A NUMERICAL INVESTIGATION OF THE MODIFIED SHERMAN SYSTEMS

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The aim of this paper was to demonstrate that it is possible to control the chaos into the Sherman system by linear feedback of own signals. After introducing of the parameter 'α' in the z-equation ($\alpha \rightarrow \alpha + \alpha_1\, x(t) + \alpha_2\, y(t) + \alpha_3\, z(t)$), we study how the global dynamics can be altered in a desired direction (α_n are considered as free parameters). We make a detailed bifurcation investigation of the modified Sherman systems by varying the parameters α_n. Finally, we calculate the maximal Lyapunov exponent, where the chaotic motion of modified Sherman systems exists.

Keywords: *chaotic dynamics, Sherman system, bifurcation, maximal Lyapunov exponent*

1. Introduction

Autonomous nonlinear three-dimensional differential equation systems can display a rich diversity of periodic and chaotic solutions dependent upon the specific values of one or more bifurcation (control) parameters. It is well-known that in these systems the only possible spectra, and the attractors, are as follows: a strange attractor, a two-torus, a limit cycle, a fixed point [1–4]. For the experimentalist, it is of great importance to know if any large deviation due to a change of parameter, occurs in his or her system. A better understanding of typical bifurcations is therefore required [5–9].

Chaotic motions are based on homoclinic (heteroclinic) structures which instability accompanied by local divergence and global contraction. Meanwhile, the transition from stability to instability requires the vanishing of stable equilibrium states and of stable periodic motions or sufficiently large increase in the periodic ones [10–12]. The stability loss can be repeated many times, forming an infinite period-doubling (tripling) bifurcation series.

In the meantime on a 1961, Lorenz was shown a model of convective motion in a fluid heated from below and cooled from above. After that, the Lorenz system has been studied in detail because it is a treasure trove of interesting phenomena. It was the first widely known chaotic system from a set of differential equations [13, 14]. In 1963 S. Sherman [15], carried out a system of three first-order ordinary differential equations describing the behavior of a nuclear spin generator (NSG). This system, not much referred to in literature, displays a larger variety of behaviors (both regular and chaotic) than the Lorenz system [16, 17].

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Following [18, 19], now NSG is called the Sherman system (SS). This system has the form

$$\dot{x} = -\beta\, x + y \ , \tag{1}$$

$$\dot{y} = -x - \beta\, y\,(1 - k\, z) \ , \tag{2}$$

$$\dot{z} = \beta\,[\alpha\,(1 - z) - k\, y^2] \ , \tag{3}$$

where x, y and z are the components of the nuclear magnetization vector in the X, Y, Z-directions, and α, β and k are positive parameters, respectively. The stationary points of the system (1)–(3) are

$$\bar{x}_1 = \bar{y}_1 = \bar{z}_1 = 1 \qquad \text{for all parametric values} \tag{4}$$

and if the condition $k > (1 + \beta^2)/\beta^2$ is valid, the system possesses two additional stationary points

$$\bar{x}_{2,3} = \pm\frac{\sqrt{\alpha\,[\beta^2\,(k-1) - 1]}}{k\,\beta^2}\ , \quad \bar{y}_{2,3} = \pm\frac{\sqrt{\alpha\,[\beta^2\,(k-1) - 1]}}{k\,\beta}\ , \quad \bar{z}_{2,3} = \frac{1 + \beta^2}{k\,\beta^2}\ . \tag{5}$$

The divergence of the flow (1)–(3) is

$$D_3 = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -\beta\,(\alpha + 2 - k\, z)\ . \tag{6}$$

The system (1)–(3) is dissipative, when $D_3 < 0$, i.e. $(\alpha + 2)/k > z$.

For different choice of parameters α, β and k the regular and chaotic solutions (and transition to chaos) of SS was investigated by [15, 16, 20]. For example, when $\alpha = 0.15$, $\beta = 0.75$ and $k = 21.5$ in [14], the chaotic motion of the system (1)–(3) is obtained.

In this study, our main goal is to investigate in what limits SS is structurally unsensitive. One of the know approaches to this problem consist in the introduction of feedback of own signals into the system's parameters (α, β and k) assuming their linear dependence on $x(t)$, $y(t)$ and $z(t)$. Applied to the Lorenz system and Rossler system, this approach yields very rich global dynamical behaviors [21–23]. In practice, it is often desired that chaos be avoided and/or that the system performance be improved or changed in some way. Given a chaotic attractor, one approach might be to make some large and possibly costly alteration in the system which completely changes its dynamics in such a way as to achieve the desired behavior [24]. In other words, here, having (1)–(3) as starting system, we can alter considerably its global dynamical behavior with minimum changes of the system's mathematical structure. Following [22], this means some kind of control of chaos in SS.

We tried various combinations. It is easy to see that the most appropriate for 'intervention' appears to be the z-equation in (3). Thus, we are going to present here some analytical and numerical results when parameter α is replaced by $\alpha(x, y, z)$ as linear function:

$$\alpha \to \alpha + \alpha_1\, x(t) + \alpha_2\, y(t) + \alpha_3\, z(t)\ , \tag{7}$$

where α_n $(n = 1 \div 3)$ are free parameters (which can be positive, zero or negative).

The scheme of the present paper is as follows. In section two we present analytical and numerical (graphical) results concerning the system (1)–(3) for different values of α_n when

(i) $\alpha = 0.15$, $\beta = 0.75$ and $k = 2.5$ (regular case – see [16, 20]); (ii) $\alpha = 0.15$, $\beta = 0.75$ and $k = 21.5$ (chaotic case – see [16, 20]). In section three we discuss and summarize our results.

2. Analytical and numerical results

In this section, we consider the system (1)–(3), which present an autonomous dynamical model, when the parameter α is:

Case 1 ($\alpha_2 = \alpha_3 = 0$). After substitution of (7) into (3), the z-equation becomes

$$\dot{z} = \beta\,[(\alpha + \alpha_1\,x)\,(1 - z) - k\,y^2]\ . \tag{8}$$

In this case, the equilibrium (steady state) points of the system (1)-(2)-(8) are:

$$\begin{gathered}\bar{x} = \frac{1}{\beta}\,\bar{y}\ , \qquad \bar{z} = \frac{1}{k}\left(\frac{1}{\beta^2} + 1\right)\ , \\ \bar{y}^2 - \frac{\alpha_1}{k\,\beta}\left[1 - \frac{1}{k}\left(\frac{1}{\beta^2} + 1\right)\right]\bar{y} - \frac{\alpha}{k}\left[1 - \frac{1}{k}\left(\frac{1}{\beta^2} + 1\right)\right] = 0\end{gathered} \tag{9}$$

when the inequalities

$$1 - \frac{1 + \beta^2}{k\,\beta^2} > 0\ , \qquad \frac{\alpha_1^2}{k\,\beta^2}\left(1 - \frac{1 + \beta^2}{k\,\beta^2}\right) + 4\,\alpha > 0 \tag{10}$$

or

$$1 - \frac{1 + \beta^2}{k\,\beta^2} < 0\ , \qquad \frac{\alpha_1^2}{k\,\beta^2}\left(1 - \frac{1 + \beta^2}{k\,\beta^2}\right) + 4\,\alpha < 0 \tag{11}$$

are valid. If case (10) or (11) are not valid, the equilibrium point is only one with values $\bar{x} = \bar{y} = 0$, $\bar{z} = 1$. The condition (10) is valid at $k > 2.778$, i.e. always. The condition (11) is valid at $\alpha_1 > 2.756$ (when $\alpha = 0.15$, $\beta = 0.75$ and $k = 2.5 < 2.778$).

In Figure 1, we fix $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$ and vary the parameter $\alpha_1 \in [5, 15]$ (see Eq. (8)). Here we note that at these values of the parameters, the relation (11) is valid. We show in Figure 1 (a) and (b) periodic orbits for $\alpha_1 = 5$ and $\alpha_1 = 10$. As the parameter α_1 is increased to $\alpha_1 = 15$, a periodic solution also take place. This result is shown in Figure 1 (c).It is seen also that in Figure 1 (c) the chaotic-like transient regime is very long.

Figure 2 shows the bifurcation diagrams for the system (1)-(2)-(8): values of z coordinate, z_n, are plotted against α_1 regarded as a continuously varying bifurcation (control) parameter for $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$ (chaotic case). We see that at $\alpha_1 \in [0.01, 1]$ the system (1)-(2)-(8) has chaotic solutions (Fig. 2b). When $\alpha_1 \in [1, 3.8]$, the inverse bifurcations occur. As α_1 increased further $\alpha_1 \in [3.8,\ 3.9]$ the period-doubling bifurcations take place. It is interesting to note that after $\alpha_1 = 3.9$ (till the end of the interval) the inverse bifurcations also occur. The similar behavior of the system (1)-(2)-(8) we have at $\alpha_1 \in [-4.16, -0.01]$. It is seen that the bifurcations diagrams in Fig. 2a and 2b are symmetrical.

Case 2 ($\alpha_1 = \alpha_3 = 0$). Following the same procedure, after substitution of (7) into (3), for the equation (3), we can write

$$\dot{z} = \beta\,[(\alpha + \alpha_2\,y)\,(1 - z) - k\,y^2]\ . \tag{12}$$

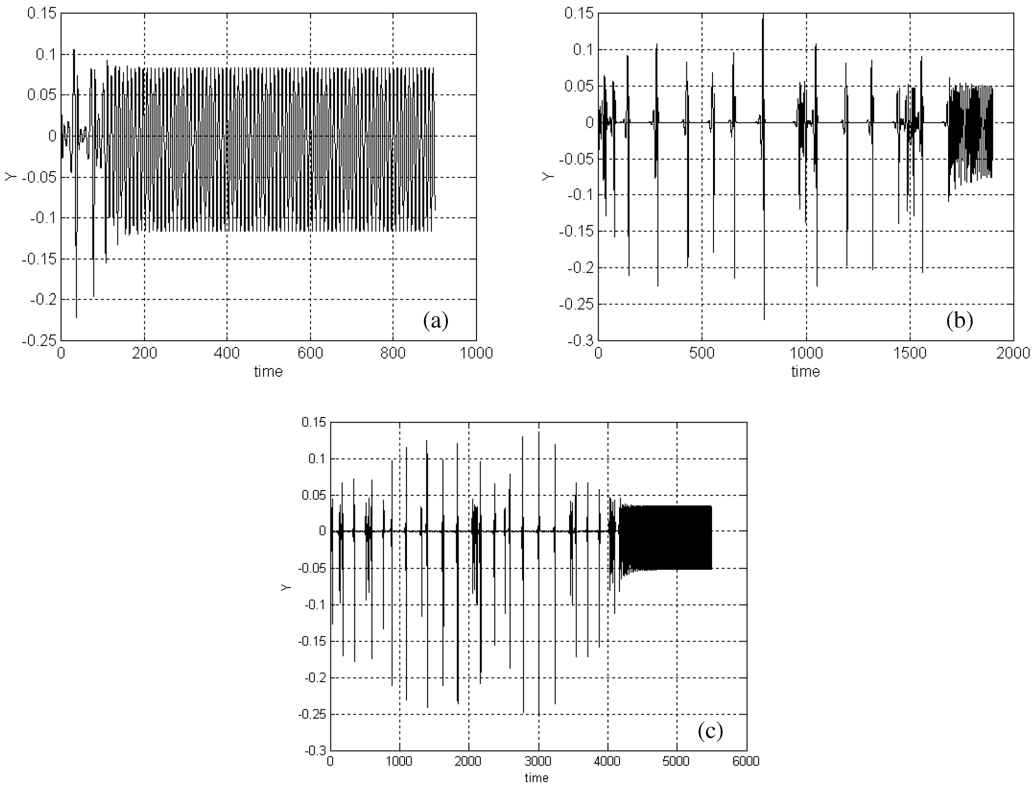

Fig.1: Periodic solutions of system (1)-(2)-(8) at $\alpha_1 = 5$ (a), $\alpha_1 = 10$ (b) and $\alpha_1 = 15$ (c)

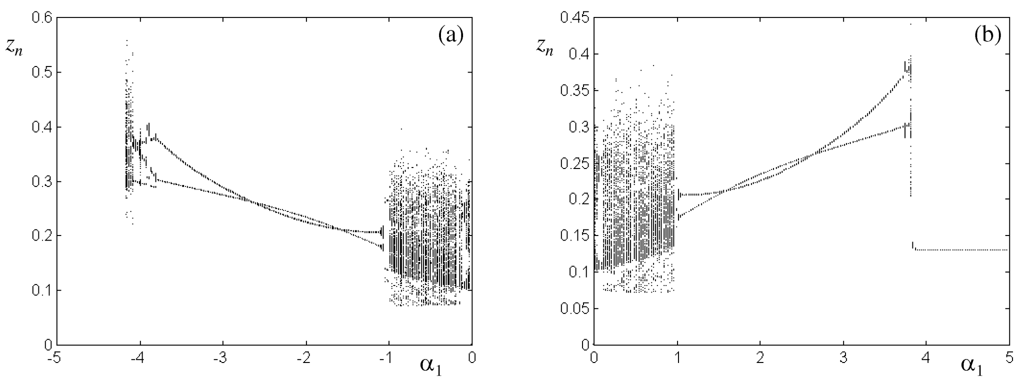

Fig.2: Bifurcation diagrams z_n versus α_1 generated by computer solutions of the system (1)-(2)-(8) computed with the parameters: $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_1 \in [-4.16, -0.01]$ (a) and $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_1 \in [0.01, 5]$ (b)

In this case the system (1)-(2)-(12) has three fixed points

$$\begin{gathered} \bar{x} = \frac{1}{\beta}\,\bar{y}\,, \qquad \bar{z} = \frac{1}{k}\left(\frac{1}{\beta^2}+1\right)\,, \\ \bar{y}^2 - \frac{\alpha_2}{k}\left[1-\left(\frac{1}{\beta^2}+1\right)\right]\bar{y} - \frac{\alpha}{k}\left[1-\left(\frac{1}{\beta^2}+1\right)\right] = 0\,. \end{gathered} \tag{13}$$

if the inequalities

$$1 - \frac{1+\beta^2}{k\,\beta^2} > 0\ , \qquad \frac{\alpha_2^2}{k}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) + 4\,\alpha > 0 \tag{14}$$

or

$$1 - \frac{1+\beta^2}{k\,\beta^2} < 0\ , \qquad \frac{\alpha_2^2}{k}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) + 4\,\alpha < 0 \tag{15}$$

are valid. The condition (14) is always valid at $k > 2.778$, and the condition (15) is valid at $\alpha_2 > 3.674$ (when $\alpha = 0.15$, $\beta = 0.75$ and $k = 2.5 < 2.778$).

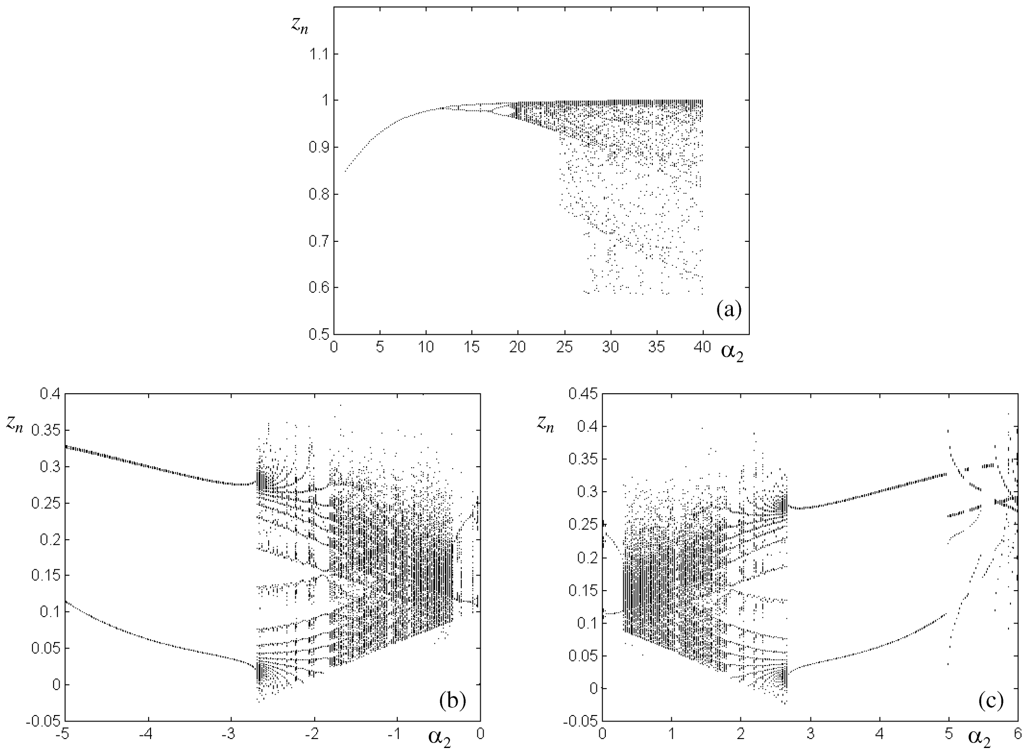

Fig.3: Bifurcation diagrams z_n versus α_2 generated by computer solutions of the system (1)-(2)-(12) computed with the parameters: $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$, $\alpha_2 \in [3.7, 40]$ (a); $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_2 \in [-5, -0.01]$ (b); and $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_2 \in [0.01, 6]$ (c)

In Fig. 3 the bifurcation diagrams of the system (1)-(2)-(12) are shown. It can be seen that at $\alpha_2 > 25$ chaotic solution occurs (Fig. 3a). In Figs. 3b and 3c, we illustrate the bifurcation behavior of the system (1)-(2)-(12) for $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$ and vary the parameter α_2. It is seen that when bifurcation parameter α_2 increase, the system passes from chaotic regime to regular one. In analogy with the previous case, the bifurcation diagrams in Figs. 3b and 3c are also symmetrical. It is important to note that an apparent sudden collapse in the size of a chaotic attractor occurs at a value of the control parameter $\alpha_2 \approx \pm 2.8$. Following [13, 25 and references therein], we conclude that such a sudden qualitative change in a chaotic attractor is known as interior crisis. Specifically, at an interior crisis, the attractor jumps discontinuously in size. As bifurcation parameter passes the critical one, the attractor collides with a saddle fixed point or periodic point **p** and suddenly incorporates the outward branch of the unstable manifold of **p**. Here we note that the modified system (1)-(2)-(12) has a new geometry (compared to Fig. 4a which is obtained for original system (1)–(3)) of the strange attractor (see Fig. 4b).

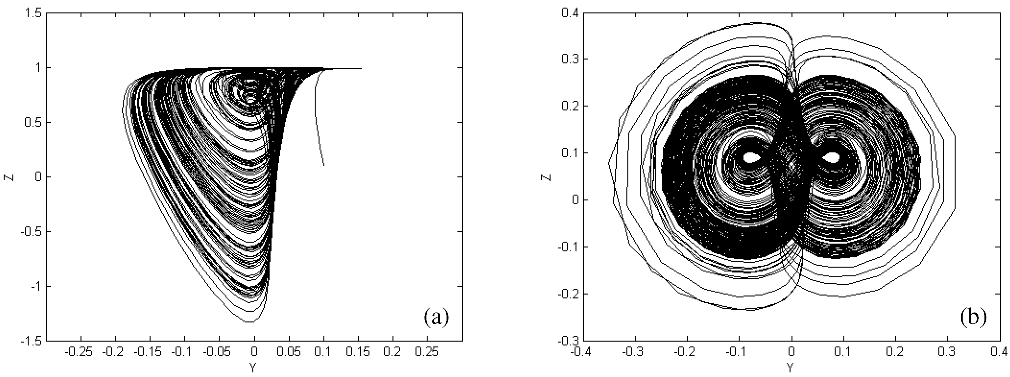

Fig.4: Chaotic attractors of the system (1)-(2)-(12) at $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$, $\alpha_2 = 27$ (a); and of the system (1)–(3) at $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$ (b)

Case 3 ($\alpha_1 = \alpha_2 = 0$). Replacing equation (7) into (3), we obtain for z-equation

$$\dot{z} = \beta \left[(\alpha + \alpha_3 z)(1 - z) - k y^2 \right] . \tag{16}$$

The equilibrium points of the system (1)-(2)-(16) are:

$$\begin{gathered} \bar{x} = \frac{1}{\beta}\,\bar{y}\,, \qquad \bar{z} = \frac{1}{k}\left(\frac{1}{\beta^2} + 1\right)\,, \\ \bar{y}^2 - \frac{1}{k}\left[\alpha + \alpha_3\left(\frac{1}{\beta^2} + 1\right)\right]\left[1 - \left(\frac{1}{\beta^2} + 1\right)\right] = 0\,. \end{gathered} \tag{17}$$

if the inequalities

$$1 - \frac{1 + \beta^2}{k\,\beta^2} > 0\,, \qquad \alpha + \alpha_3\left(1 - \frac{1 + \beta^2}{k\,\beta^2}\right) > 0 \tag{18}$$

or

$$1 - \frac{1 + \beta^2}{k\,\beta^2} < 0\,, \qquad \alpha + \alpha_3\left(1 - \frac{1 + \beta^2}{k\,\beta^2}\right) < 0 \tag{19}$$

are valid. The condition (18) is always valid at $k > 2.778$, and the condition (19) is valid at $\alpha_3 < -\alpha\, k\, \beta^2/(1 + \beta^2) = -0.135$, when $\alpha = 0.15$, $\beta = 0.75$ and $k = 2.5 < 2.778$.

Firstly, we compute the case when $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$, $\alpha_3 < -0.135$ and we obtain that here the system (1)-(2)-(16) has only periodic solutions (see Fig. 5). Therefore in this case the system lies in the region of regularity of its parametric space. In Figs. 6a and 6b, we show the bifurcation diagrams of the system (1)-(2)-(16) for $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$ (chaotic regime) and vary the parameter α_3. It is seen that after introduction of α_3, the system passes from chaos to regular motion. It is interesting to note that in Fig. 6b two symmetrical regular branches take place.

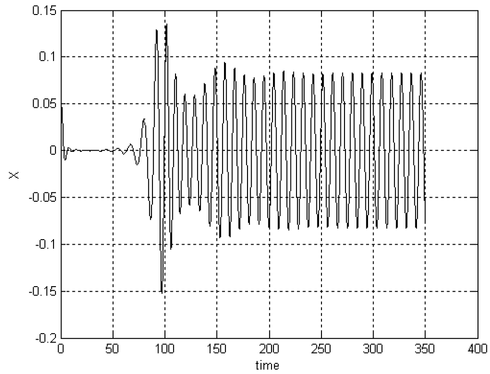

Fig.5: Periodic solution of the system (1)-(2)-(16) at $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$, $\alpha_3 = -0.2$

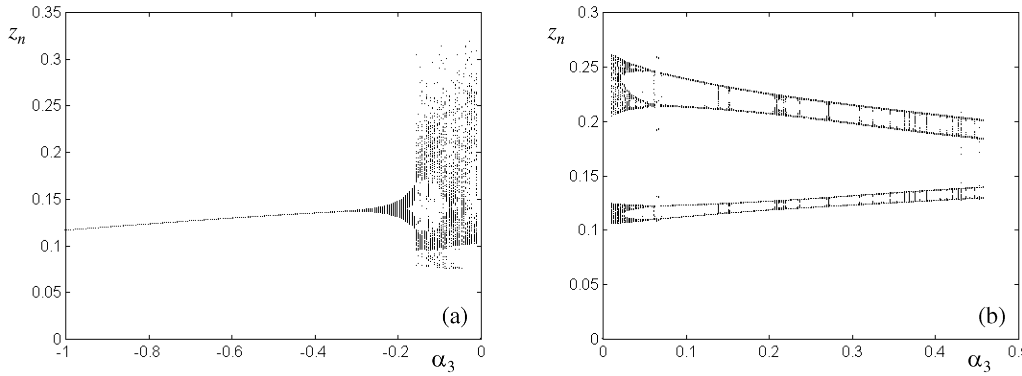

Fig.6: Bifurcation diagrams of the system (1)-(2)-(16) at $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_3 \in [-1, -0.01]$ *(a),* $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_3 \in [0.01, 0.47]$ *(b)*

Case 4 ($\alpha_3 = 0$). Here, after replacing (7) into (3), we obtain for z equation

$$\dot{z} = \beta[(\alpha + \alpha_1\, x + \alpha_2\, y)\,(1 - z) - k\, y^2]\ . \tag{20}$$

Now the $\bar{y}$ equation is

$$\bar{y}^2 - \frac{1}{k}\left(\frac{\alpha_1}{\beta} + \alpha_2\right)\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right)\bar{y} - \frac{\alpha}{k}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) = 0\ . \tag{21}$$

The system (1)-(2)-(20) has three fixed points if inequalities

$$1 - \frac{1+\beta^2}{k\,\beta^2} > 0\ , \qquad \frac{(\alpha_1 + \beta\,\alpha_2)^2}{k\,\beta^2}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) + 4\,\alpha > 0 \tag{22}$$

or

$$1 - \frac{1+\beta^2}{k\,\beta^2} < 0\ , \qquad \frac{(\alpha_1 + \beta\,\alpha_2)^2}{k\,\beta^2}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) + 4\,\alpha < 0 \tag{23}$$

are valid. Therefore, similar to (1)–(3), two real roots exist. Condition (22) is valid at $k > 2.778$, i.e. always. Condition (23) is valid at $\alpha_1 + 0.75\,\alpha_2 > 2.76$, when $\alpha = 0.15$, $\beta = 0.75$ and $k = 2.5 < 2.778$.

The numerical procedure used here is similar to previous three cases. Now, we have two new bifurcation parameters α_1 and α_2. In Figure 7a, the bifurcation diagram of the system (1)-(2)-(20) is shown for different values of the parameter α_2 (and α_1 is fixed). Here we note that in this case $k = 2.5$ (regular case). It is seen that for these values of $\alpha_2 \in [7, 35]$ and $\alpha_1 = -0.5$ the new system has regular and chaotic behavior. Following [13], two types of bifurcation are most basic. In the first, called a saddle-node bifurcation, fixed points are born. The second is called a period-doubling bifurcation, where a fixed point loses its stability and simultaneously, a new orbit of doubled period is created. Here we see that this is an example of the period-doubling route to chaos. Figure 7a shows that period-doubling cascade in this case begins at $\alpha_2 \approx 11$. Note that inverse bifurcations also occur.

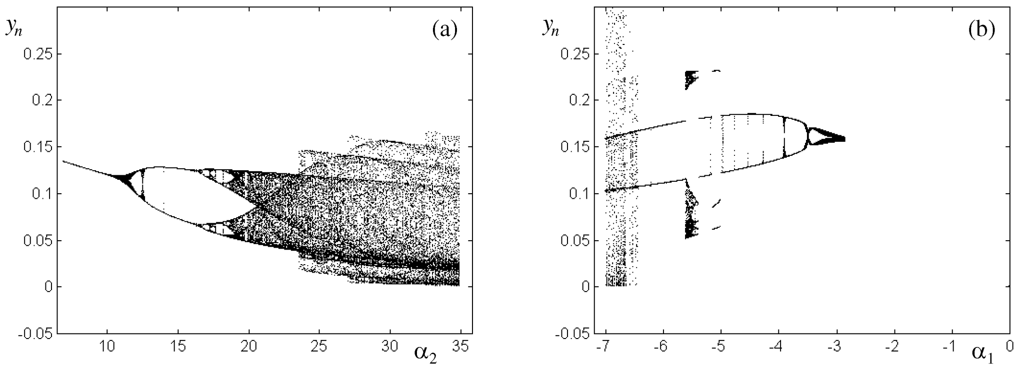

Fig.7: Bifurcations diagrams of system (1)-(2)-(20) computed with the parameters: $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$, when (a) $\alpha_2 \in [7, 35]$, $\alpha_1 = -0.5$ and (b) $\alpha_1 \in [-7, -2.3]$, $\alpha_2 = 10$

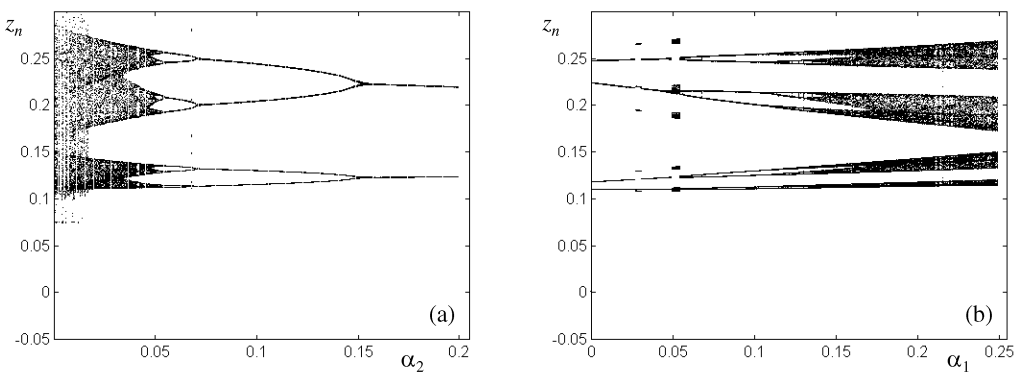

Fig.8: Bifurcations diagrams of system (1)-(2)-(20) computed with the parameters: $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, when (a) $\alpha_2 \in [0.01, 0.2]$, $\alpha_1 = 0.15$ and (b) $\alpha_1 \in [0.01, 0.25]$, $\alpha_2 = 0.05$

The bifurcation diagram in Fig. 7b show that an apparent sudden collapse in the size of a chaotic occurs at a value of the bifurcation parameter $\alpha_2 \approx -6.5$. Therefore, similar to previous Figs. 3b and 3c it is example for interior crisis. It is interesting to note here that near $\alpha_2 = -5.8$ the second interior crisis occur.

Diagram in Fig. 8a depicts the case when $\alpha_1 = 0.15$, $k = 21.5$ and $\alpha_2 \in [0.01, 0.2]$. It is evident that the chaos occurs after period-doubling bifurcations, starting from regular solutions with period two. Comparing Fig. 8a and Fig. 8b, we conclude that behavior of the system (1)-(2)-(20) is not similar. In Fig. 8b, we see four different orbits in which the period-doubling bifurcations occur. Note that interior crises also take place for $\alpha_1 \approx 0.03$ and $\alpha_1 = 0.05$.

Case 5 ($\alpha_2 = 0$). This case is solved identically. Here we obtain

$$\dot{z} = \beta\,[(\alpha + \alpha_1\,x + \alpha_3\,z)\,(1-z) - k\,y^2]\ , \tag{24}$$

where

$$\bar{y}^2 - \frac{\alpha_1}{k\,\beta}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right)\bar{y} - \frac{1}{k}\left(\alpha + \alpha_3\,\frac{1+\beta^2}{k\,\beta^2}\right)\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) = 0\ . \tag{25}$$

Here, the system (1)-(2)-(24) has three fixed point if inequalities

$$1 - \frac{1+\beta^2}{k\,\beta^2} > 0\ , \qquad \frac{\alpha_1^2}{k^2\,\beta^2}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) + \frac{4}{k}\left(\alpha + \alpha_3\,\frac{1+\beta^2}{k\,\beta^2}\right) > 0 \tag{26}$$

or

$$1 - \frac{1+\beta^2}{k\,\beta^2} < 0\ , \qquad \frac{\alpha_1^2}{k^2\,\beta^2}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) + \frac{4}{k}\left(\alpha + \alpha_3\,\frac{1+\beta^2}{k\,\beta^2}\right) < 0 \tag{27}$$

are valid. The condition (26) is valid for $k > 2.778$ and $\alpha_1^2 + 7.1773\,\alpha_3 > -8.3328$ (when $k = 21.5 > 2.778$). The condition (27) is valid for $k < 2.778$ and $\alpha_1^2 - 56.2322\,\alpha_3 > 7.5912$ (when $k = 2.5 < 2.778$).

First, in Figure 9a a bifurcation diagram of the system (1)-(2)-(24) is presented. Here, we consider α_1 to be bifurcation (control) parameter and $\alpha_3 = 0.2$ is fixed. The system has periodic solutions (with period two) at the beginning and in the end of the interval. Near $\alpha_1 = -0.23$ the first interior crisis occur and near $\alpha_1 = -1.52$ the second. In mid interval the system has very rich bifurcation behavior where straight and inverse bifurcations take place. Comparing Fig. 9a and Fig. 9b we conclude that when α_3 is control parameter (see Fig. 9b) only the inverse bifurcations occur (beginning from $\alpha_3 = -0.01$). Here we note that in the end of the intervals for α_1 and α_3 the system has similar behavior (periodic solutions with period two).

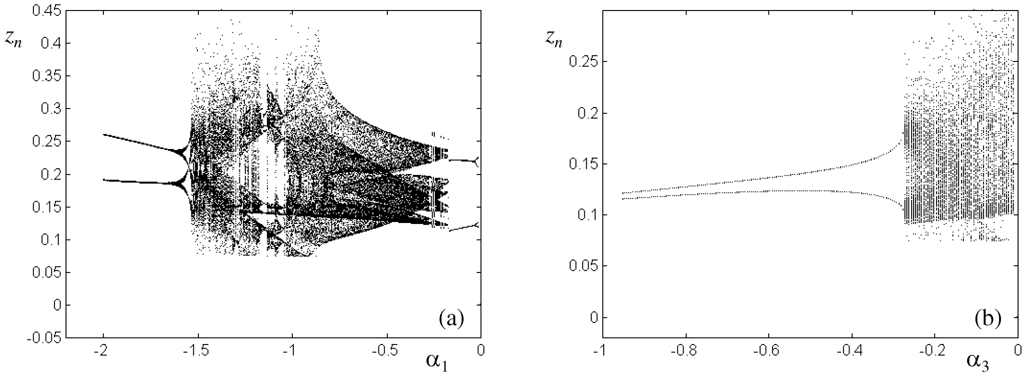

Fig.9: Bifurcation diagrams obtained by integrating system (1)-(2)-(24), (a) for $\alpha_1 \in [-2, -0.01]$, $\alpha_3 = 0.2$ and (b) for $\alpha_3 \in [-0.95, -0.01]$, $\alpha_1 = 0.05$

Second, we compute the case when $k = 2.5 < 2.778$ and we obtain that here system (1)-(2)-(24) has only stable or periodic solutions. Therefore, in this case the system is un-sensitive to linear feedback. That's why, we not illustrate these results.

Case 6 ($\alpha_1 = 0$). Substituting Eq. (7) into Eq. (3), we get the following differential equation

$$\dot{z} = \beta\left[(\alpha + \alpha_2\, y + \alpha_3\, z)\,(1 - z) - k\, y^2\right] \tag{28}$$

and for $\bar{y}$ we write

$$\bar{y}^2 - \frac{\alpha_2}{k}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right)\bar{y} - \frac{1}{k}\left(\alpha + \alpha_3\,\frac{1+\beta^2}{k\,\beta^2}\right)\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) = 0\ . \tag{29}$$

If the inequalities

$$1 - \frac{1+\beta^2}{k\,\beta^2} > 0\ , \qquad \frac{\alpha_2^2}{k^2}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) + \frac{4}{k}\left(\alpha + \alpha_3\,\frac{1+\beta^2}{k\,\beta^2}\right) > 0 \tag{30}$$

or

$$1 - \frac{1+\beta^2}{k\,\beta^2} < 0\ , \qquad \frac{\alpha_2^2}{k^2}\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) + \frac{4}{k}\left(\alpha + \alpha_3\,\frac{1+\beta^2}{k\,\beta^2}\right) < 0 \tag{31}$$

are valid, the system (1)-(2)-(28) has three equilibrium (steady state) points. If case (30) or (31) are not valid the system has only one equilibrium point with coordinates $\bar{x} = \bar{y} = 0$, $\bar{z} = 1$. Condition (30) is valid at $k > 2.778$ and $\alpha_2^2 + 12.76\,\alpha_3 > -14.814$ (when $k = 21.5 > 2.778$). Condition (31) is valid at $\alpha_2^2 - 99.968\,\alpha_3 > 13.469$ (when $k = 2.5 < 2.778$).

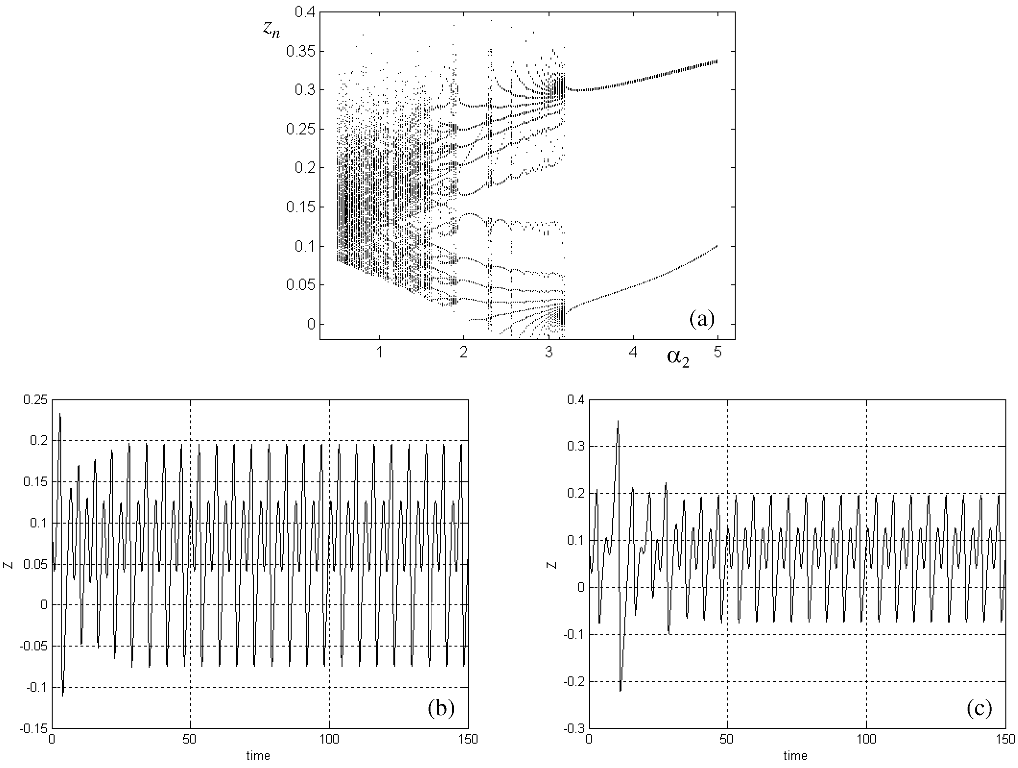

Fig.10: Bifurcation diagram of the system (1)-(2)-(28) at $\alpha = 0.15$, $\beta = 0.75$, $\alpha_3 = 0.05$, $k = 21.5$ and $\alpha_2 \in [0.5, 5]$ (a); periodic solutions at $\alpha = 0.15$, $\beta = 0.75$, $\alpha_2 = 0.2$, $k = 21.5$, $\alpha_3 \in [0.01, 0.5]$ (b), and at $\alpha = 0.15$, $\beta = 0.75$, $\alpha_2 = -0.2$, $k = 21.5$, $\alpha_3 \in [0.01, 0.5]$ (c)

In Figure 10a the bifurcation diagram of system (1)-(2)-(28) for $\alpha = 0.15$, $\beta = 0.75$, $\alpha_3 = 0.05$, $k = 21.5$ and $\alpha_2 \in [0.5, 5]$ is shown. Here we see that when bifurcation parameter α_2 increases, the system passes from chaotic regime to regular one. It is interesting to note that near $\alpha_3 = 1.8$ the interior crisis occur and two symmetrical 'grape like' branches appear till the end of the interval. For $\alpha_3 \in [3.2, 5]$ the system has regular solution with period two. Comparing Fig. 10a and Fig. 3c we see that they resemble each other.

In Figures 10b and 10c we fix $\alpha = 0.15$, $\beta = 0.75$, $\alpha_2 = \pm 0.2$, $k = 21.5$ and vary the parameter $\alpha_3 \in [0.01, 0.5]$. Here, the system has only periodic solutions i.e. after control the modified system also has regular behavior.

By analogy with the previous case (for $k = 2.5$), here the system (1)-(2)-(28) has only stable or regular solutions with period one or two. Therefore, in this case the system is also unsensitive to linear feedback Because of that we not show the numerical results for this case.

Case 7. Finally, replacing (7) into (3), we obtain for z-equation

$$\dot{z} = \beta \left[(\alpha + \alpha_1 x + \alpha_2 y + \alpha_3 z)(1 - z) - k y^2 \right] . \tag{32}$$

Now $\bar{y}$ is

$$\bar{y}^2 - \frac{1}{k}\left(\frac{\alpha_1}{\beta} + \alpha_2\right)\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right)\bar{y} - \frac{1}{k}\left(\alpha + \alpha_3\,\frac{1+\beta^2}{k\,\beta^2}\right)\left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) = 0 . \tag{33}$$

Here the system also has three equilibrium points if inequalities

$$1 - \frac{1+\beta^2}{k\,\beta^2} > 0 , \qquad \left(\frac{\alpha_1}{\beta} + \alpha_2\right)^2 \left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) + 4\,k\left(\alpha + \alpha_3\,\frac{1+\beta^2}{k\,\beta^2}\right) > 0 \tag{34}$$

or

$$1 - \frac{1+\beta^2}{k\,\beta^2} < 0 , \qquad \left(\frac{\alpha_1}{\beta} + \alpha_2\right)^2 \left(1 - \frac{1+\beta^2}{k\,\beta^2}\right) + 4\,k\left(\alpha + \alpha_3\,\frac{1+\beta^2}{k\,\beta^2}\right) < 0 \tag{35}$$

are valid. By analogy with previous cases condition (34) is valid if $k > 2.778$ and $(\alpha_1 + 0.75\,\alpha_2)^2 + 7.1775\,\alpha_3 > -8.3328$ (when $k = 21.5$). Condition (35) is valid if $k < 2.778$ and $(\alpha_1 + 0.75\,\alpha_2)^2 - 56.2322\,\alpha_3 > 7.5912$ (when $k = 2.5$).

The divergence of the flow (1)-(2)-(32) is

$$D_3 = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -\beta[\alpha + \alpha_3 + 2 + \alpha_1 x + \alpha_2 y + (2\,\alpha_3 - k)\,z] . \tag{36}$$

In this case we have three bifurcation (control) parameters α_1, α_2 and α_3. What we see in this general case?

In Figure 11a the bifurcation diagram of the system (1)-(2)-(32) at $\alpha = 0.15$, $\beta = 0.75$, $\alpha_2 = 0.1$, $\alpha_3 = -0.1$, $k = 21.5$ and $\alpha_1 \in [0.01, 2]$ is shown. For $\alpha_1 \in [0.01, 0.68]$ the system has chaotic (see the result for maximal Lyapunov exponent below in the text) behavior. As α_1 increased from 0.68 there are very fast inverse bifurcations. After $\alpha_1 = 1$ (till the end of the interval), the new system is regular with period two. Here we note that chaotic zone in this case is longer than these obtained in Fig. 11b for bifurcation parameter α_2.

Comparing results obtained in Figs. 11a, 11b and 11c, we conclude that the case, when α_3 is bifurcation parameter, is more interesting from dynamical point of view than these with bifurcation parameters α_1 and α_2. Discussing the results shown in Fig. 11c, it is seen that for $\alpha = 0.15$, $\beta = 0.75$, $\alpha_1 = 0.3$, $\alpha_2 = 0.1$, $k = 21.5$ and $\alpha_3 \in [-0.75, -0.63]$ the system has periodic solutions with period two. After that (as α_3 increased) in result of interior crisis the system suddenly passes to chaotic regime. Here we note that at $\alpha_3 \in [-0.27, -0.24]$, $\alpha_3 \in [-0.17, -0.13]$ and $\alpha_3 \in [-0.075, -0.01]$ the inverse bifurcations take place. It is interesting that in the end of the interval the new system has regular behavior with period four.

We note here that, when $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$ (regular case) are fixed and vary α_1, α_2, α_3, the new system has only stable or regular solutions with period one.

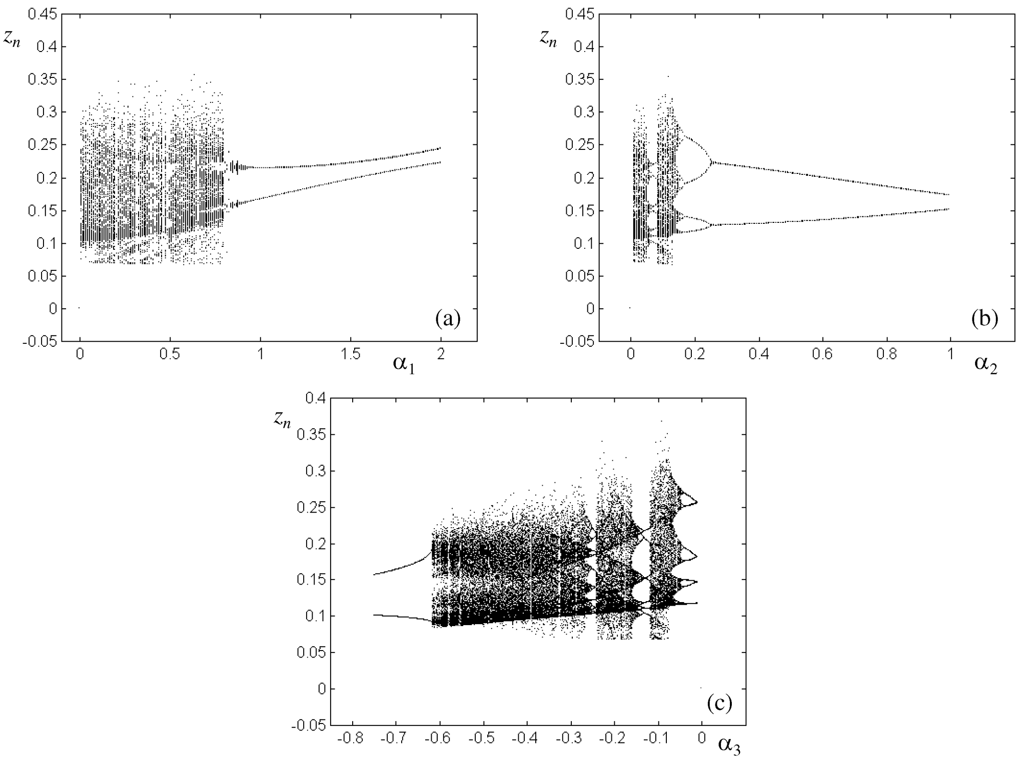

Fig.11: Bifurcation diagrams obtained by integrating system (1)-(2)-(32), (a) for $\alpha_1 \in [0.01, 0.2]$, $\alpha_2 = 0.1$, $\alpha_3 = -0.1$; (b) for $\alpha_2 \in [0.01, 1]$, $\alpha_1 = 0.3$, $\alpha_3 = -0.1$ and (c) for $\alpha_3 \in [-0.75, -0.01]$, $\alpha_1 = 0.3$, $\alpha_2 = 0.1$

The Lyapunov exponents describe the action of the dynamics defining the evolution of trajectories. In chaotic evolutions nearby trajectories separate exponentially. For small enough length scales and short enough time scales the initial effect of the dynamics will be to distort the evolving spheroid into an ellipsoidal shape, with some directions being stretched and others contracted. The longest axis of this ellipsoid will correspond to the most unstable direction of the flow. The asymptotic rate of expansion of this axis is measured by the largest (maximal) Lyapunov exponent. In details, if the infinitesimal radius of the initial fiducial volume is called $r(0)$, and the length of the ith principal axis at time t is called $l_i(t)$, then the ith Lyapunov exponent can be defined as [26, 27, 29] :

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log \frac{l_i(t)}{r(0)} \ . \tag{37}$$

The Lyapunov exponents are always ordered and it is common to use a decreasing ordering in the spectrum of Lyapunov exponents, $\lambda_1 > \lambda_2 > \lambda_3 > \dots$. Here we note that for maps of dimension k and for continuous-time dynamical systems this spectrum of Lyapunov exponents is defined similarly. More precisely information and theorems can be found in [30].

The maximal Lyapunov exponent $\lambda_{\max}$ shows the kind of motion on the phase space: (i) if $\lambda_{\max} < 0$ the motion is a stable fixed point; (ii) if $\lambda_{\max} = 0$ the motion is a stable limit cycle; (iii) if $0 < \lambda_{\max} < \infty$ the motion is chaotic and (iv) if $\lambda_{\max} = \infty$ the motion is noise [26]. Following [26], the maximal Lyapunov exponent for a given data set can be calculated by means of the sum

$$S(\Delta n) = \frac{1}{N} \sum_{n_0}^{N} \ln\left(\frac{1}{|\Psi(\beta_{n_0})|} \sum_{\beta_{n_0}\in\Psi} |s_{n_0+\Delta n} - s_{n+\Delta n}| \right) , \tag{38}$$

where reference points β_{n_0} are embedding vectors, $\Psi(\beta_{n_0})$ is the neighborhood of β_{n_0} with diameter ε, s_{n_0} is the last element of β_{n_0} and $s_{n_0+\Delta n}$ is outside the time span covered by the delay vector β_{n_0}. Since a priori one might neither know the minimal embedding dimension m nor the optimal distance ε, one should compute $S(\Delta n)$ for a variety of both values. The size of the neighbourhood should be as small as possible, but large enough such that on average each reference point has at least a few neighbours. Otherwise one might systematically ignore certain parts of the attractor and thus compute a wrong value [26, 27].

For the numerical calculation of $\lambda_{\max}$ we use the TISEAN software package [28]. The obtained maximal Lyapunov exponents (per unit time) are represented in Appendix.

Comparing these results, we conclude that in Case 2 (a), modified Sherman system is more chaotic than the rest, but is smaller chaotic than original Sherman system (see [16, 20]). Opposite, in case 5 (b) the maximal Lyapunov exponent is the smallest.

3. Summary and conclusions

The paper studies how the dynamics and the global behavior of Sherman's system vary, introducing linear feedback $\alpha \to \alpha + \alpha_1\, x(t) + \alpha_2\, y(t) + \alpha_3\, z(t)$ in the equation for z (see (3)). Assuming successively that some of the coefficients α_n ($n = 1, 2, 3$) be zero, we find seven different modifications of Sherman's system. The governing equations were solved numerically using MATLAB (The MathWorks, Inc., Natick, MA, USA).

Firstly, we summarize :

A. Case at $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$.

The system in case 1 (i.e. $\alpha \to \alpha + \alpha_1\, x(t)$) has three fixed points, i.e. a number of fixed points is equal to that of the original system. Yet, it is seen from numerical results (see Fig. 1) that the modified system has also regular behavior after long transitional period. The modified system in case 2 attains a chaotic state after continuous doubling of the period. This is a result of the gradual increase of the bifurcation parameter α_2. Here we note that the geometry of the chaotic attractor is different from the geometry of the chaotic attractor of the original system ([16, 18, 20] and Fig. 4). The modified system in cases 3, 5, 6, 7 has only stable or regular (with period one or two) solutions. We should note also here that:

– considering all the cases, we find such values of the parameters, α_1, α_2 and α_3, where the system is simultaneously structurally stable and has real fixed points;
– the parameters were fixed as $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$, while initial conditions $x_0 = 0.01$, $y_0 = z_0 = 0.1$.

B. Case at $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$.

Here we find such values of the coefficients α_1, α_2 and α_3 (different from these in A) where the system has regular solutions with different period, and real fixed points. For all simulations the initial conditions were also $x_0 = 0.01$, $y_0 = z_0 = 0.1$. Here we note that at symmetrical intervals of change of the bifurcation parameter, the modified systems have symmetrical bifurcation diagrams, i.e. symmetrical behavior. We find also that maximal Lyapunov exponents for symmetrical intervals are approximately equal.

Finally, the proposed study is a first step to the profound and fill analysis of the modifications thus found. We hope that the approach taken here will find application in the practice (for example in areas of Electro-mechanics and Mechatronics) in which chaos, plays a role.

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Appendix: Calculation of the maximal Lyapunov exponents

for **Case 1**

(a) bifurcation parameter (BP) α_1: $\lambda_{\max} = 0.173 \pm 0.021$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_1 = -0.5$,

(b) BP α_1: $\lambda_{\max} = 0.1685 \pm 0.0225$ when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_1 = 0.5$,

for **Case 2**

(a) BP α_2: $\lambda_{\max} = 0.2869 \pm 0.0241$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$, $\alpha_2 = 27$,

(b) BP α_2: $\lambda_{\max} = 0.2445 \pm 0.03$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_2 = 0.9$,

(c) BP α_2: $\lambda_{\max} = 0.1935 \pm 0.0201$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_2 = -0.9$,

for **Case 3**

BP α_3: $\lambda_{\max} = 0.1335 \pm 0.0144$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_3 = -0.03$,

for **Case 4**

(a) BP α_2: $\lambda_{\max} = 0.1459 \pm 0.004$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 2.5$, $\alpha_1 = -0.5$, $\alpha_2 = 25$,

(b) BP α_2: $\lambda_{\max} = 0.183 \pm 0.02$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_1 = 0.15$, $\alpha_2 = 0.018$,

for **Case 5**

(a) BP α_1: $\lambda_{\max} = 0.1013 \pm 0.012$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_1 = -0.8$, $\alpha_3 = 0.2$,

(b) BP α_3: $\lambda_{\max} = 0.0373 \pm 0.00134$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_1 = 0.05$, $\alpha_3 = -0.2$,

for **Case 6**

BP α_2: $\lambda_{\max} = 0.1535 \pm 0.0175$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_2 = 0.8$, $\alpha_3 = 0.05$,

for **Case 7**

(a) BP α_1: $\lambda_{\max} = 0.1197 \pm 0.0132$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_1 = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = -0.1$,

(b) BP α_2: $\lambda_{\max} = 0.10097 \pm 0.0099$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_1 = 0.3$, $\alpha_2 = 0.03$, $\alpha_3 = -0.1$,

(c) BP α_3: $\lambda_{\max} = 0.216 \pm 0.0161$, when $\alpha = 0.15$, $\beta = 0.75$, $k = 21.5$, $\alpha_1 = 0.3$, $\alpha_2 = 0.1$, $\alpha_3 = -0.35$.

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