ANALYTICAL SOLUTION TO INFINITE FATIGUE LIFE OF MACHINE PARTS UNDER HARMONIC LOADING

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The article presents an analytical solution to deterministic problems of infinite fatigue life (determining the safety factor, predicting the carrying capacity and proposing the cross-section dimensions) of machine parts subjected to arbitrary pure harmonic loading. Presented mathematical models do not require to construct any fatigue strength diagram adopting it only as a basis of graphical solving. Finally, these models may serve as a starting point formulating analytical solution to deterministic problems of infinite fatigue life of machine parts under combined harmonic loading.

Keywords: *infinite fatigue life, harmonic loading, safety factor, carrying capacity, cross-section dimensions*

1. Introduction

An oldest and frequently used procedure of machine parts dimensioning subjected to fatigue loading is dimensioning on infinite fatigue life, i.e. dimensioning under the endurance limit [1–4]. The procedure adopts a safety factor related to this limit. Regarding fatigue data dispersion the safety factor is to be prescribed bigger than the static safety factor. This implies the bigger robustness of structures and reduces specific power of machines.

Regardless of it, designing methods on infinite fatigue life preserve their practical importance till today [5] : they enable structural design for infinite fatigue life and the working competence of machine parts designed according to this approach may be reached during the whole period of physical life of a whole structure. Further methods of designing machine parts subjected to oscillating loading have therefore been developed in many universities and research laboratories during recent decades, e.g., [6]. Dimensioning on limited or timed strength is to be mentioned which makes it possible to design machine parts with increased specific power.

The classical conception of infinite fatigue life ($N_{\mathrm{C}} \geq 10^7$) assumes the nominal stress state in characteristic cross-section of a machine part and its determination according to elementary elasticity. The fundamental parameter for determining the limit state of fatigue is the unnotched fully reversed fatigue limit, σ_{C} (or τ_{C}) of a material. Completing these basic assumptions with Goodman's linear approximation of the effect of loading cycles asymmetry, the simplest classical version of the infinite fatigue life may be adopted. The graphical solving of deterministic problem of infinite fatigue life of machine parts subjected to pure harmonic loading is widely spread, especially as fare as determining the safety factor is concerned. The graphical solving of other problems of infinite fatigue life of machine parts within the

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frame of the classical approach is either cumbersome or cannot be applied. In addition, the graphical solving excludes the automatic plotting of appropriate nomograms.

This article presents an analytical solution to deterministic problems of infinite fatigue life (determining the safety factor, predicting the carrying capacity and proposing the cross-section dimensions) of machine parts subjected to arbitrary pure harmonic loading. Presented mathematical models do not require to construct any fatigue strength diagram adopting it as a basis of graphical solving. In addition, these models make it possible to solve the problems in question inclusive of problems which are not solvable graphically. They facilitate the plotting of nomograms which are not available by using neither graphical nor discrete methods [7, 8]. Finally, the presented mathematical models may serve as a starting point for formulating analytical solution to deterministic problems of infinite fatigue life of machine parts under arbitrary combined harmonic loading.

2. Input data for analytical models of infinite fatigue life

Input data according to Figs. 1 and 2 may be classified in the following groups:

– mechanical properties of a material:

$$\begin{aligned} \sigma_\mathrm{F}, \tau_\mathrm{F} &= \text{fictive strength (usually identified with ultimate strength)}\ , \\ \sigma_\mathrm{K}, \tau_\mathrm{K} &= \text{yield strength*}\ , \\ \sigma_\mathrm{C}, \tau_\mathrm{C} &= \text{unnotched fully reversed fatigue limit*}\ , \end{aligned} \tag{1}$$

– the fatigue notch factors:

$$\beta_\sigma = 1 + (\alpha_\sigma - 1)\, q\ , \qquad \beta_\tau = 1 + (\alpha_\tau - 1)\, q\ , \tag{2}$$

where α_σ and α_τ are the elastic stress concentration factors and q is the notch sensitivity of a material,

– or, as the case may be, reduced fatigue notch factors

$$\beta_{\sigma\mathrm{r}} = \frac{\beta_\sigma}{v_\sigma\, \varphi_\sigma}\ , \qquad \beta_{\tau\mathrm{r}} = \frac{\beta_\tau}{v_\tau\, \varphi_\tau}\ , \tag{3}$$

where v_σ, v_τ are the size factors and φ_σ, φ_τ stand for the fatigue strength surface condition factors,

– fatigue limits of a notched part are

$$\sigma^*_\mathrm{C} = \frac{v_\sigma\, \varphi_\sigma}{\beta_\sigma}\, \sigma_\mathrm{C} \quad \Rightarrow \quad \sigma^*_\mathrm{C} = \frac{\sigma_\mathrm{C}}{\beta_{\sigma\mathrm{r}}}\ , \tag{4}$$

or

$$\tau^*_\mathrm{C} = \frac{v_\tau\, \varphi_\tau}{\beta_\tau}\, \tau_\mathrm{C} \quad \Rightarrow \quad \tau^*_\mathrm{C} = \frac{\tau_\mathrm{C}}{\beta_{\tau\mathrm{r}}}\ , \tag{5}$$

– the magnitude of loading (the maximum of loading cycle):

$$\text{at bending:} \qquad M^\mathrm{P}_\mathrm{oh}\ , \tag{6}$$

$$\text{at tension-compression:} \qquad N^\mathrm{P}_\mathrm{h}\ , \tag{7}$$

$$\text{at torsion:} \qquad M^\mathrm{P}_\mathrm{kh}\ , \tag{8}$$

* For pure bending, specific values of yield strength and fatigue limit (both higher then values for tension-compression) should be used.

where $M_{\mathrm{oh}}^{\mathrm{P}}$, $N_{\mathrm{h}}^{\mathrm{P}}$ and $M_{\mathrm{kh}}^{\mathrm{P}}$ denote the generalized internal forces,

– the slope of load line:

at bending or tension-compression: $$\eta_\sigma = \frac{\sigma_{\mathrm{h}}^{\mathrm{P}}}{\sigma_{\mathrm{m}}^{\mathrm{P}}} \in \langle 1;\infty) \ , \tag{9}$$

at torsion: $$\eta_\tau = \frac{\tau_{\mathrm{h}}^{\mathrm{P}}}{\tau_{\mathrm{m}}^{\mathrm{P}}} \in \langle 1;\infty) \ , \tag{10}$$

– the size of a characteristic cross-section of a structure part (e.g. the size of a diameter d of the shaft cross-section),

– the safety factor related to the endurance limit allowing for the linear approximation of the effect of loading cycles asymmetry, corresponding to the type of loading: at a bending or tension-compression:

$$k_\sigma = \frac{\sigma_{\mathrm{h}}^{\mathrm{M}\sigma}}{\sigma_{\mathrm{h}}^{\mathrm{P}}} \ , \tag{11}$$

where

$$\sigma_{\mathrm{h}}^{\mathrm{P}} = \frac{M_{\mathrm{oh}}^{\mathrm{P}}}{W_{\mathrm{o}}} \quad \text{or} \quad \sigma_{\mathrm{h}}^{\mathrm{P}} = \frac{N_{\mathrm{h}}^{\mathrm{P}}}{S} \tag{12}$$

or at a torsion:

$$k_\tau = \frac{\tau_{\mathrm{h}}^{\mathrm{M}\tau}}{\tau_{\mathrm{h}}^{\mathrm{P}}} \ , \tag{13}$$

where

$$\tau_{\mathrm{h}}^{\mathrm{P}} = \frac{M_{\mathrm{kh}}^{P}}{W_{\mathrm{k}}} \tag{14}$$

with S, W_{o} and W_{k} standing for the cross-section properties.

3. Deriving mathematical models of infinite fatigue life

We will derive mathematical models of infinite fatigue life following the classical Smith's diagram. The estimated Smith's diagram for a part subjected to pure harmonic axial stress due to bending or tension-compression is in Fig. 1. Similarly, the Smith's diagram for a part under pure harmonic shear stress due to torsion is in Fig. 2. Derivation of analytical expressions of the limiting curve A^*D^*G of a fatigue failure consists in deriving relations for ordinates $\sigma_{\mathrm{h}}^{M_{\mathrm{C}}}$ and $\tau_{\mathrm{h}}^{M_{\mathrm{C}}}$ of points $M_{\mathrm{C}} \subset \overline{A^*D^*}$ in Smith's diagrams according to Figs. 1a and 2a. A magnitude of the stress components is determined by magnitude of the upper quantity, $\sigma_{\mathrm{h}}^{\mathrm{P}}$ or $\tau_{\mathrm{h}}^{\mathrm{P}}$, of loading cycles. Equations of straight lines OM_{C} and A^*M_{C} are as follows:

– an equation of the straight line OM_{C}:

$$\sigma_{\mathrm{h}}^{M_{\mathrm{C}}} = \frac{\sigma_{\mathrm{h}}^{\mathrm{P}}}{\sigma_{\mathrm{m}}^{\mathrm{P}}}\,\sigma_{\mathrm{m}}^{M_{\mathrm{C}}} = \left(1 + \frac{\sigma_{\mathrm{a}}^{\mathrm{P}}}{\sigma_{\mathrm{m}}^{\mathrm{P}}}\right)\sigma_{\mathrm{m}}^{M_{\mathrm{C}}} \ , \tag{15}$$

– an equation of the straight line A^*M_{C}:

$$\sigma_{\mathrm{h}}^{M_{\mathrm{C}}} = \frac{\sigma_{\mathrm{F}} - \sigma_{\mathrm{C}}^*}{\sigma_{\mathrm{F}}}\,\sigma_{\mathrm{m}}^{M_{\mathrm{C}}} + \sigma_{\mathrm{C}}^* = \left(1 - \frac{\sigma_{\mathrm{C}}^*}{\sigma_{\mathrm{F}}}\right)\sigma_{\mathrm{m}}^{M_{\mathrm{C}}} + \sigma_{\mathrm{C}}^* \ . \tag{16}$$

Thus, regarding Eqs. (15) and (16) we get

$$\left(1+\frac{\sigma_{\mathrm{a}}^{\mathrm{P}}}{\sigma_{\mathrm{m}}^{\mathrm{P}}}\right)\sigma_{\mathrm{m}}^{M_{\mathrm{C}}}=\left(1-\frac{\sigma_{\mathrm{C}}^{*}}{\sigma_{\mathrm{F}}}\right)\sigma_{\mathrm{m}}^{M_{\mathrm{C}}}+\sigma_{\mathrm{C}}^{*}, \tag{17}$$

from which

$$\sigma_{\mathrm{m}}^{M_{\mathrm{C}}}=\frac{\sigma_{\mathrm{C}}^{*}}{\dfrac{\sigma_{\mathrm{a}}^{\mathrm{P}}}{\sigma_{\mathrm{m}}^{\mathrm{P}}}+\dfrac{\sigma_{\mathrm{C}}^{*}}{\sigma_{\mathrm{F}}}}\ . \tag{18}$$

By substituting Eq. (18) into Eqs. (15) or (16) we get

$$\sigma_{\mathrm{h}}^{M_{\mathrm{C}}}=\frac{1+\dfrac{\sigma_{\mathrm{a}}^{\mathrm{P}}}{\sigma_{\mathrm{m}}^{\mathrm{P}}}}{\dfrac{\sigma_{\mathrm{a}}^{\mathrm{P}}}{\sigma_{\mathrm{m}}^{\mathrm{P}}}+\dfrac{\sigma_{\mathrm{C}}^{*}}{\sigma_{\mathrm{F}}}}\,\sigma_{\mathrm{C}}^{*}\ . \tag{19}$$

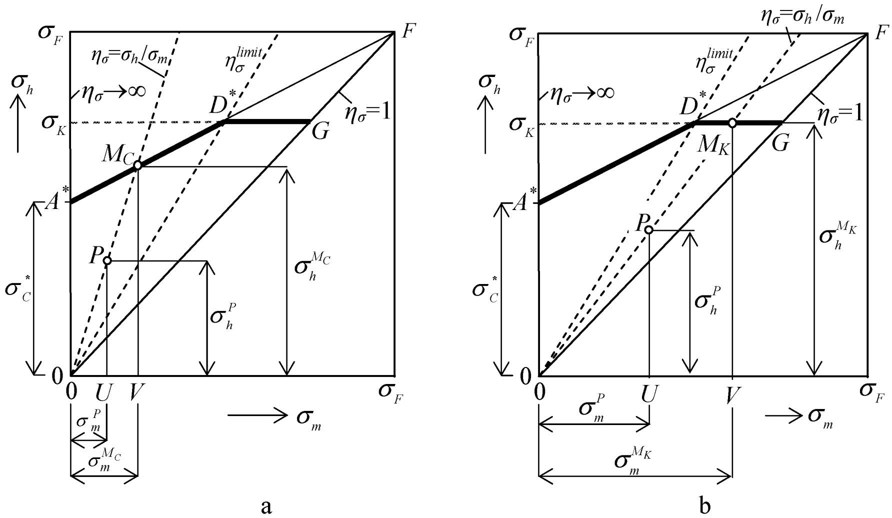

Fig.1: Smith's diagram of machine parts subjected to harmonic tension-compression or bending with indicated load lines η_σ in two characteristic positions: a) $\eta_\sigma \geq \eta_\sigma^{\mathrm{limit}}$ or b) $1 < \eta_\sigma < \eta_\sigma^{\mathrm{limit}}$ (for $\eta_\sigma^{\mathrm{limit}}$ see eq. (33))

Recalling the slope of load line

$$\frac{\sigma_{\mathrm{h}}}{\sigma_{\mathrm{m}}}=\eta_\sigma \in \langle 1,\infty)\ , \tag{20}$$

which implies

$$\eta_\sigma=\frac{\sigma_{\mathrm{m}}+\sigma_{\mathrm{a}}}{\sigma_{\mathrm{m}}}=1+\frac{\sigma_{\mathrm{a}}}{\sigma_{\mathrm{m}}}\ , \tag{21}$$

the Eqs. (18) and (19) read

$$\sigma_{\mathrm{m}}^{M_{\mathrm{C}}}=\frac{\sigma_{\mathrm{C}}^{*}}{\eta_\sigma+\dfrac{\sigma_{\mathrm{C}}^{*}}{\sigma_{\mathrm{F}}}-1} \tag{22}$$

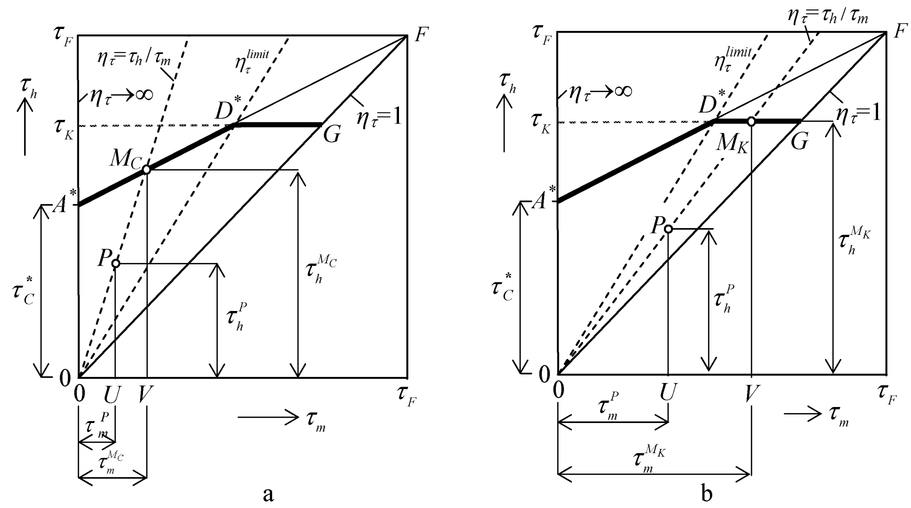

Fig.2: Smith's diagram of machine parts subjected to harmonic torsion with indicated load lines η_τ in two characteristic positions: a) $\eta_\tau \geq \eta_\tau^{\rm limit}$ or b) $1 < \eta_\tau < \eta_\tau^{\rm limit}$ (for $\eta_\tau^{\rm limit}$ see eq. (36))

and

$$\sigma_{\rm h}^{M_{\rm C}} = \frac{\sigma_{\rm C}^*}{\eta_\sigma + \dfrac{\sigma_{\rm C}^*}{\sigma_{\rm F}} - 1}\,\eta_\sigma = \sigma_{\rm m}^{M_{\rm C}}\,\eta_\sigma\ . \tag{23}$$

Equations of straight lines, $OM_{\rm K}$ and D^*G, are

$$\sigma_{\rm h}^{M_{\rm K}} = \frac{\sigma_{\rm h}^{\rm P}}{\sigma_{\rm m}^{\rm P}}\,\sigma_{\rm m}^{M_{\rm K}} = \eta_\sigma\,\sigma_{\rm m}^{M_{\rm K}} \tag{24}$$

and

$$\sigma_{\rm h}^{M_{\rm K}} = \sigma_{\rm K}\ . \tag{25}$$

Substituting this identity into Eq. (24) we arrive at

$$\sigma_{\rm m}^{M_{\rm K}} = \frac{\sigma_{\rm K}}{\eta_\sigma}\ . \tag{26}$$

Referring to a notation in Figs. 2a and 2b, we can derive the coordinates of points $M_{\rm C}$ and $M_{\rm K}$ in a Smith's diagram for infinite fatigue life of machine parts under pure torsional harmonic loading following Eqs. (22), (23) and (25), (26), respectively:

$$\tau_{\rm m}^{M_{\rm C}} = \frac{\tau_{\rm C}^*}{\eta_\tau + \dfrac{\tau_{\rm C}^*}{\tau_{\rm F}} - 1}\ , \tag{27}$$

$$\tau_{\rm h}^{M_{\rm C}} = \frac{\tau_{\rm C}^*}{\eta_\tau + \dfrac{\tau_{\rm C}^*}{\tau_{\rm F}} - 1}\,\eta_\tau = \tau_{\rm m}^{M_{\rm C}}\,\eta_\tau \tag{28}$$

or

$$\tau_{\rm h}^{M_{\rm K}} = \tau_{\rm K}\ , \tag{29}$$

$$\tau_{\rm m}^{M_{\rm K}} = \frac{\tau_{\rm K}}{\eta_\tau}\ . \tag{30}$$

Substituting for $\sigma_{\rm h}^{M_{\rm C}}$ from Eq. (23) a condition

$$\sigma_{\rm h}^{M_{\rm C}} = \sigma_{\rm K} \tag{31}$$

will read

$$\frac{\sigma_{\rm C}^*}{\eta_\sigma + \dfrac{\sigma_{\rm C}^*}{\sigma_{\rm F}} - 1}\,\eta_\sigma = \sigma_{\rm K} \ , \tag{32}$$

from which

$$\eta_\sigma = \eta_\sigma^{\rm limit} = \frac{1 - \dfrac{\sigma_{\rm C}^*}{\sigma_{\rm F}}}{1 - \dfrac{\sigma_{\rm C}^*}{\sigma_{\rm K}}} \ . \tag{33}$$

Similarly, substituting for $\tau_{\rm h}^{M_{\rm C}}$ from Eq. (28) a condition

$$\tau_{\rm h}^{M_{\rm C}} = \tau_{\rm K} \tag{34}$$

implies

$$\frac{\tau_{\rm C}^*}{\eta_\tau + \dfrac{\tau_{\rm C}^*}{\tau_{\rm F}} - 1}\,\eta_\tau = \tau_{\rm K} \ , \tag{35}$$

yielding

$$\eta_\tau = \eta_\tau^{\rm limit} = \frac{1 - \dfrac{\tau_{\rm C}^*}{\tau_{\rm F}}}{1 - \dfrac{\tau_{\rm C}^*}{\tau_{\rm K}}} \ . \tag{36}$$

The quantities $\eta_\sigma^{\rm limit}$ and $\eta_\tau^{\rm limit}$ according to (33) and (36) represent limit magnitudes of the factors of a loading cycle asymmetry for bending, tension-compression and torsion, respectively. Recalling denotation according to Figs. 1 and 2, the following relations may be assumed:

$$\sigma_{\rm h}^{M_\sigma} = \begin{cases} \sigma_{\rm h}^{M_{\rm C}} = \dfrac{\eta_\sigma}{\eta_\sigma + \dfrac{\sigma_{\rm C}^*}{\sigma_{\rm F}} - 1}\,\sigma_{\rm C}^* & \text{if} \quad \eta_\sigma \geq \eta_\sigma^{\rm limit} \ , \\ \sigma_{\rm h}^{M_{\rm K}} = \sigma_{\rm K} & \text{if} \quad \eta_\sigma < \eta_\sigma^{\rm limit} \ , \end{cases} \tag{37}$$

$$\tau_{\rm h}^{M_\tau} = \begin{cases} \tau_{\rm h}^{M_{\rm C}} = \dfrac{\eta_\tau}{\eta_\tau + \dfrac{\tau_{\rm C}^*}{\tau_{\rm F}} - 1}\,\tau_{\rm C}^* & \text{if} \quad \eta_\tau \geq \eta_\tau^{\rm limit} \ , \\ \tau_{\rm h}^{M_{\rm K}} = \tau_{\rm K} & \text{if} \quad \eta_\tau < \eta_\tau^{\rm limit} \ . \end{cases} \tag{38}$$

Thus, the ordinates $\sigma_{\rm h}^{M_\sigma}$ and $\tau_{\rm h}^{M_\tau}$ of points $M_{\rm C}$ and $M_{\rm K}$ of limit curves A^*D^*G of infinite fatigue life represent the fundamental relations for formulating appropriate mathematical models of infinite fatigue life of machine parts subjected to arbitrary pure harmonic loading.

4. Solving basic problems of infinite fatigue life of machine parts subjected to pure harmonic loading

Assuming a pure harmonic loading in bending or tension-compression, the safety factor, k_σ, may be defined as

$$k_\sigma = \frac{\sigma_\mathrm{h}^{M_\sigma}}{\sigma_\mathrm{h}^\mathrm{P}} \ , \tag{39}$$

where $\sigma_\mathrm{h}^{M_\sigma}$ is covered by (37) and

$$\sigma_\mathrm{h}^\mathrm{P} = \frac{M_\mathrm{oh}^\mathrm{P}}{W_\mathrm{o}} \qquad \text{or} \qquad \sigma_\mathrm{h}^\mathrm{P} = \frac{N_\mathrm{h}^\mathrm{P}}{S} \ , \tag{40}$$

in which, e.g. for machine parts with circular cross-section of the diameter d,

$$W_\mathrm{o} = \frac{\pi\, d^3}{32} \qquad \text{or} \qquad S = \frac{\pi\, d^2}{4} \ . \tag{41}$$

Similarly, at pure harmonic loading in torsion the safety factor, k_τ, is to be defined as

$$k_\tau = \frac{\tau_\mathrm{h}^{M_\tau}}{\tau_\mathrm{h}^\mathrm{P}} \ , \tag{42}$$

where for $\tau_\mathrm{h}^{M_\tau}$ the relation (38) is valid and

$$\tau_\mathrm{h}^\mathrm{P} = \frac{M_\mathrm{kh}^\mathrm{P}}{W_\mathrm{k}} \ , \tag{43}$$

where

$$W_\mathrm{k} = \frac{\pi\, d^3}{16} \ . \tag{44}$$

Substituting (40) in (39) and (43) in (42) we get for the safety factors k_σ and k_τ

$$k_\sigma = \frac{W_\mathrm{o}}{M_\mathrm{oh}^\mathrm{P}}\, \sigma_\mathrm{h}^{M_\sigma} \qquad \text{or} \qquad k_\sigma = \frac{S}{N_\mathrm{h}^\mathrm{P}}\, \sigma_\mathrm{h}^{M_\sigma} \tag{45}$$

and

$$k_\tau = \frac{W_\mathrm{k}}{M_\mathrm{kh}^\mathrm{P}}\, \tau_\mathrm{h}^{M_\tau} \ . \tag{46}$$

Hence, the upper magnitude of harmonic cycles of loading read

$$M_\mathrm{oh}^\mathrm{P} = \frac{W_\mathrm{o}}{k_\sigma}\, \sigma_\mathrm{h}^{M_\sigma} \qquad \text{or} \qquad N_\mathrm{h}^\mathrm{P} = \frac{S}{k_\sigma}\, \sigma_\mathrm{h}^{M_\sigma} \tag{47}$$

and

$$M_\mathrm{kh}^\mathrm{P} = \frac{W_\mathrm{k}}{k_\tau}\, \tau_\mathrm{h}^{M_\tau} \tag{48}$$

and for magnitude of cross-section properties W_o or S and W_k,

$$W_\mathrm{o} = \frac{M_\mathrm{oh}^\mathrm{P}\, k_\sigma}{\sigma_\mathrm{h}^{M_\sigma}} \qquad \text{or} \qquad S = \frac{N_\mathrm{h}^\mathrm{P}\, k_\sigma}{\sigma_\mathrm{h}^{M_\sigma}} \tag{49}$$

and

$$W_\mathrm{k} = \frac{M^\mathrm{P}_\mathrm{kh}\, k_\tau}{\tau_\mathrm{h}^{M_\tau}} \; . \tag{50}$$

Thus, the relations (45)–(50) and (37), (38) enable analytical solving to all deterministic problems of infinite fatigue life of machine parts subjected to arbitrary pure harmonic loading:

– determining the safety factor,
– predicting the limit carrying capacity, and
– dimensioning the cross-section.

For mechanical properties σ_F, σ_K and σ_C of a material subjected to bending loading M_o or tension-compression N and for mechanical properties τ_F, τ_K and τ_C of a material subjected to torsion moment M_k the following notation will be introduced as follows:

$$\sigma_{\mathrm{F}M_\mathrm{o}} = (\sigma_\mathrm{F})_{M_\mathrm{o}} \; , \qquad \sigma_{\mathrm{K}M_\mathrm{o}} = (\sigma_\mathrm{K})_{M_\mathrm{o}} \; , \qquad \sigma_{\mathrm{C}M_\mathrm{o}} = (\sigma_\mathrm{C})_{M_\mathrm{o}} \; , \tag{51}$$

$$\sigma_{\mathrm{F}N} = (\sigma_\mathrm{F})_{N} \; , \qquad \sigma_{\mathrm{K}N} = (\sigma_\mathrm{K})_{N} \; , \qquad \sigma_{\mathrm{C}N} = (\sigma_\mathrm{C})_{N} \; , \tag{52}$$

$$\tau_{\mathrm{F}M_\mathrm{k}} = (\tau_\mathrm{F})_{M_\mathrm{k}} \; , \qquad \tau_{\mathrm{K}M_\mathrm{k}} = (\tau_\mathrm{K})_{M_\mathrm{k}} \; , \qquad \tau_{\mathrm{C}M_\mathrm{k}} = (\tau_\mathrm{C})_{M_\mathrm{k}} \; , \tag{53}$$

$$(\sigma_\mathrm{h}^{M_\sigma})_{M_\mathrm{o}} = \sigma_\mathrm{h}^{M_\sigma} \quad \text{according to (37) at the harmonic bending, } M_\mathrm{o} \; , \tag{54}$$

$$(\sigma_\mathrm{h}^{M_\sigma})_{N} = \sigma_\mathrm{h}^{M_\sigma} \quad \text{according to (37) at the harmonic tension-compression, } N \; , \tag{55}$$

$$(\tau_\mathrm{h}^{M_\tau})_{M_\mathrm{k}} = \tau_\mathrm{h}^{M_\tau} \quad \text{according to (38) at the harmonic torsion, } M_\mathrm{k} \; , \tag{56}$$

$$\eta^\mathrm{limit}_{\sigma M_\mathrm{o}} = (\eta^\mathrm{limit}_\sigma)_{M_\mathrm{o}} = \left(\frac{1 - \sigma^*_\mathrm{C}/\sigma_\mathrm{F}}{1 - \sigma^*_\mathrm{C}/\sigma_\mathrm{K}} \right)_{M_\mathrm{o}} = \frac{1 - \sigma^*_{\mathrm{C}M_\mathrm{o}}/\sigma_{\mathrm{F}M_\mathrm{o}}}{1 - \sigma^*_{\mathrm{C}M_\mathrm{o}}/\sigma_{\mathrm{K}M_\mathrm{o}}} \; , \tag{57}$$

$$\eta^\mathrm{limit}_{\sigma N} = (\eta^\mathrm{limit}_\sigma)_{N} = \left(\frac{1 - \sigma^*_\mathrm{C}/\sigma_\mathrm{F}}{1 - \sigma^*_\mathrm{C}/\sigma_\mathrm{K}} \right)_{N} = \frac{1 - \sigma^*_{\mathrm{C}N}/\sigma_{\mathrm{F}N}}{1 - \sigma^*_{\mathrm{C}N}/\sigma_{\mathrm{K}N}} \; , \tag{58}$$

$$\eta^\mathrm{limit}_{\tau M_\mathrm{k}} = (\eta^\mathrm{limit}_\tau)_{M_\mathrm{k}} = \left(\frac{1 - \tau^*_\mathrm{C}/\tau_\mathrm{F}}{1 - \tau^*_\mathrm{C}/\tau_\mathrm{K}} \right)_{M_\mathrm{k}} = \frac{1 - \tau^*_{\mathrm{C}M_\mathrm{k}}/\tau_{\mathrm{F}M_\mathrm{k}}}{1 - \tau^*_{\mathrm{C}M_\mathrm{k}}/\tau_{\mathrm{K}M_\mathrm{k}}} \; , \tag{59}$$

where

$$\sigma^*_{\mathrm{C}M_\mathrm{o}} = \frac{\sigma_{\mathrm{C}M_\mathrm{o}}}{\beta_{\sigma\mathrm{r}}} \; , \qquad \sigma^*_{\mathrm{C}N} = \frac{\sigma_{\mathrm{C}N}}{\beta_{\sigma\mathrm{r}}} \; , \qquad \tau^*_{\mathrm{C}M_\mathrm{k}} = \frac{\tau_{\mathrm{C}M_\mathrm{k}}}{\beta_{\tau\mathrm{r}}} \; , \tag{60}$$

4.1. Safety factor at a pure harmonic loading by a bending moment M_o, tension-compression N and a torsion moment M_k (problems B_M$_\mathrm{o}$, B_N, B_M$_\mathrm{k}$)

After substituting from (37) and (38), the Eqs. (45) and (46) read:

– at a loading by bending moment M^P_oh (problem **B_M$_\mathrm{o}$**):

$$k_\sigma = \frac{W_\mathrm{o}}{M^\mathrm{P}_\mathrm{oh}} (\sigma_\mathrm{h}^{M_\sigma})_{M_\mathrm{o}} = \begin{cases} \dfrac{W_\mathrm{o}}{M^\mathrm{P}_\mathrm{oh}} \dfrac{\eta_\sigma}{\eta_\sigma + \dfrac{\sigma^*_{\mathrm{C}M_\mathrm{o}}}{\sigma_{\mathrm{F}M_\mathrm{o}}} - 1} \sigma^*_{\mathrm{C}M_\mathrm{o}} & \text{if} \quad \eta_\sigma \ge \eta^\mathrm{limit}_{\sigma M_\mathrm{o}} \; , \\[2ex] \dfrac{W_\mathrm{o}}{M^\mathrm{P}_\mathrm{oh}} \sigma_{\mathrm{K}M_\mathrm{o}} & \text{if} \quad \eta_\sigma < \eta^\mathrm{limit}_{\sigma M_\mathrm{o}} \; , \end{cases} \tag{61}$$

– at a loading by axial tension-compression $N_{\mathrm{h}}^{\mathrm{P}}$ (problem **B_N**):

$$k_\sigma = \frac{S}{N_{\mathrm{h}}^{\mathrm{P}}}\,(\sigma_{\mathrm{h}}^{M_\sigma})_N = \begin{cases} \dfrac{S}{N_{\mathrm{h}}^{\mathrm{P}}}\,\dfrac{\eta_\sigma}{\eta_\sigma + \dfrac{\sigma_{\mathrm{C}N}^*}{\sigma_{\mathrm{F}N}} - 1}\,\sigma_{\mathrm{C}N}^* & \text{if} \quad \eta_\sigma \geq \eta_{\sigma N}^{\mathrm{limit}}\,, \\ \dfrac{S}{N_{\mathrm{h}}^{\mathrm{P}}}\,\sigma_{\mathrm{K}N} & \text{if} \quad \eta_\sigma < \eta_{\sigma N}^{\mathrm{limit}}\,, \end{cases} \tag{62}$$

– at a loading by torsion moment M_{kh}^{P} (problem **B_M$_\mathbf{k}$**):

$$k_\tau = \frac{W_{\mathrm{k}}}{M_{\mathrm{kh}}^{\mathrm{P}}}\,(\tau_{\mathrm{h}}^{M_\tau})_{M_{\mathrm{k}}} = \begin{cases} \dfrac{W_{\mathrm{k}}}{M_{\mathrm{kh}}^{\mathrm{P}}}\,\dfrac{\eta_\tau}{\eta_\tau + \dfrac{\tau_{\mathrm{C}M_{\mathrm{k}}}^*}{\tau_{\mathrm{F}M_{\mathrm{k}}}} - 1}\,\tau_{\mathrm{C}M_{\mathrm{k}}}^* & \text{if} \quad \eta_\tau \geq \eta_{\tau M_{\mathrm{k}}}^{\mathrm{limit}}\,, \\ \dfrac{W_{\mathrm{k}}}{M_{\mathrm{kh}}^{\mathrm{P}}}\,\tau_{\mathrm{K}M_{\mathrm{k}}} & \text{if} \quad \eta_\tau < \eta_{\tau M_{\mathrm{k}}}^{\mathrm{limit}}\,. \end{cases} \tag{63}$$

4.2. Limiting carrying capacity at a pure harmonic loading by bending moment M_o, tension-compression N and torsion moment M_k (problems U_M$_\mathbf{o}$, U_N, U_M$_\mathbf{k}$)

After substituting from (37) and (38), the Eqs. (47) and (48) read:

– at a loading by bending moment M_{oh}^{P} (problem **U_M$_\mathbf{o}$**):

$$M_{\mathrm{oh}}^{\mathrm{P}} = \frac{W_{\mathrm{o}}}{k_\sigma}\,(\sigma_{\mathrm{h}}^{M_\sigma})_{M_{\mathrm{o}}} = \begin{cases} \dfrac{W_{\mathrm{o}}}{k_\sigma}\,\dfrac{\eta_\sigma}{\eta_\sigma + \dfrac{\sigma_{\mathrm{C}M_{\mathrm{o}}}^*}{\sigma_{\mathrm{F}M_{\mathrm{o}}}} - 1}\,\sigma_{\mathrm{C}M_{\mathrm{o}}}^* & \text{if} \quad \eta_\sigma \geq \eta_{\sigma M_{\mathrm{o}}}^{\mathrm{limit}}\,, \\ \dfrac{W_{\mathrm{o}}}{k_\sigma}\,\sigma_{\mathrm{K}M_{\mathrm{o}}} & \text{if} \quad \eta_\sigma < \eta_{\sigma M_{\mathrm{o}}}^{\mathrm{limit}}\,, \end{cases} \tag{64}$$

– at a loading by axial tension-compression $N_{\mathrm{h}}^{\mathrm{P}}$ (problem **U_N**):

$$N_{\mathrm{h}}^{\mathrm{P}} = \frac{S}{k_\sigma}\,(\sigma_{\mathrm{h}}^{M_\sigma})_N = \begin{cases} \dfrac{S}{k_\sigma}\,\dfrac{\eta_\sigma}{\eta_\sigma + \dfrac{\sigma_{\mathrm{C}N}^*}{\sigma_{\mathrm{F}N}} - 1}\,\sigma_{\mathrm{C}N}^* & \text{if} \quad \eta_\sigma \geq \eta_{\sigma N}^{\mathrm{limit}}\,, \\ \dfrac{S}{k_\sigma}\,\sigma_{\mathrm{K}N} & \text{if} \quad \eta_\sigma < \eta_{\sigma N}^{\mathrm{limit}}\,, \end{cases} \tag{65}$$

– at a loading by torsion moment M_{kh}^{P} (problem **U_M$_\mathbf{k}$**):

$$M_{\mathrm{kh}}^{\mathrm{P}} = \frac{W_{\mathrm{k}}}{k_\tau}\,(\tau_{\mathrm{h}}^{M_\tau})_{M_{\mathrm{k}}} = \begin{cases} \dfrac{W_{\mathrm{k}}}{k_\tau}\,\dfrac{\eta_\tau}{\eta_\tau + \dfrac{\tau_{\mathrm{C}M_{\mathrm{k}}}^*}{\tau_{\mathrm{F}M_{\mathrm{k}}}} - 1}\,\tau_{\mathrm{C}M_{\mathrm{k}}}^* & \text{if} \quad \eta_\tau \geq \eta_{\tau M_{\mathrm{k}}}^{\mathrm{limit}}\,, \\ \dfrac{W_{\mathrm{k}}}{k_\tau}\,\tau_{\mathrm{K}M_{\mathrm{k}}} & \text{if} \quad \eta_\tau < \eta_{\tau M_{\mathrm{k}}}^{\mathrm{limit}}\,. \end{cases} \tag{66}$$

4.3. Dimensioning at a pure harmonic loading by a bending moment $M_\mathbf{o}$, tension-compression N and a torsion moment $M_\mathbf{k}$ (problems D_M$_\mathbf{o}$, D_N, D_M$_\mathbf{k}$)

After substituting from (37) and (38), the Eqs. (49) and (50) read:

– at a loading by bending moment M_{oh}^{P} (problem **D_M$_\mathbf{o}$**) :

$$W_{\mathrm{o}} = \frac{M_{\mathrm{oh}}^{\mathrm{P}}\, k_\sigma}{(\sigma_{\mathrm{h}}^{M_\sigma})_{M_{\mathrm{o}}}} = \begin{cases} M_{\mathrm{oh}}^{\mathrm{P}}\, k_\sigma \dfrac{\eta_\sigma + \dfrac{\sigma_{\mathrm{C}M_{\mathrm{o}}}^{*}}{\sigma_{\mathrm{F}M_{\mathrm{o}}}} - 1}{\eta_\sigma\, \sigma_{\mathrm{C}M_{\mathrm{o}}}^{*}} & \text{if} \quad \eta_\sigma \geq \eta_{\sigma M_{\mathrm{o}}}^{\mathrm{limit}} \,, \\ M_{\mathrm{oh}}^{\mathrm{P}}\, k_\sigma \dfrac{1}{\sigma_{\mathrm{K}M_{\mathrm{o}}}} & \text{if} \quad \eta_\sigma < \eta_{\sigma M_{\mathrm{o}}}^{\mathrm{limit}} \,, \end{cases} \tag{67}$$

– at a loading by an axial tension-compression $N_{\mathrm{h}}^{\mathrm{P}}$ (problem **D_N**):

$$S = \frac{N_{\mathrm{h}}^{\mathrm{P}}\, k_\sigma}{(\sigma_{\mathrm{h}}^{M_\sigma})_{N}} = \begin{cases} N_{\mathrm{h}}^{\mathrm{P}}\, k_\sigma \dfrac{\eta_\sigma + \dfrac{\sigma_{\mathrm{C}N}^{*}}{\sigma_{\mathrm{F}N}} - 1}{\eta_\sigma\, \sigma_{\mathrm{C}N}^{*}} & \text{if} \quad \eta_\sigma \geq \eta_{\sigma N}^{\mathrm{limit}} \,, \\ N_{\mathrm{h}}^{\mathrm{P}}\, k_\sigma \dfrac{1}{\sigma_{\mathrm{K}N}} & \text{if} \quad \eta_\sigma < \eta_{\sigma N}^{\mathrm{limit}} \,, \end{cases} \tag{68}$$

– at a loading by torsion moment M_{kh}^{P} (problem **D_M$_\mathbf{k}$**):

$$W_{\mathrm{k}} = \frac{M_{\mathrm{kh}}^{\mathrm{P}}\, k_\tau}{(\tau_{\mathrm{h}}^{M_\tau})_{M_{\mathrm{k}}}} = \begin{cases} M_{\mathrm{kh}}^{\mathrm{P}}\, k_\tau \dfrac{\eta_\tau + \dfrac{\tau_{\mathrm{C}M_{\mathrm{k}}}^{*}}{\tau_{\mathrm{F}M_{\mathrm{k}}}} - 1}{\eta_\tau\, \tau_{\mathrm{C}M_{\mathrm{k}}}^{*}} & \text{if} \quad \eta_\tau \geq \eta_{\tau M_{\mathrm{k}}}^{\mathrm{limit}} \,, \\ M_{\mathrm{kh}}^{\mathrm{P}}\, k_\tau \dfrac{1}{\tau_{\mathrm{K}M_{\mathrm{k}}}} & \text{if} \quad \eta_\tau < \eta_{\tau M_{\mathrm{k}}}^{\mathrm{limit}} \,, \end{cases} \tag{69}$$

5. Case studies

Case 5.1. A pull-rod of a circular cross-section is to be subjected to a harmonic tension-compression loading, N, within the scope of infinite fatigue life (the problem **B_N**).

Given: Material: mild steel 11 500.1:

- $\sigma_{\mathrm{F}N} = 550\,\mathrm{MPa}$, $\sigma_{\mathrm{K}N} = 310\,\mathrm{MPa}$, $\sigma_{\mathrm{C}N} = 180\,\mathrm{MPa}$,
- $N_{\mathrm{h}}^{\mathrm{P}} \in \langle N_{\mathrm{h\,min}}^{\mathrm{P}}; N_{\mathrm{h\,max}}^{\mathrm{P}} \rangle = \langle 0; 300 \rangle\,\mathrm{kN}$ (limits of maximum loading),
- $d \in \langle 0; 50 \rangle\,\mathrm{mm}$ (limits of the cross-section diameter),
- $\beta_{\sigma\mathrm{r}} = 2,3$ (reduced fatigue notch factor),
- $\eta_\sigma = 1,4$ (the slope of load line),
- $p = 6$ (number of prescribed values of the safety factor k_σ),
- $k_\sigma = 1, 2, 4, 6, 9, 12$ (prescribed values of the safety factor k_σ),
- $S = \pi\, d^2/4$ (the cross-section area),
- $n = 500$ (number of segments for *linspace* $x = N_{\mathrm{h}}^{\mathrm{P}}$ and $y = d$).

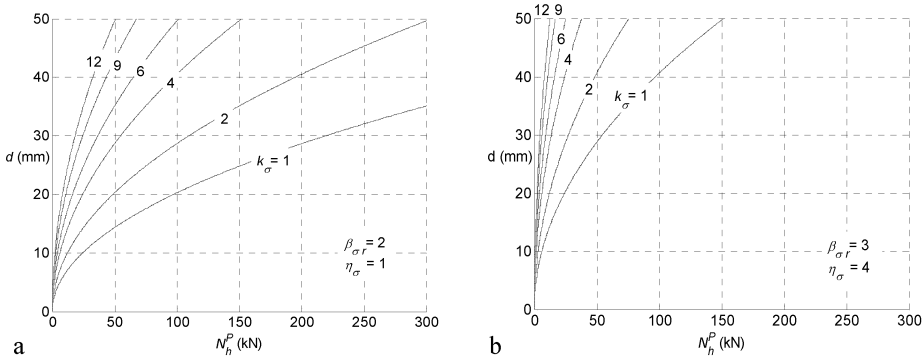

Fig.3: Theoretical variation of the safety factor, k_σ, referring to given machine parts which are to be subjected to given harmonic tension-compression causing infinite fatigue life and allowing for influence of the reduced fatigue notch factor, $\beta_{\sigma\mathrm{r}}$, and the slope of load line, η_σ, a) $\beta_{\sigma\mathrm{r}} = 2$, $\eta_\sigma = 1$, b) $\beta_{\sigma\mathrm{r}} = 3$, $\eta_\sigma = 4$

Case 5.2. A connective notched shift of a circular cross-section is to be subjected to a harmonic loading by bending moment, M_o, causing infinite fatigue life (the problem $\mathbf{U\_M_o}$).

Given : Material : mild steel 11 500.1 :

- $\sigma_{\mathrm{F}M_\mathrm{o}} = 880\,\mathrm{MPa}$, $\sigma_{\mathrm{K}M_\mathrm{o}} = 370\,\mathrm{MPa}$, $\sigma_{\mathrm{C}M_\mathrm{o}} = 240\,\mathrm{MPa}$,
- $M^\mathrm{P}_\mathrm{oh} \in \langle M^\mathrm{P}_{\mathrm{oh\,min}}; M^\mathrm{P}_{\mathrm{oh\,max}}\rangle = \langle 0; 300\rangle\,\mathrm{Nm}$ (limits of maximum loading),
- $d \in \langle 0; 50\rangle$ mm (limits of the cross-section diameter),
- $\beta_{\sigma\mathrm{r}} = 2, 3$ (reduced fatigue notch factor),
- $\eta_\sigma = 1, 4$ (the slope of load line),
- $p = 6$ (number of prescribed values of the safety factor k_σ),
- $k_\sigma = 1, 2, 4, 6, 9, 12$ (prescribed values of the safety factor k_σ),
- $W_\mathrm{o} = \pi\, d^3/32$ (property of the circular cross section),
- $n = 500$ (number of segments for *linspace* $x = M^\mathrm{P}_\mathrm{oh}$ and $y = d$).

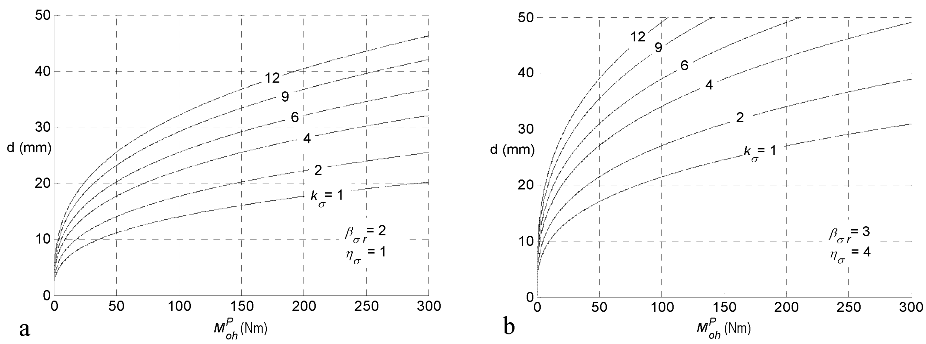

Fig.4: Theoretical variation of the safety factor, k_σ, referring to given machine parts which are to be subjected to given harmonic bending loading causing infinite fatigue life and allowing for influence of the reduced fatigue notch factor, $\beta_{\sigma\mathrm{r}}$, and the factor of the loading cycle asymmetry, η_σ, a) $\beta_{\sigma\mathrm{r}} = 2$, $\eta_\sigma = 1$, b) $\beta_{\sigma\mathrm{r}} = 3$, $\eta_\sigma = 4$

Case 5.3. A propeller shaft is to be subjected to a harmonic torsional loading caused by torsional moment, M_k, causing infinite fatigue life (the problem $\mathbf{D\_M_k}$)

Given: Material: mild steel 11 500.1:

- $\tau_{\mathrm{F}M_\mathrm{k}} = 440\,\mathrm{MPa}$, $\tau_{\mathrm{K}M_\mathrm{k}} = 190\,\mathrm{MPa}$, $\tau_{\mathrm{C}M_\mathrm{k}} = 140\,\mathrm{MPa}$,
- $M^\mathrm{P}_\mathrm{kh} \in \langle M^\mathrm{P}_\mathrm{kh\,min}; M^\mathrm{P}_\mathrm{kh\,max}\rangle = \langle 0; 300\rangle$ Nm (limits of maximum loading),
- $d \in \langle 0; 50\rangle$ mm (limits of the cross-section diameter),
- $\beta_{\tau\mathrm{r}} = 2,3$ (reduced fatigue notch factor),
- $\eta_\tau = 1,4$ (the slope of load line),
- $p = 6$ (number of prescribed values of the safety factor k_τ),
- $k_\tau = 1, 2, 4, 6, 9, 12$ (prescribed values of the safety factor k_τ),
- $W_\mathrm{k} = \pi\, d^3/16$ (property of the circular cross section),
- $n = 500$ (number of segments for *linspace* $x = M^\mathrm{P}_\mathrm{kh}$ and $y = d$).

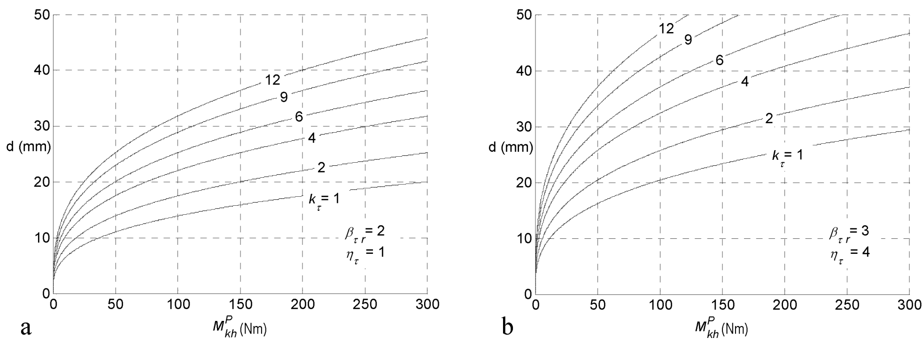

Fig. 5: Theoretical variation of the safety factor, k_τ, referring to given machine parts which are to be subjected to given harmonic torsion loading causing infinite fatigue life and allowing for influence of the reduced fatigue notch factor, $\beta_{\tau\mathrm{r}}$, and the slope of load line, η_τ, a) $\beta_{\tau\mathrm{r}} = 2$, $\eta_\tau = 1$, b) $\beta_{\tau\mathrm{r}} = 3$, $\eta_\tau = 4$

6. Conclusions

Presented analysis makes it possible to solve deterministic problems of infinite fatigue of notched components subjected to arbitrary pure harmonic loading. Graphical presentation of calculated results of three examples concerning determination of the safety factor, predicting the carrying capacity and proposing the cross-section dimensions are enclosed in Figs. 3 to 5.

Derived mathematical models do not require to construct any fatigue strength diagram adopting it only as a basis of graphical solving. Finally, these models may serve as a starting point for formulating analytical solution to deterministic problems of infinite fatigue life of machine parts under combined harmonic loading.

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