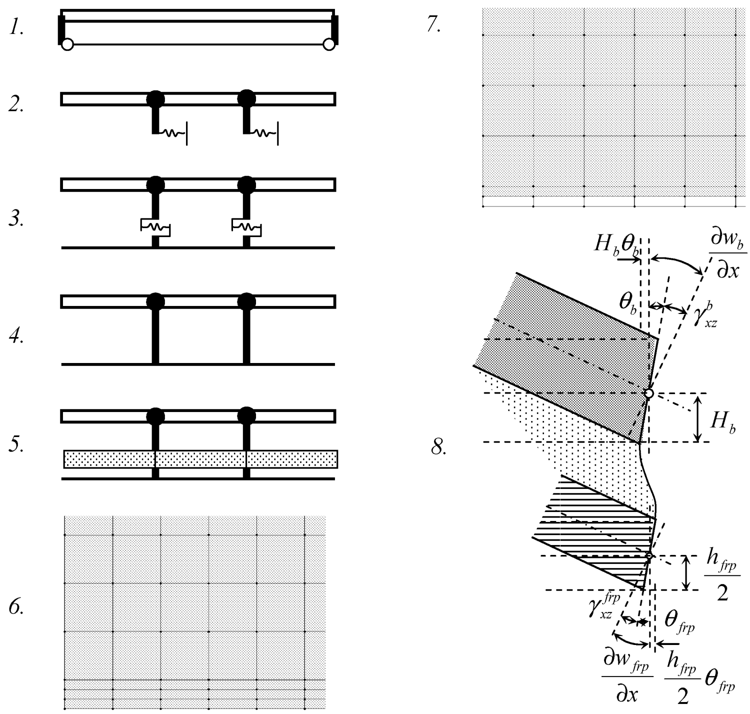

Fig.4: Test models

The comparative models for verification of the proposed element are follows (Fig. 4):

– The beam is tied at both ends by a FRP strip;
– The beam is meshed by 16 frame elements. Intermediate nodes are supported by horizontal springs with equivalent stiffness obtained from the shear module of the adhesive. These springs are at the level of the adhesive layer;
– The beam and the FRP strip are meshed by 16 frame elements. The interaction between the two adherents is modelled by longitudinal springs with equivalent stiffness obtained from the shear module of the adhesive;
– The beam and the FRP strip are meshed by 16 elements and the pairs are rigidly connected;
– The beam and the strengthening strip are meshed by 320 frame elements, the adhesive layer is modelled by using four-nodes plane stress elements;
– The beam, the strengthening strip and the adhesive are represented by four-nodes plane stress elements. The adhesive layer is modelled by one element for thickness, the FRP strip – by two elements and the beam – by 40 elements. In longitudinal direction the whole system is meshed by 320 elements;
– The beam and the adhesive layer are represented by plane stress elements, the FRP strip by frame elements. The mesh size is the same as in the previous case;
– The modified closed-form high-order theory introduced in [5].

In all tests the transversal displacement, the bending moment and the axial force are observed for a control cross-section, located in the middle of the span. Maximum shear stress is calculated for a cross-section located near the left (right) support. The results are compared to those, obtained by solving the problem with two finite elements.

The compared results are shown in table 1.

Models	Displacement mm		Bending moment kN mm		Axial force kN		Shear stress GPa	
	w	%	M	%	N	%	$\sigma_{\max}$	%
Proposed element	0.753	0	21 792.7	0	19.04	0	0.00215	0
1	0.775	2.94	22 866.49	4.93	13.4	29.61	0	0
2	0.844	12.1	24 361.81	11.79	6.16	67.64	0.0023	6.64
3	0.85	13.0	24 572.36	12.75	5.036	73.55	0.00127	41.06
4	0.746	0.88	21 462.95	1.51	20.28	6.53	0	0
5	0.743	1.3	21 365.84	1.96	20.76	9.05	0.00222	3.05
6	**0.754**	**0.17**	**21 350.77**	**2.03**	**20.83**	**9.44**	**0.00213**	**1.13**
7	0.754	0.17	21 350.85	2.03	20.83	9.44	0.00213	1.13
8	0.739	1.79	21 256.04	2.46	21.29	11.85	0.00012	94.44

Tab.1: Test results

The proposed finite element is also verified with the test experiments presented in [10]. The report deals with the determination of the bending capacity of beams, strengthened by various schemes of FRP strips. The difference between the vertical displacement of the controlled cross-section for the tested specimen before the first significant cracking and the vertical displacement obtained by the proposed finite element is within 7.6 %. The differences are due to an inexact support modelling, strengthening schemes and etc.

4. Conclusions

The numerical tests prove the applicability of the proposed finite element. Let us assume that the results of the solution of model 6, built with 13 760 four-node plane elements, are sufficiently accurate. Then the comparison with the results obtained with other models leads to the following conclusions:

- The application of the proposed finite element is efficient in terms of computer time and resources;
- The proposed element is acceptably accurate – the transversal displacements are smaller than those obtained with model 6 by 0.17 %. The difference in the bending moment is 2.03 %, and the difference in the axial force is 9.44 %;
- The element can be used for the analysis of thin or moderately thick beams;
- The model, which presumes only shear stiffness in the adhesive, does not adequately describe the behaviour of the strengthened system. The normal stiffness of the adhesive and the bending stiffness of the FRP strips have to be taken into account;
- The numerical solutions by 16 elements in models 1, 2 and 3 are inappropriate – the axial force is between 29 % and 73 % less than that obtained in model 6;
- The 'design' model 4 is useful for a global analysis of strengthened beams. The model cannot be applied to study the stresses in the adhesive layer;
- For a detailed analysis of the strengthened system it is advisable to use either a model with plane or solid finite elements with fine meshing or special elements. A suitable approach is to solve differential equations of analytical models such as those proposed in [5, 14, 16, 21].

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