# SENSITIVITY ANALYSIS OF GEARBOX TORSIONAL VIBRATIONS

Michal Hajžman, Pavel Polach*

*The paper deals with the modelling of a real gearbox used in cement mill applications and with the sensitivity analysis of its eigenfrequencies with respect to design parameters. The torsional model (including a motor and couplings) based on the finite element method implemented in an in-house MATLAB application is described. The sensitivity analysis of gearbox eigenfrequencies is performed in order to avoid possible dangerous resonance states of the gearbox. The parameters chosen with respect to the sensitivity analysis are used for tuning the gearbox eigenfrequencies outside resonance areas. Two approaches (analytical and numerical) to the sensitivity calculation are discussed.*

Keywords: *flexible shafts, mesh gear, modal analysis, eigenfrequencies, resonance, tuning*

## 1. Introduction

The computational investigation of gearbox dynamic properties is an important step in the gearbox design. Mainly the resonance analysis (the comparison of system eigenfrequencies with possible excitation frequencies) is performed in order to avoid dangerous resonance states. The successive activity in case of possible resonances should lead to the tuning of chosen eigenfrequencies out of the resonance areas. The sensitivity analysis of the gearbox eigenfrequencies and eigenmodes is the most suitable tool in this case. It can be used for the identification of design parameters which are proper to influence chosen eigenfrequencies.

The first necessary step of the introduced methodology is the modelling method of the gearbox vibration. The most often used methods are based on the finite element analysis of various complexity. Solid finite elements can be used in order to describe three-dimensional flexible bodies. However, considering limited computer capacity, rotation of the components and its mutual connections, the most efficient method is based on the torsional modelling [16] of an interior rotating gearbox system. The dynamic model of the real Wikov Gear gearbox (Figure 1) with a motor and couplings (not shown in Figure 1) is investigated in this paper. The model (see Figure 2) is based on the finite element method combined with discrete mass and flexible elements and it is implemented in an in-house MATLAB application. The possible extension of the torsional modelling methodology for the nonlinear systems was shown e.g. in [4].

On the other hand design sensitivity calculation is a very useful tool motivated mainly by potential usage in optimization tasks [2]. Sensitivities with respect to the design parameters are important also in many complex multidisciplinary applications, e.g. [6]. Short review

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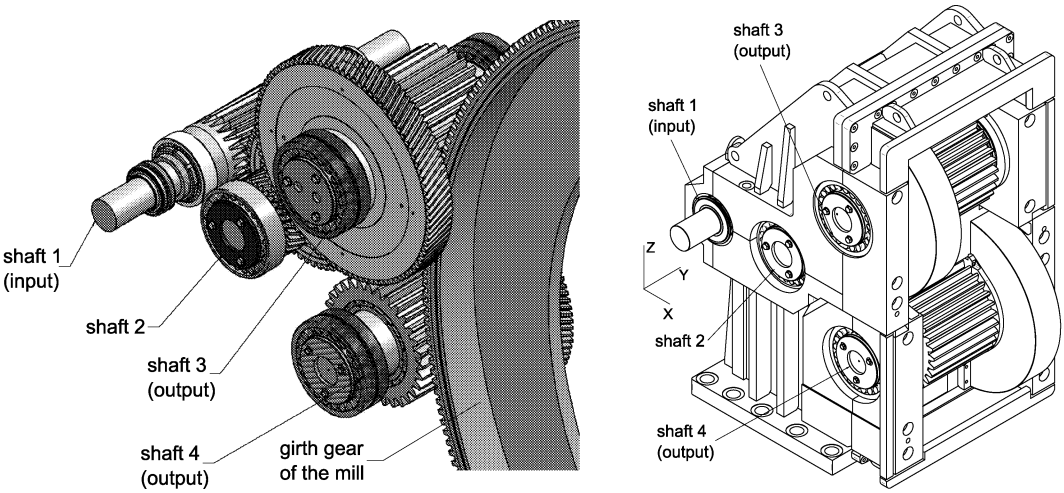


*Fig.1: Gearbox developed and produced by Wikov Gear s.r.o. Company*

of various approaches can be found in [1]. Authors of [7] present general methods for the calculation of the first order sensitivities of eigenvalues and eigenvectors of asymmetric damped systems. For the sake of the usage in several optimization methods the second order derivatives of eigenvalues for the same type of mechanical systems is shown in [8]. The approaches suitable for the sensitivity analysis of mixed rigid-flexible multibody systems are developed in [5]. The dynamic response has to be solved by symbolic methods (i.e. analytical formulation using symbolic mathematics, not numerics) and then the derivatives can be calculated. Sub-structuring of decomposed systems can be advantageously used (see [12]) in order to efficiently perform sensitivity analysis of various variables representing system dynamic properties.

Analytical and numerical approaches to the eigenfrequencies' sensitivity calculation are presented in this paper. Due to the decomposition of the whole system the calculation can be fast and can be used in the optimization tasks. The sensitivity analysis results are used in order to tune the eigenfrequency that can be in possible resonance with excitation frequencies.

## 2. Gearbox modelling and modal analysis

The investigated gearbox (see Figures 1 and 2) with the motor and couplings was divided into four subsystems [11]. The main parts of the subsystems are flexible shafts with mounted gear wheels. The first subsystem (input shaft 1) was extended by a drive engine, by a coupling between the drive and the shaft and by a coupling between the shaft and an auxiliary drive (it is intended for the gearbox manipulation and it is decoupled during the operation). Only a torsional motion of all subsystems was considered in the analysis, although the vibration of the gearbox shafts is spatial. It is known, based on the mill operator experience, that the torsional vibration is the most dominant one and it is almost not affected by other kinds of motions. The girth gear wheel of the ball mill was considered to be mounted to the rigid frame due to the huge mill moment of inertia.

The flexible shafts were modelled using two-node shaft finite elements with one torsional degree of freedom in each node (see Figure 3). It is supposed that element $e$ of length $l$ has

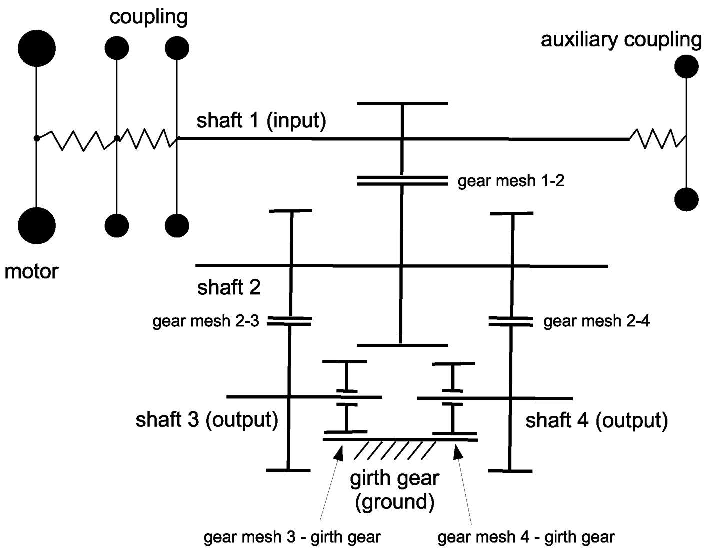


*Fig.2: Scheme of the gearbox model*

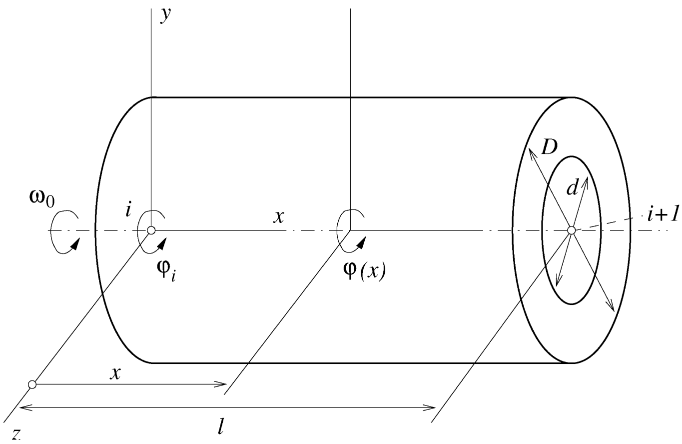


*Fig.3: Scheme of the torsional shaft finite element of length $l$, outer diameter $D$, inner diameter $d$ and with nodes $i$ and $i+1$*

two nodes $i$ and $i+1$ with torsional deformations (angles) $\varphi_i$ and $\varphi_{i+1}$ superimposed on the rotational motion with angular velocity $\omega_0$. In order to derive the shaft torsional finite element matrices the longitudinal coordinate $x$ of the cross-sectional infinitesimal element defined by area $A(x)$ and width $\mathrm{d}x$ rotating at angular velocity $\omega_0 + \dot{\varphi}(x)$ is considered. The kinetic energy can be then formulated in the form

$$E_{\mathrm{k}}^{(\mathrm{e})} = \frac{1}{2} \int_0^l J_{\mathrm{p}}(x) \left(\omega_0 + \dot{\varphi}(x)\right)^2 \rho \, \mathrm{d}x \,, \tag{1}$$

where $J_{\mathrm{p}}(x)$ is the polar moment of inertia and $\rho$ is the material density. Similarly the element potential (deformation) energy is

$$E_{\mathrm{p}}^{(\mathrm{e})} = \frac{1}{2} \int_{0}^{l} \int_{(A(x))} G \left( \gamma_{\mathrm{xy}}^2 + \gamma_{\mathrm{xz}}^2 \right) \mathrm{d}A(x)\, \mathrm{d}x \ , \tag{2}$$

where $G$ is the shear modulus and strain $\gamma_{xy}$ and $\gamma_{xz}$ can be expressed [3], [15] using

$$\gamma_{\mathrm{xy}} = -z\, \varphi'(x) \ , \qquad \gamma_{\mathrm{xz}} = y\, \varphi'(x) \ , \qquad (\cdot)' = \frac{\partial(\cdot)}{\partial x} \ . \tag{3}$$

Expression (2) for the potential energy can be rewritten as

$$E_{\mathrm{p}}^{(\mathrm{e})} = \frac{1}{2} \int_{0}^{l} \int_{(A(x))} G\, \varphi'^2(x)\, (y^2 + z^2)\, \mathrm{d}A(x)\, \mathrm{d}x \ . \tag{4}$$

Torsional deformation of the arbitrary point of the finite element is approximated by linear polynomial

$$\varphi(x) = \mathbf{\Phi}(x)\, \mathbf{c}, \quad \mathbf{\Phi}(x) = [\, 1 \quad x \,] \ , \tag{5}$$

with vector $\mathbf{c}$ of unknown coefficients. After several modifications one can get the expression

$$\varphi(x) = \mathbf{\Phi}(x)\, \mathbf{S}^{-1}\, \mathbf{q}^{(\mathrm{e})} \ , \tag{6}$$

where

$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \ , \qquad \mathbf{q}^{(\mathrm{e})} = \begin{bmatrix} \varphi(0) \\ \varphi(l) \end{bmatrix} \ . \tag{7}$$

Using the identity based on the Lagrange' equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial E_{\mathrm{k}}^{(\mathrm{e})}}{\partial \dot{\mathbf{q}}^{(\mathrm{e})}} \right) - \frac{\partial E_{\mathrm{k}}^{(\mathrm{e})}}{\partial \mathbf{q}^{(\mathrm{e})}} + \frac{\partial E_{\mathrm{p}}^{(\mathrm{e})}}{\partial \mathbf{q}^{(\mathrm{e})}} = \mathbf{M}^{(\mathrm{e})} \ddot{\mathbf{q}}^{(\mathrm{e})} + \mathbf{K}^{(\mathrm{e})} \mathbf{q}^{(\mathrm{e})} \tag{8}$$

together with expressions of energies (1) and (4) and approximation (6) the torsional shaft element mass and stiffness matrices are

$$\mathbf{M}^{(\mathrm{e})} = \mathbf{S}^{-\mathrm{T}}\, \mathbf{I}_1\, \mathbf{S}^{-1} \ , \qquad \mathbf{K}^{(\mathrm{e})} = \mathbf{S}^{-\mathrm{T}}\, \mathbf{I}_2\, \mathbf{S}^{-1} \ , \tag{9}$$

where integral matrices $\mathbf{I}_1$ and $\mathbf{I}_2$ can be calculated for the prismatic finite element (area $A(x)$ and polar moment of inertia $J_{\mathrm{p}}(x)$ are constant) of axisymmetric cross-section as

$$\begin{aligned} \mathbf{I}_1 &= \int_0^l \rho\, J_{\mathrm{p}}(x)\, \mathbf{\Phi}^{\mathrm{T}}(x)\, \mathbf{\Phi}(x)\, \mathrm{d}x = \rho\, J_{\mathrm{p}}\, l \begin{bmatrix} 1 & l/2 \\ l/2 & l^2/3 \end{bmatrix} \ , \\ \mathbf{I}_2 &= \int_0^l G\, J_{\mathrm{p}}(x)\, \mathbf{\Phi}'^{\mathrm{T}}(x)\, \mathbf{\Phi}'(x)\, \mathrm{d}x = G\, J_{\mathrm{p}}\, l \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \ . \end{aligned} \tag{10}$$

The derivation of the element matrices for the axial-torsional-bending vibration of two node shaft elements with six degrees of freedom in each node can be found e.g. in [15] and [9] or more generally for more rotor cases in [3].

After decomposition into subsystems the mathematical models of the uncoupled subsystems can be written in the form

$$\mathbf{M}_s\,\ddot{\mathbf{q}}_s(t) + \mathbf{K}_s\,\mathbf{q}_s(t) = \mathbf{0}\ , \qquad s = 1, 2, \ldots, 4\ , \tag{11}$$

where $\mathbf{M}_s$ is the subsystem mass matrix and $\mathbf{K}_s$ is the subsystem stiffness matrix. They are composed of finite element matrices and of the inertia moments and stiffnesses which correspond to the rigid wheels and discrete springs. The structure of the subsystem models (Figure 2) is:

– *Subsystem 1* – is composed of the first input shaft divided by 14 finite elements and joined with the input coupling (of Pencoflex company, 2 rigid wheels with rubber elements in between represented by a torsional spring, the first part of the coupling is connected to shaft nodes 1 and 2) which is joined to the main engine (massive rigid wheel). The coupling (one rigid wheel) to the auxiliary motor is connected with the last shaft nodes (number 14 and 15).

– *Subsystem 2* – represents the second shaft discretized using 12 shaft finite elements. It is connected with shaft 1 and with shafts 3 and 4 by gear mesh.

– *Subsystem 3* – is composed of one of the output shafts divided by 10 finite elements. This shaft is coupled by the gear mesh to the second shaft and with the girth gear of the mill which is supposed to be a rigid frame. The output pinion of the shaft is mounted on the shaft using inner gear mesh.

– *Subsystem 4* – it is almost identical to Subsystem 3.

The coupling of the subsystems is realized by gear mesh stiffness matrices

$$\mathbf{K}_{\mathrm{G}z} = k_{\mathrm{G}z}\,\cos^2\alpha_z\,\cos^2\beta_z \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & r_{\mathrm{p}z}^2 & \cdots & r_{\mathrm{p}z}\,r_{\mathrm{w}z} & \cdots \\ & \vdots & & \vdots & \\ \cdots & r_{\mathrm{p}z}\,r_{\mathrm{w}z} & \cdots & r_{\mathrm{w}z}^2 & \cdots \\ & \vdots & & \vdots & \end{bmatrix}\ , \qquad z = 1, 2, \ldots, 5\ , \tag{12}$$

where $k_{\mathrm{G}z}$ is the meshing stiffness, $\alpha_z$ is the normal pressure angle, $\beta_z$ is the teeth inclination angle, $r_{\mathrm{p}z}$ is the pinion diameter and $r_{\mathrm{w}z}$ is the wheel diameter. The position of the nonzero elements in the stiffness matrix is given by global numbers of coupled nodes of various subsystems. More comments and derivation of the complex gear mesh model can be found in [10], [3].

The whole model of the gearbox is

$$\mathbf{M}\,\ddot{\mathbf{q}}(t) + \mathbf{K}\,\mathbf{q}(t) = \mathbf{0}\ , \quad \mathbf{q}(t) = [\mathbf{q}_1^{\mathrm{T}}(t),\ \mathbf{q}_2^{\mathrm{T}}(t),\ \mathbf{q}_3^{\mathrm{T}}(t),\ \mathbf{q}_4^{\mathrm{T}}(t)]^{\mathrm{T}}\ , \tag{13}$$

where the block diagonal global matrices are

$$\mathbf{M} = \mathrm{diag}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4)\ , \qquad \mathbf{K} = \mathrm{diag}(\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4) + \sum_{z=1}^{5} \mathbf{K}_{\mathrm{G}z}\ . \tag{14}$$

The model can be easily extended by a proportional damping or by a certain excitation vector on the right hand side. Derived model (13) implemented in the in-house MATLAB application was used for the mill drive modal analysis. Original eigenfrequencies (stars in Figure 4) are compared with excitation frequencies (lines in Figure 4) by gear meshing (the first three harmonic components are considered). It is clear that several eigenfrequencies are close to the excitation frequencies and therefore these eigenfrequencies should be tuned out of the excitation frequencies. The sensitivity analysis is a suitable tool for the determination of usable design parameters.

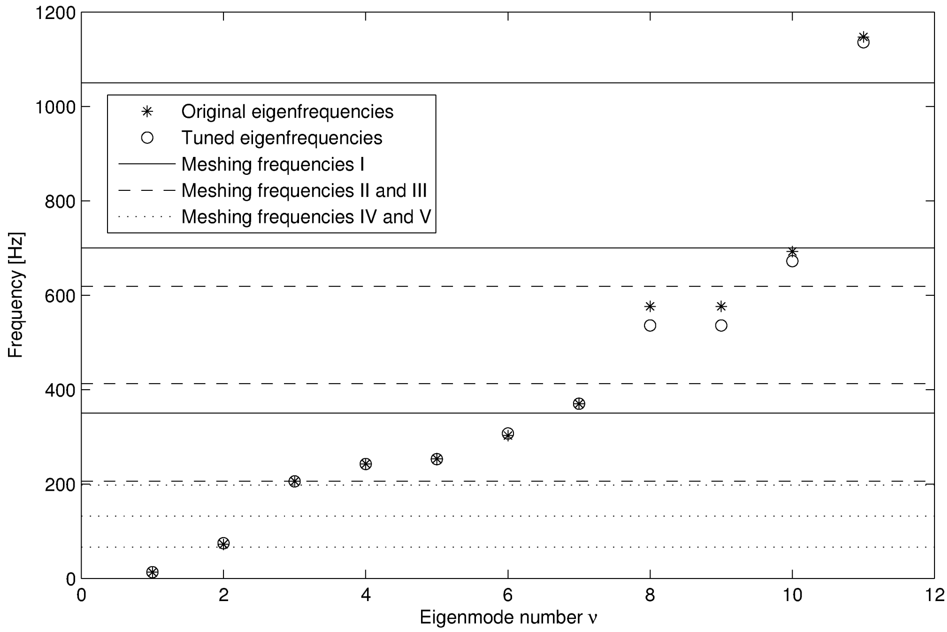


*Fig.4: Comparison of original and tuned eigenfrequencies with excitation frequencies*

## 3. Numerical approach to the sensitivity analysis

A direct and basic approach to the calculation of the sensitivity of a certain variable is based on the numerical differentiation of the variable with respect to the chosen design parameters. Moreover it is suitable to derive the sensitivities with respect to $P$ physically different parameters by special relative expressions [14]. Change $\Delta f_\nu$ of the $\nu$-th eigenfrequency $f_\nu$ can be expressed for small change $\Delta\mathbf{p} \in \mathbb{R}^P$ of initial parameter vector $\mathbf{p}_0 \in \mathbb{R}^P$ using the Taylor series

$$\Delta f_\nu = f_\nu(\mathbf{p}_0 + \Delta\mathbf{p}) - f_\nu(\mathbf{p}_0) = \sum_{j=1}^{P} \frac{\partial f_\nu(\mathbf{p}_0)}{\partial p_j}\,\Delta p_j\ . \tag{15}$$

After modification one can get

$$\frac{\Delta f_\nu}{f_\nu(\mathbf{p}_0)} = \sum_{j=1}^{P} \frac{\partial f_\nu(\mathbf{p}_0)}{\partial p_j}\,\frac{p_{j0}}{f_\nu(\mathbf{p}_0)}\,\frac{\Delta p_j}{p_{j0}}\ . \tag{16}$$

Relative sensitivity $\Delta\bar{f}_\nu^j$ with respect to the change of the $j$-th parameter $p_j$ can be derived from equation (16) as

$$\Delta\bar{f}_\nu^j = \frac{\partial f_\nu(\mathbf{p}_0)}{\partial p_j}\,\frac{p_{j0}}{f_\nu(\mathbf{p}_0)}\ . \tag{17}$$

Partial derivative is calculated in this case by finite difference

$$\frac{\partial f_\nu(\mathbf{p}_0)}{\partial p_j} = \frac{f_\nu(\mathbf{p}_0 + \Delta \mathbf{p}_j) - f_\nu(\mathbf{p}_0)}{\Delta p_j} \,, \tag{18}$$

where holds

$$\Delta \mathbf{p}_j = [\,0, \ldots, 0, \Delta p_j, 0, \ldots, 0\,]^{\mathrm{T}} \,. \tag{19}$$

The value of difference parameter $\Delta p_j$ should be chosen with respect to the convergence of the numerical difference calculation. The advantage of the numerical approach is in the fast implementation of the analysis, however the attention should be paid to the $\Delta p_j$ selection. Possible inaccuracy of the results is considered to be a drawback.

## 4. Analytical approach to the sensitivity analysis

The derivation of the comparable analytical relative sensitivities starts from the eigenvalue problem definition and its transposition

$$(\mathbf{K} - \lambda_\nu \, \mathbf{M}) \, \mathbf{v}_\nu = \mathbf{0} \,, \qquad \mathbf{v}_\nu^{\mathrm{T}} \, (\mathbf{K} - \lambda_\nu \, \mathbf{M}) = \mathbf{0}^{\mathrm{T}} \,. \tag{20}$$

The orthonormality conditions using the mass matrix are

$$\mathbf{v}_\nu^{\mathrm{T}} \, \mathbf{M} \, \mathbf{v}_\nu = 1 \quad \text{and} \quad \mathbf{v}_\mu^{\mathrm{T}} \, \mathbf{M} \, \mathbf{v}_\nu = 0 \,, \quad \text{for} \quad \mu \neq \nu \,. \tag{21}$$

Using differentiation of (20) with respect to parameter $p_j$, after multiplying the equation by $\mathbf{v}_\nu^{\mathrm{T}}$ and after modification one gets

$$\mathbf{v}_\nu^{\mathrm{T}} \left( \frac{\partial \mathbf{K}}{\partial p_j} - \lambda_\nu \, \frac{\partial \mathbf{M}}{\partial p_j} \right) \mathbf{v}_\nu + \mathbf{v}_\nu^{\mathrm{T}} \, (\mathbf{K} - \lambda_\nu \, \mathbf{M}) \, \frac{\partial \mathbf{v}_\nu}{\partial p_j} - \frac{\partial \lambda_\nu}{\partial p_j} \, \mathbf{v}_\nu^{\mathrm{T}} \, \mathbf{M} \, \mathbf{v}_\nu = 0 \,. \tag{22}$$

The expression for the sensitivity of eigenvalue $\lambda_\nu$ with respect to the change of parameter $p_j$ results from equations (20), (21), (22) in the form

$$\frac{\partial \lambda_\nu}{\partial p_j} = \mathbf{v}_\nu^{\mathrm{T}} \left( \frac{\partial \mathbf{K}}{\partial p_j} - \lambda_\nu \, \frac{\partial \mathbf{M}}{\partial p_j} \right) \mathbf{v}_\nu \,. \tag{23}$$

Considering $\lambda_\nu = \Omega_\nu^2 = (2\,\pi\, f_\nu)^2$, it can be written that

$$\frac{\partial f_\nu}{\partial p_j} = \frac{1}{8\,\pi^2 \, f_\nu} \, \frac{\partial \lambda_\nu}{\partial p_j} = \frac{1}{8\,\pi^2 \, f_\nu} \, \mathbf{v}_\nu^{\mathrm{T}} \left( \frac{\partial \mathbf{K}}{\partial p_j} - \lambda_\nu \, \frac{\partial \mathbf{M}}{\partial p_j} \right) \mathbf{v}_\nu \tag{24}$$

and finally relative sensitivity is

$$\frac{\partial \bar{f}_\nu(\mathbf{p}_0)}{\partial p_j} = \frac{\partial f_\nu}{\partial p_j} \, \frac{p_{j0}}{f_\nu(\mathbf{p}_0)} \,. \tag{25}$$

The advantage of the presented analytical approach is in the fast and accurate sensitivity calculation. As the system is composed of several subsystems the first order derivatives in expression (24) can be calculated very quickly. Each derivative is composed of block diagonal zero matrices and only one nonzero matrix. It depends on the type of the design parameter. The particular form of the derivatives of finite element matrices, discrete element matrices or

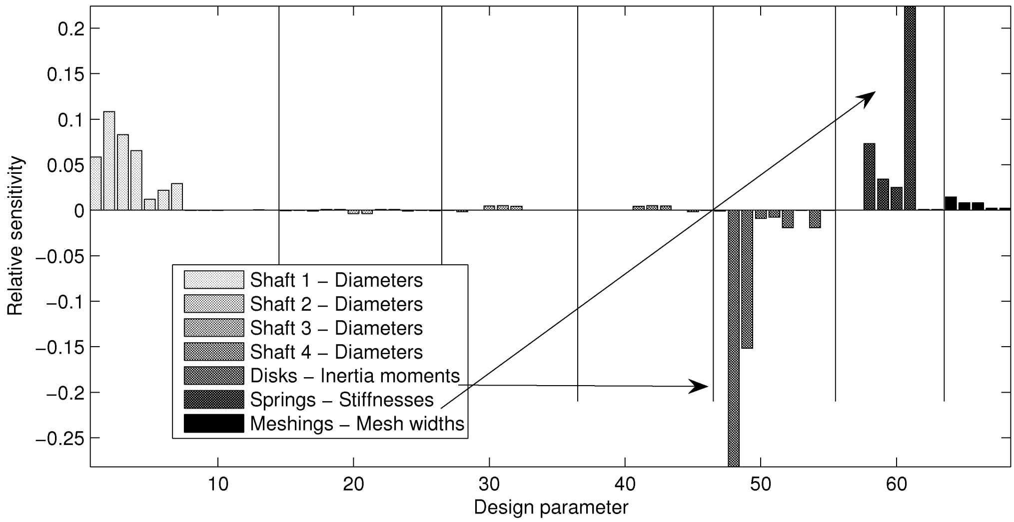


*Fig.5: Relative sensitivities of the third eigenfrequency with respect to the change of various design parameters*

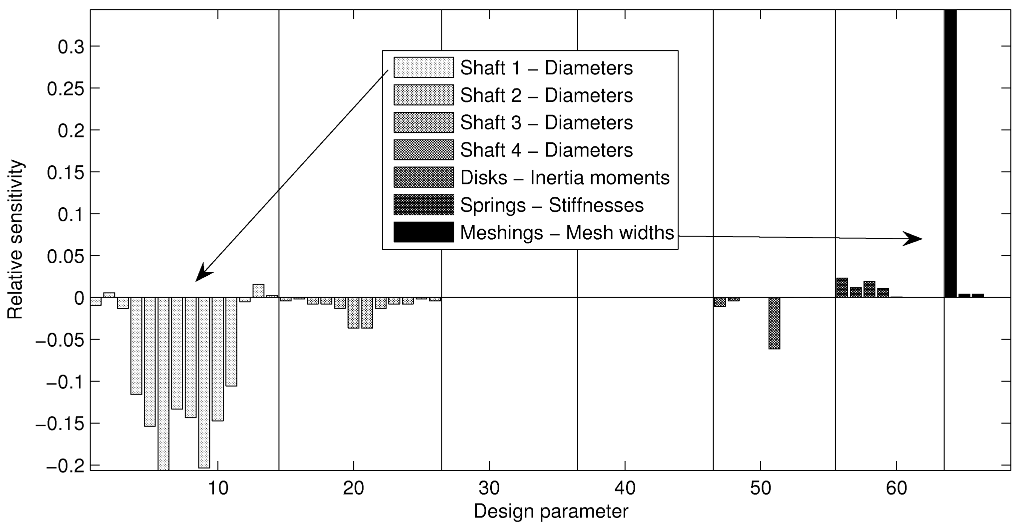


*Fig.6: Relative sensitivities of the tenth eigenfrequency with respect to the change of various design parameters*

gear mesh stiffness matrix is obvious. More general derivation of eigenvalue and eigenvector sensitivities for various non-rotating and rotating systems is given in [3]. The analytical approach in case of multiple (repeated) eigenvalues was published in [13].

The sensitivity analysis results were tested for both numerical and analytical expressions (17) and (25). A good agreement for proper selection of $\Delta p_j$ can be found. The illustration of the sensitivity values for the Wikov Gear gearbox is in Figures 5 and 6. There are the relative sensitivities of the third and the tenth gearbox eigenfrequencies that can cause possible resonance states (see Figure 4). The third eigenfrequency (and similarly the second eigenfrequency) is characterized by the dominant influence of the Pencoflex coupling parameters (its moments of inertia and its stiffness). Since these are not the gearbox design parameters the selection of the coupling is based on the mill operator decision. On the other hand the meshing stiffness of the first gear mesh, which affects the tenth eigenfrequency (Figure 6), can be changed by the designers.

## 5. Conclusions

The model suitable for the solution of the torsional vibration of the real gearbox produced by Wikov Gear s.r.o. including the motor and the couplings is presented in this paper. It can be used for the most important analysis of the system eigenvalues and it can help to determine possible dangerous resonances. Further two approaches to the sensitivity analysis of eigenfrequencies were introduced. Design parameters chosen with respect to the sensitivity analysis were used for tuning the gearbox eigenfrequencies out of the excitation frequencies. Based on the relative sensitivities (Figure 6) the width of the first gear mesh between shaft 1 and shaft 2 was chosen in order to tune the tenth eigenfrequency out of the resonant state. The tuned eigenfrequencies are denoted in Figure 4 by circles. The second and the third eigenfrequencies, which are also close to the excitation frequency, cannot be affected by gearbox design parameters because they are characterized by a dominant vibration of input equipment (mainly by coupling, see Figure 5). Therefore they were not tuned out.

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## References

[1] Bogomolni M., Kirsch U., Sheinman I.: Efficient design sensitivities of structures subjected to dynamic loading, International Journal of Solids and Structures 43 (2006) 5485–5500

[2] Bozca M.: Torsional vibration model based optimization of gearbox geometric design parameters to reduce rattle noise in an automotive transmission, Mechanism and Machine Theory 45 (2010) 1583–1598

[3] Byrtus M., Hajžman M., Zeman V.: Dynamika rotujících soustav, University of West Bohemia 2010, Plzeň

[4] Crowther A.R., Zhang N.: Torsional finite elements and nonlinear numerical modelling in vehicle powertrain dynamics, Journal of Sound and Vibration 284 (2005) 825–849

[5] Dias J.M.P., Pereira M.S.: Sensitivity Analysis of Rigid-Flexible Multibody Systems, Multibody System Dynamics 1 (1997) 303–322

[6] Dong J. et al.: Parametric design sensitivity analysis of high-frequency structuralůacoustic problems using energy finite element method, International Journal for Numerical Methods in Engineering 62 (2005) 83–121

[7] Guedria N., Smaoui H., Chouchane M.: A direct algebraic method for eigensolution sensitivity computation of damped asymmetric systems, International Journal for Numerical Methods in Engineering 68 (2006) 674–689

[8] Guedria N., Chouchane M., Smaoui H.: Second-order eigensensitivity analysis of asymmetric damped systems using Nelson's method, Journal of Sound and Vibration 300 (2007) 974–992

[9] Hajžman M., Šašek J., Zeman V.: Modelling of flexible rotor vibrations in the rotating coordinate system, Modelling, Simulation and Control of Nonlinear Engineering Dynamical Systems. State-of-the-Art, Perspectives and Applications, Springer 2009, Berlin, pp. 277–288

[10] Hajžman M., Zeman V.: Influence of rotors unbalance and gears eccentricity on stator vibration and noise, In: 7th IFToMM-Conference on Rotor Dynamics, eds H. Springer, H. Ecker, Vienna University of Technology, Vienna 2006, full paper on CD-ROM

[11] Hajžman M.: Rozšířený model torzního kmitání převodovky Wikov Gear SideDrive 6MW a jeho citlivostní analýza, Research report VYZ 1270/2010, ŠKODA VÝZKUM s.r.o. 2010, Plzeň

[12] Lallemand B., Level P., Duveau H., Mahieux B.: Eigensolutions sensitivity analysis using a sub-structuring method, Computers and Structures 71 (1999) 257–265

[13] Pešek L.: An extension of the inverse sensitivity method to systems with repeated eigenvalues, Journal of Sound and Vibration 182 (1995) 623–635

[14] Polach P., Hajžman M.: Multibody simulations of trolleybus vertical dynamics and influence of spring-damper structural elements, Applied and Computational Mechanics 2 (2008) 101–112

[15] Slavík J., Stejskal V., Zeman, V.: Základy dynamiky strojů, CTU Publishing House 1997, Praha

[16] Walker D.: Torsional Vibration of Turbomachinery, McGraw-Hill 2003, New York

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