B-SPLINE FILTRATION OF CONVEX DATA

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A method for B-spline filtration of data measured on a very precisely manufactured sphere is described in this paper. The method has been developed to decrease the uncertainty of measurement which can be obtained by least squares method commonly used when processing the measured data. The B-spline filtration is realised as a Bspline surface approximating the data measured on 3D coordinate measuring machine and transformed into the parametric space of the sphere. This transformation eliminates the undesirable consequences of convex hull property of B-spline surfaces when processing convex data. Furthermore, the B-spline representation of the measured sphere can be considered as a certain replacement of the original sphere. Subsequently, the B-spline representation can be used for a new more precise measuring strategy in iterative measuring process.

Keywords: B-spline, measured data, uncertainty of measurement, least squares method, filtration, transformation, reverse transformation

1. Introduction

A sphere as a basic geometrical element is widely used in dimensional metrology, especially when calibrating coordinate measuring machines (CMM). To determine the volumetric error of CMM, a tactile probe of spherical shape and measuring standards such as individual calibration sphere or sets of calibration spheres (so called ball plates [3,7]) as well as special calibration artefacts containing spherical surfaces [1,8] are used. Therefore, the sphere measurement and the data measured on the sphere processing belong to the basic problem in dimensional metrology.

The purpose of measurement is to provide information about a quantity of interest – a measurand. No measurement is exact. When a quantity is measured, the outcome depends on the measuring system, the measurement procedure, the skill of the operator, the environment, and other effects [9]. In general, the result of the measurement is only an approximation of the precise value. Therefore, the measurand, and thus the measurement result is complete only when accompanied by a quantitative statement of its uncertainty. The data measured on the sphere are considered to be distorted by two types of errors: random error caused by the accuracy limit of the measuring instrument and systematic error caused by incorrect calibration of the measuring instrument.

To process the data measured on the sphere means to determine its centre **S**, characteristic radius r, radius r_{max} of the circumscribed sphere and radius r_{min} of the inscribed sphere. The uncertainty zone Δr is given by $\Delta r = |r_{\text{max}} - r_{\text{min}}|$. Usually, the least squares method

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(LSM) and some type of filters [5] are implemented in metrological software to obtain these measurands. If topography of radii is necessary to know, the values obtained by LSM are insufficient and alternative methods of real sphere surface reconstruction are developed [2].

In this article, the special application of approximation bicubic B-spline surface [6] as a filtration of data measured on a very precise manufactured sphere is presented. Approximation B-spline surface is used as a filter rather then interpolation one because the data measured on the sphere are not accurate. Therefore, the 'variation diminishing property' of B-spline basic functions (the total variation of B-spline basic functions is less or equal to the total variation of the original approximated function) can be applied. Based on this property, the B-spline basic functions have a blending effect to the approximated data.

2. B-spline surface

Approximation B-spline surface (B-spline surface) is determined by $(m + 1) \times (n + 1)$ control points arranged into a spatial mesh. The resulted surface does not pass through the control points but its shape is influenced by the configuration of control points. B-spline surface is a *piecewise* surface which consists of continuously joined regular elements of the surface – segments. The continuity of all segments joining is C^{p-1} in the direction of uparameter and C^{q-1} in the direction of v parameter, where p is the degree of surface in the direction of u parameter and q is the degree of the surface in the direction of v parameter.

The vector equation of B-spline surface is given by

$$\mathbf{S}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{P}_{i,j} N_{i,j}(u,v) , \quad u \in [u_p, u_{r-p}] , \quad v \in [v_q, v_{s-q}] , \quad (1)$$

where $\mathbf{S}(u,v) = (x(u,v), y(u,v), z(u,v))$ is radius vector of the surface point, $\mathbf{P}_{i,j} = [x_{i,j}, y_{i,j}, y_{i,j}]$, i = 0, 1, ..., m, j = 0, 1, ..., n are control points and $N_{i,j}(u,v)$, i = 0, 1, ..., m, j = 0, 1, ..., n are B-spline basic functions of two parameters. B-spline basic functions of two parameters are created as a tenzor product of B-spline basic functions of one parameter

$$N_{i,j}(u,v) = N_{i,p}(u) N_{j,q}(v) , \quad i = 0, 1, \dots, m , \quad j = 0, 1, \dots, n .$$
(2)

B-spline basic functions of p-th degree of parameter u, and B-spline basic functions of q-th degree of parameter v are defined by recurrent formula:

$$N_{i,0}(u) = \begin{cases} 1 & u \in [u_i, u_{i+1}), \\ 0 & u \notin [u_i, u_{i+1}), \\ 0 & u \notin [u_i, u_{i+1}), \end{cases}$$
(3)

$$N_{i,k}(u) = \frac{u - u_i}{u_{i+k} - u_i} N_{i,k-1}(u) + \frac{u_{i+k+1} - u}{u_{i+k+1} - u_{i+1}} N_{i+1,k-1}(u), \quad k = 1, 2, \dots, p, \\ N_{j,0}(v) = \begin{cases} 1 & v \in [v_j, v_{j+1}), \\ 0 & v \notin [v_j, v_{j+1}), \\ 0 & v \notin [v_j, v_{j+1}), \\ N_{j,k}(v) = \frac{v - v_j}{v_{j+k} - v_j} N_{j,k-1}(v) + \frac{v_{j+k+1} - v}{v_{j+k+1} - v_{j+1}} N_{j+1,k-1}(v), \quad k = 1, 2, \dots, q, \end{cases}$$
(4)

where

$$U = (u_0, u_1, \dots, u_r) ,$$

$$V = (v_0, v_1, \dots, v_s)$$
(5)

are knot vectors. The members of knot vectors – knots – create a nondecreasing sequence. The number of knot spans is

$$r = m + p + 1$$
, $s = n + q + 1$. (6)

The domain of parameterization of B-spline basic functions is the whole range of knot vectors, i.e. $u \in [u_0, u_r]$, $v \in [v_0, v_s]$. However, the domain of parameterization of B-spline surface is the reduced range of knot vectors $u \in [u_p, u_{r-p}]$, $v \in [v_q, v_{s-q}]$, so called *active* domain of parameterization [4]. The first and the last p knot spans of U knot vector and the first and the last q knot spans of V knot vector create so called *passive* domains of parameterization [4]. Usually, the normalized active domain of parameterization is considered, i.e. $[u_p, u_{r-p}] = [0, 1], [v_q, v_{s-q}] = [0, 1].$

If the knots are equally spaced, the knot vector is called *uniform*. Here, the uniform normalized knot vectors in the following form are used

$$U = \left(\underbrace{-p\,\Delta u, -(p-1)\,\Delta u, \dots}_{\text{passive domain}}, \underbrace{0, \Delta u, 2\,\Delta u, \dots, 1}_{\text{active domain}}, \underbrace{\dots, (r-p)\,\Delta u}_{\text{passive domain}}\right),$$

$$V = \left(\underbrace{-q\,\Delta v, -(q-1)\,\Delta v, \dots}_{\text{passive domain}}, \underbrace{0, \Delta v, 2\,\Delta v, \dots, 1}_{\text{active domain}}, \underbrace{\dots, (s-q)\,\Delta v}_{\text{passive domain}}\right),$$
(7)

where the knot spans are given by [4]

$$\Delta u = u_i - u_{i-1} = \frac{1}{m - p + 1}, \quad i = 1, 2, \dots, r ,$$

$$\Delta v = v_j - v_{j-1} = \frac{1}{n - q + 1}, \quad j = 1, 2, \dots, s .$$
(8)

It is possible to proof that B-spline basic functions (3) and (4) are total positive and that their sum for the arbitrary knot span from the active domain of parameterization $u \in [u_p, u_{r-p}], v \in [v_q, v_{s-q}]$ is unit. Consequently, the whole B-spline surface is lying in the convex hull of its control mesh.

2.1. Bicubic B-spline surface

The B-spline basic functions degree influences the shape of the resulted B-spline surface in the following way: the higher degree is chosen the more intensive blending effect on the filtered data is indicated. Simultaneously, the original shape information of the filtered data is lost. Then, it is necessary to accept a compromise. The bicubic B-spline surfaces, where p = q = 3, have been proved the most successful in practice. The continuity of these surfaces is C^2 in the direction of both parameters, which is sufficient in the majority of applications. The control mesh of bicubic B-spline surface has to be created by sixteen control points at minimal, i.e. $m \leq 3$, $n \leq 3$.

The example of B-spline basic functions of 3^{rd} degree is shown in Fig. 1. On the left side, there are depicted B-spline basic functions of one parameter acc. (3) and (4) for m = 3 and n = 5. Their tenzor product, i.e. the basic functions of two parameters acc. (2), is depicted

on the right side. The control mesh with four rows (m = 3) and six columns (n = 5) and the corresponding bicubic B-spline surface is depicted in Fig. 2.

Note that the basic functions in Fig. 1 are depicted in the whole domain of parameterization, i.e. $u \in [u_0, u_7]$ and $v \in [v_0, v_9]$. Whereas, the resulted bicubic B-spline surface in Fig. 2 is depicted in the reduced active domain of parameterization $u \in [u_3, u_4]$ and $v \in [v_3, v_6]$.



Fig.1: B-spline basic functions of one parameter (left) and their tenzor product (right)



Fig.2: Control mesh and its approximation B-spline surface

2.2. Open and closed bicubic B-spline surface

The surface in Fig. 2 is the so called *open* bicubic B-spline surface. The open surface does not pass through the control points and it is created approximately around the inner part of control mesh without end polygons of control points. Therefore, the set of input data must be larger than the surface which characteristic dimensions have to be determined.

The bicubic B-spline surface closed in the direction u(v) arises when repeating three beginning rows (columns) at the end of control mesh. The C^2 continuity is preserved along the joined boundary. The increasing number of control points leads to increasing number of knots in the corresponding knot vector (6).

2.3. B-spline filtration of convex data

The main aim of this work is to design the method for B-spline filtration of points measured on a very precisely manufactured sphere. The configuration of these points is shown in Fig. 3. The measured points cannot be considered as the control points of B-spline surface (see Fig. 4), because the obtained result is absolutely distorted (see Fig. 5).



Fig.3: Configuration of measured points



Fig.4: Control mesh created by measured points

Fig.5: Direct filtration of measured points

The reason of this distortion is that the sphere is a convex surface and the points measured on the very precisely manufactured sphere create just convex hull of B-spline surface inside which the B-spline surface lies. It cannot be assumed that the measured points deviations from the ideal sphere are so big to disrupt the convexity.

The above mentioned B-spline filtration disadvantage can be eliminated by transformation of Cartesian coordinates x, y, z of the measured points into the spherical coordinates φ, θ, ρ :

$$\varphi = \arctan \frac{y}{x} ,$$

$$\theta = \arctan \frac{\sqrt{x^2 + y^2}}{z} ,$$

$$\rho = \sqrt{x^2 + y^2 + y^2} .$$
(9)

Following the substitution control points $\mathbf{P}_{i,j} = [\varphi_{i,j}, \theta_{i,j}, \rho_{i,j}], i = 0, 1, \dots, m, j = 0, 1, \dots, n$ in (1), the B-spline surface $\mathbf{S}(u, v) = (\varphi(u, v), \theta(u, v), \rho(u, v))$ is obtained.

The ideal sphere in the space (φ, θ, ρ) is mapped into the plane perpendicular to the ρ axis and passing through the point $\rho = r$, where r is the radius of the ideal sphere (see Fig. 6). The points on the ideal sphere are mapped into the points in that plane. The measured points are mapped into the points above and below that plane due to the deviations. Thereby, the input data completely loses its convex character and B-spline filtration provides the relevant results.



Fig.6: Points on ideal sphere and ideal sphere in the space (φ, θ, ρ)

After B-spline filtration of the data in the space (φ, θ, ρ) , the reverse transformation into the space (x, y, z) has to be done

$$\begin{aligned} x(u,v) &= \rho(u,v) \cos(\varphi(u,v)) \sin(\theta(u,v)) ,\\ y(u,v) &= \rho(u,v) \sin(\varphi(u,v)) \sin(\theta(u,v)) ,\\ z(u,v) &= \rho(u,v) \cos(\theta(u,v)) , \end{aligned}$$
(10)

to be able to draw and evaluate the points on B-spline surface $\mathbf{S}(u, v) = (\varphi(u, v), \theta(u, v), \rho(u, v))$ in the common way.

2.4. Minimal and maximal radii of the sphere determination

The minimal radius r_{\min} and the maximal radius r_{\max} of the sphere are determined as the global extremes of the coordinate function $\rho(u, v)$ from the vector equation of the surface $\mathbf{S}(u, v) = (\varphi(u, v), \theta(u, v), \rho(u, v))$:

$$r_{\min} = \min(\rho(u, v)) , \qquad r_{\max} = \max(\rho(u, v)) . \tag{11}$$

2.5. Characteristic radius of the sphere determination

The characteristic radius r can be determined as the altitude of the prism with the base Ω in the plane (φ, θ) (top view of the B-spline surface $\mathbf{S}(u, v) = (\varphi(u, v), \theta(u, v), \rho(u, v)))$ and with the volume V

$$V = \iint_{\Omega} \rho(\varphi, \theta) \,\mathrm{d}\varphi \,\mathrm{d}\theta \,\,. \tag{12}$$

The volume V must be the some as is the volume of the prismatic solid with the base Ω in the plane (φ, θ) (top view of the B-spline surface) and bounded by B-spline surface from above. This approach is correct due to the fact that the ideal sphere in the space (φ, θ, ρ) is mapped into the plane perpendicular to the ρ axis and passing through the point $\rho = r$.

The characteristic radius r is calculated according to the following formula

$$r = \frac{V}{S_{\Omega}} , \qquad (13)$$

where S_{Ω} is an area of the base Ω in the plane (φ, θ) .

3. Preparation phase of filtration

The set of measured data contained the information about the diameter of measuring probe $d_{\rm S} = 4.9997 \,\mathrm{mm}$ (i.e. radius $r_{\rm S} = 2.49985 \,\mathrm{mm}$) and Cartesian coordinates of 101 measured points – centres of the measuring probe. The center **M** of the measuring probe is shown in Fig. 7. Point of contact **T** between the sphere and the measuring probe lies on the connecting line **SM**, therefore only the centres of measuring probe **M** will be considered in the following procedure.



Fig.7: Point of contact between sphere and probe

3.1. Centre of measured sphere determination

It is necessary to know the coordinates of the centre **S** of the measured sphere to be able to determine the point of contact **T**. The least squares method has been used for finding the best-fitting sphere (centre $\mathbf{S} = [m, n, p]$ and radius r) to the set of measured points \mathbf{M}_i , $i = 0, 1, \ldots, 100$. The following values have been found by means of Solver in MS Excel (by minimizing the sum of the squares of the points deviations from the sphere):

m = 0.001439, n = 0.001495, p = 0.001372, r = 17.499886

and the following correction of the measured points has been done:

$$\mathbf{M}_{i} = [x_{i} = x_{i} - m, y_{i} = y_{i} - n, z_{i} = z_{i} - p], \quad i = 0, 1, \dots, 100.$$
(14)

After that, the centre \mathbf{S} of the measured sphere is identical with the coordinate system origin.

3.2. Points of contact determination

Point of contact \mathbf{T}_i , i = 0, 1, ..., 100 lies on the radius vector of point \mathbf{M}_i , i = 0, 1, ..., 100 (14) in the distance $r_{\rm S} = 2.49985 \,\mathrm{mm}$ from the point \mathbf{M}_i , i = 0, 1, ..., 100. The coordinates of point \mathbf{T}_i , i = 0, 1, ..., 100 are given by

$$\mathbf{T}_{i} = \mathbf{M}_{i} - \mathbf{U}_{i} r_{\mathrm{S}} , \quad i = 0, 1, \dots, 100 ,$$
 (15)

where U_i , i = 0, 1, ..., 100 is unit radius vector of the point M_i , i = 0, 1, ..., 100,

$$\mathbf{U}_{i} = (u_{i}^{1}, u_{i}^{2}, u_{i}^{3}) = \left(\frac{x_{i}}{r_{i}}, \frac{y_{i}}{r_{i}}, \frac{z_{i}}{r_{i}}\right) , \quad i = 0, 1, \dots, 100 , \qquad (16)$$

where

$$r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}, \quad i = 0, 1, \dots, 100$$
 (17)

is the size of radius vector of the point \mathbf{M}_i , $i = 0, 1, \dots, 100$.

3.3. Transformation into the space (φ, θ, ρ)

The points of contact $\mathbf{T}_i = [x_i, y_i, z_i]$, i = 0, 1, ..., 100 are transformed into the space (φ, θ, ρ) acc. (9). The points of contact $\mathbf{T}_i = [\varphi_i, \theta_i, \rho_i]$, i = 0, 1, ..., 100 in the space (φ, θ, ρ) are obtained by reverse transformation

$$\mathbf{T}_{i} = \left[\arctan \frac{y_{i}}{x_{i}}, \arctan \frac{\sqrt{x_{i}^{2} + y_{i}^{2}}}{z_{i}}, \sqrt{x_{i}^{2} + y_{i}^{2} + z_{i}^{2}} \right], \quad i = 0, 1, \dots, 100.$$
(18)

The points of contact calculated acc. (18) create the base for control mesh of B-spline surface.

3.4. Control mesh of B-spline surface

The measured data is approximated by bicubic B-spline surface. This surface has to be closed in u direction (corresponds to the φ direction) and open in v direction (corresponds to the θ direction). The points of contact (18) are placed into the control mesh in such a way



Fig.8: Control mesh in the space (φ, θ, ρ)

to fulfil not only the requirement to close the surface in u direction, but also to ensure the extension of the control mesh in v direction.

This extension guarantees overlapping of input data required in section 2.2 Open and closed bicubic B-spline surface. The coordinates of control points $\mathbf{P}_{i,j} = [\varphi_{i,j}, \theta_{i,j}, \rho_{i,j}]$, $i = 0, 1, \ldots, 12, j = 0, 1, \ldots, 12$ in control mesh are calculated from the coordinates of points of contact (18) according to the rules which are obvious from Tab. 1. The control mesh in the space (φ, θ, ρ) is shown in Fig. 8.

	${\rm Coordinate}\;\varphi$												
i, j	0	1	2	3	4	5	6	7	8	9	10	11	12
0	$2\pi - \varphi_{11}$	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	$2\pi + \varphi_2$	$2\pi + \varphi_3$
1	$2\pi - \varphi_{11}$	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	$2\pi + \varphi_2$	$2\pi + \varphi_3$
2	$2\pi - \varphi_{11}$	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	$2\pi + \varphi_2$	$2\pi + \varphi_3$
3	$2\pi - \varphi_{21}$	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}	$arphi_{17}$	φ_{18}	$arphi_{19}$	φ_{20}	φ_{21}	$2\pi + \varphi_{12}$	$2\pi + \varphi_{13}$
4	$2\pi - \varphi_{31}$	φ_{22}	φ_{23}	φ_{24}	φ_{25}	φ_{26}	φ_{27}	φ_{28}	φ_{29}	$arphi_{30}$	φ_{31}	$2\pi + \varphi_{22}$	$2\pi + \varphi_{23}$
5	$2\pi - \varphi_{41}$	φ_{32}	$arphi_{33}$	φ_{34}	$arphi_{35}$	$arphi_{36}$	φ_{37}	$arphi_{38}$	$arphi_{39}$	φ_{40}	φ_{41}	$2\pi + \varphi_{32}$	$2\pi + \varphi_{33}$
6	$2\pi - \varphi_{51}$	φ_{42}	φ_{43}	φ_{44}	φ_{45}	φ_{46}	φ_{47}	φ_{48}	φ_{49}	φ_{50}	φ_{51}	$2\pi + \varphi_{42}$	$2\pi + \varphi_{43}$
7	$2\pi - \varphi_{61}$	φ_{52}	φ_{53}	φ_{54}	φ_{55}	φ_{56}	φ_{57}	$arphi_{58}$	$arphi_{59}$	φ_{60}	φ_{61}	$2\pi + \varphi_{52}$	$2\pi + \varphi_{53}$
8	$2\pi - \varphi_{71}$	φ_{62}	φ_{63}	φ_{64}	φ_{65}	φ_{66}	$arphi_{67}$	φ_{68}	$arphi_{69}$	φ_{70}	φ_{71}	$2\pi + \varphi_{62}$	$2\pi + \varphi_{63}$
9	$2\pi - \varphi_{81}$	φ_{72}	φ_{73}	φ_{74}	φ_{75}	φ_{76}	φ_{77}	φ_{78}	φ_{79}	φ_{80}	φ_{81}	$2\pi + \varphi_{72}$	$2\pi + \varphi_{73}$
10	$2\pi - \varphi_{91}$	φ_{82}	φ_{83}	φ_{84}	φ_{85}	φ_{86}	φ_{87}	φ_{88}	φ_{89}	$arphi_{90}$	φ_{91}	$2\pi + \varphi_{82}$	$2\pi + \varphi_{83}$
11	$2\pi - \varphi_{101}$	φ_{92}	$arphi_{93}$	φ_{94}	$arphi_{95}$	$arphi_{96}$	$arphi_{97}$	$arphi_{98}$	$arphi_{99}$	φ_{100}	φ_{101}	$2\pi + \varphi_{92}$	$2\pi + \varphi_{93}$
12	$2\pi - \varphi_{91}$	φ_{82}	φ_{83}	φ_{84}	φ_{85}	φ_{86}	φ_{87}	φ_{88}	φ_{89}	$arphi_{90}$	φ_{91}	$2\pi + \varphi_{82}$	$2\pi + \varphi_{83}$
Coordinate θ													
i, j	0	1	2	3	4	5	6	7	8	9	10	11	12
0	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_2	θ_3	θ_{11}	θ_2	θ_3	θ_4	θ_5
1	θ_1	θ_1	θ_1	θ_1	θ_1	θ_1	θ_1	θ_1	θ_1	θ_1	θ_1	θ_1	θ_1
2	θ_{11}	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_2	θ_3
3	θ_{21}	θ_{12}	θ_{13}	θ_{14}	θ_{15}	θ_{16}	θ_{17}	θ_{18}	θ_{19}	θ_{20}	θ_{21}	θ_{12}	θ_{13}
4	θ_{31}	θ_{22}	θ_{23}	θ_{24}	θ_{25}	θ_{26}	θ_{27}	θ_{28}	θ_{29}	θ_{30}	θ_{31}	θ_{22}	θ_{23}
5	θ_{41}	θ_{32}	θ_{33}	θ_{34}	θ_{35}	θ_{36}	θ_{37}	θ_{38}	θ_{39}	θ_{40}	θ_{41}	θ_{32}	θ_{33}
6	θ_{51}	θ_{42}	θ_{43}	θ_{44}	θ_{45}	θ_{46}	θ_{47}	θ_{48}	θ_{49}	θ_{50}	θ_{51}	θ_{42}	θ_{43}
7	θ_{61}	θ_{52}	θ_{53}	θ_{54}	θ_{55}	θ_{56}	θ_{57}	θ_{58}	θ_{59}	θ_{60}	θ_{61}	θ_{52}	θ_{53}
8	θ_{71}	θ_{62}	θ_{63}	θ_{64}	θ_{65}	θ_{66}	θ_{67}	θ_{68}	θ_{69}	θ_{70}	θ_{71}	θ_{62}	θ_{63}
9	θ_{81}	θ_{72}	θ_{73}	θ_{74}	θ_{75}	θ_{76}	θ_{77}	θ_{78}	θ_{79}	θ_{80}	θ_{81}	θ_{72}	θ_{73}
10	θ_{91}	θ_{82}	θ_{83}	θ_{84}	θ_{85}	θ_{86}	θ_{87}	θ_{88}	θ_{89}	θ_{90}	θ_{91}	θ_{82}	θ_{83}
11	θ_{101}	θ_{92}	θ_{93}	θ_{94}	θ_{95}	θ_{96}	θ_{97}	θ_{98}	θ_{99}	θ_{100}	θ_{101}	θ_{92}	θ_{93}
12	$\pi - \theta_{91}$	$\pi - \theta_{82}$	$\pi - \theta_{83}$	$\pi - \theta_{84}$	$\pi - \theta_{85}$	$\frac{\pi-\theta_{86}}{2}$	$\pi - \theta_{87}$	$\pi - \theta_{88}$	$\pi - \theta_{89}$	$\pi - \theta_{90}$	$\pi - \theta_{91}$	$\pi - \theta_{82}$	$\pi - \theta_{83}$
i i j													
i, j	0	1	2	3	4	0	0	1	0	9	10	11	12
1	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}	ρ_{11}	ρ_2	ρ_3	ρ_{11}	ρ_2	ρ_3	ρ_4	ρ_5
2	<i>p</i> ₁	ρ_1	ρ_1	ρ_1	ρ_1	ρ_1	ρ_1	ρ_1	ρ_1	ρ_1	ρ_1	ρ_1	ρ_1
2	<i>P</i> 11	<i>p</i> ₂	<i>P</i> 3	ρ_4	ρ_5	ρ_{6}	<i>P</i> 7	<i>P</i> 8	<i>P</i> 9	ρ_{10}	ρ ₁₁	<i>P</i> 2	<i>P</i> 3
1	ρ ₂₁	<i>P</i> ₁₂	<i>P</i> 13	ρ_{14}	ρ ₁₅	ρ_{16}	ρ ₁₇	<i>P</i> 18	ρ ₁₉	<i>P</i> 20	ρ ₂₁	<i>P</i> 12	<i>P</i> 13
5	<i>P</i> 31	022	P23	P24 024	P25	<i>P</i> 26	027	P28	P29 020	<i>P</i> 30	<i>P</i> 31	P22 022	<i>μ</i> 23
6	051	P32	P33	P34	P35	P36	047	P38	P39	P40	P41 051	P32	P33
7	0e1	0E2	P43 0=2	0=4	P40 0FF	P40 056	0=7	P48 0=0	0=0	P50 060	061	050	0E2
8	071	P52 062	P 53	P 34	P 55	Dee	067	P38 069	P 59	070	071	062	P 33
9	091	P62 079	P03	074	P03	P00 076	077	P08	070	000	091	072	P03
10	<i>P</i> 81 <i>0</i> 01	000	000	094	00E	096	007	000	000	P80 000	P81 001	082	002
11	<i>P</i> 91 <i>0</i> 101	P82 002	P83 002	P84 004	P85 005	P80 006	007	P88 000	1289 1000	P90 0100	0101	P82 002	P83 002
$\frac{11}{12}$	001	082	P 93 083	084	P 95 085	P 90 086	087	P 90 088	<i>P</i> 33 <i>0</i> 80	000	001	082	083
	P 91	P04	100	P 04	100	100	104	100	109	P90	191	P04	100

Tab.1: Coordinates φ , θ , ρ of points of contact in the control mesh of B-spline surface

4. Filtration

The number of knot spans acc. (6) is

$$r = s = 12 + 3 + 1 = 16 \tag{19}$$

because m = n = 12. The size of knot spans acc. (8) is

$$\Delta u = \Delta v = \frac{1}{12 - 3 + 1} = 0.1 \tag{20}$$

and knot vectors acc. (7) are

$$U = (u_0, u_1, \dots, u_{16}) = (-0.3, -0.2, \dots, 1.3) ,$$

$$V = (v_0, v_1, \dots, v_{16}) = (-0.3, -0.2, \dots, 1.3) .$$
(21)

First, we express the B-spline basic functions (3) and (4) for the knot vectors (21). After that, we calculate their tenzor product (2) and then, together with the coordinates of control points acc. (18) we substitute them in (1). Thus, we obtain the mathematical description of the measured data – bicubic B-spline surface $\mathbf{S}(u, v) = (\varphi(u, v), \theta(u, v), \rho(u, v))$. This mathematical description is too large – the resulting surface consists of 100 segments which are C^2 continuously joined in the direction of both parameters. Each segment is described by a different vector equation containing three coordinate functions of two parameters. Therefore, this description is not written here.



Fig.9: B-spline surface $\mathbf{S}(u, v)$ as a filter of measured data in the space (φ, θ, ρ)

The resulting bicubic B-spline surface (in the front view, top view, profile view and axonometric view) is depicted in Fig. 9; the control mesh is included.

4.1. Minimal and maximal radii of the sphere

The global extremes of the coordinate function $\rho(u, v)$ (11) have been calculated numerically by means of grid method [10] and the following values have been obtained

 $r_{\min} = 14.999791 \,\mathrm{mm}$, $r_{\max} = 15.000365 \,\mathrm{mm}$.

4.2. Characteristic radius of the sphere

Double integral in (12) has been calculated numerically by means of rectangle rule for multivariate integration and the obtained value of characteristic radius of the sphere is

$$r = 15.000028 \,\mathrm{mm}$$
.

The sphere with characteristic radius r = 15.000028 mm is mapped into the plane perpendicular to the ρ axis and passing through the point $\rho = 15.000028$. In front and profile view, this plane is projected into the line perpendicular to the ρ axis, see thick black line in Fig. 10.



Fig.10: Characteristic radius of the sphere determination

The measured points after reverse transformation (10) into the space (x, y, z) are shown in Fig. 11 together with the sphere with characteristic radius r = 15.000028 mm. The control mesh acc. Tab. 1 and bicubic B-spline surface after reverse transformation (10) into the space (x, y, z) are shown in Fig. 12.

5. Refining of measured data via B-spline filtration

The filtered data can be considered as a certain replacement of the original data measured on the sphere. Then, the B-spline representation of the sphere, especially the free-form surface acc. Fig. 12, has to be taken into account instead of the original sphere.



Fig.11: Measured points and sphere with characteristic radius in the space (x, y, z)(the scale of distance between measured points and sphere is 10 000:1)



Fig.12: B-spline surface $\mathbf{S}(u, v)$ as a filter of measured data in the space (x, y, z)(the scale of distance between measured points and the sphere is 10 000:1)

The measuring probe is moved in the normal direction to the measured surface at the specific point which has to be measured. The software of measuring machine calculates the normal direction to the basic geometrical shapes (e.g. to the sphere, conical or cylindrical surface) automatically according to the known formulas from differential geometry. This calculation respects the chosen strategy of measuring – distribution and density of measured points. Therefore, in the case of free-form measuring it is necessary to know both – the coordinates of measured points and coordinates of normal vectors at these points.

5.1. Normal vector at the point of B-spline surface

The direction of normal vector at the point of B-spline surface is perpendicular to the tangent plane at this point. The tangent plane at the point of B-spline surface is determined by a pair of tangent vectors: tangent vector to the parametric u-curve and tangent vector to the parametric v-curve passing through this point.

For B-spline surface (1), the parametric *u*-curve $\mathbf{S}(u,\beta)$ and the parametric *v*-curve $\mathbf{S}(\alpha, v)$ are obtained via substitution the constant value of parameter $u = \alpha, \alpha \in [u_p, u_{r-p}]$ and $v = \beta, \beta \in [v_q, v_{s-q}]$ in (1):

$$\mathbf{S}(u,\beta) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{P}_{i,j} N_{i,j}(u,\beta) , \quad u \in [u_p, u_{r-p}]$$
(22)

and

$$\mathbf{S}(\alpha, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{P}_{i,j} N_{i,j}(\alpha, v) , \quad v \in [v_q, v_{s-q}] , \qquad (23)$$

where p is the degree of the surface in u-direction, q is the degree of the surface in v-direction, and r and s are numbers of knot spans of knot vectors (5).

The first partial derivative

$$\mathbf{S}^{u}(u,v) = \frac{\partial \mathbf{S}(u,v)}{\partial u} \quad \text{and} \quad \mathbf{S}^{v}(u,v) = \frac{\partial \mathbf{S}(u,v)}{\partial v}$$
(24)

is a vector function. This vector function determines for (α, β) the tangent vector of parametric *u*-curve (22) and tangent vector of parametric *v*-curve (23). The cross product of tangent vectors (24) is the normal vector $\mathbf{n}(\alpha, \beta)$ at the point (α, β) of B-spline surface

$$\mathbf{n}(\alpha,\beta) = \mathbf{S}^u(\alpha,\beta) \times \mathbf{S}^v(\alpha,\beta) \ . \tag{25}$$

The unit normal vectors are more useful for graphical representation. The unit normal vector is determined by

$$\mathbf{n}(\alpha,\beta) = \left(\frac{n_{\mathbf{x}}(\alpha,\beta)}{||\mathbf{n}(\alpha,\beta)||}, \frac{n_{\mathbf{y}}(\alpha,\beta)}{||\mathbf{n}(\alpha,\beta)||}, \frac{n_{\mathbf{z}}(\alpha,\beta)}{||\mathbf{n}(\alpha,\beta)||}\right) , \qquad (26)$$

where

$$||\mathbf{n}(\alpha,\beta)|| = \sqrt{n_{\mathbf{x}}(\alpha,\beta)^2 + n_{\mathbf{y}}(\alpha,\beta)^2 + n_{\mathbf{z}}(\alpha,\beta)^2}$$
(27)

is the magnitude of the normal vector (25).

5.2. New measured data determination

Firstly, the coordinates of points $\mathbf{D}_{i,j} = [\varphi_{i,j}, \theta_{i,j}, \rho_{i,j}], i = 0, 1, \dots, 10, j = 0, 1, \dots, 10$ in the space (φ, θ, ρ) on bicubic B-spline surface are obtained by substitution of suitable uniformly chosen values of parameters u and v

$$u = 0, 0.1, \dots, 1$$
 and $v = 0, 0.1, \dots, 1$ (28)





Fig.13: New input data for measuring – points in the space (φ, θ, ρ)

Secondly, the coordinates of tangent vectors (24) and normal vectors (25) in the space (φ, θ, ρ) for the values of parameters (28) are calculated. After that, the reverse transformation (10) is done and the Cartesian coordinates of points $\mathbf{D}_{i,j} = [x_{i,j}, y_{i,j}, z_{i,j}]$, $i = 0, 1, \ldots, 10, j = 0, 1, \ldots, 10$ on B-spline surface and normal vectors $\mathbf{n}_{i,j} = [x_{i,j}, y_{i,j}, z_{i,j}]$, $i = 0, 1, \ldots, 10, j = 0, 1, \ldots, 10$ in the space (x, y, z) are calculated.

The measured points after the reverse transformation (10) into the space (x, y, z) are depicted together with the sphere in Fig. 14. The radius of the sphere in Fig. 14 is equal to the characteristic radius calculated in the section 4.2 Characteristic radius of the sphere. To keep the readability of Fig. 14, the scale 10 000:1 is used for the distance of calculated points from the sphere.



Fig.14: New input data for measuring obtained by B-spline filtration : points and normal vectors in the space (x, y, z) (the scale of distance between calculated points and sphere is 10 000:1)



Fig.15: New input data for measuring obtained by B-spline filtration : points and normal vectors in the space (x, y, z) (the scale of distance between calculated points and sphere is 10 000:1, the scale of unit normal vectors length is 2:1)

The bicubic B-spline surface in the space (x, y, z) with the points $\mathbf{D}_{i,j} = [x_{i,j}, y_{i,j}, z_{i,j}]$, $i = 0, 1, \ldots, 10, j = 0, 1, \ldots, 10$ is shown in Fig. 15. The scale of distance between calculated points and sphere is 10 000:1 (therefore, the direction of depicted normal vectors is distorted), the scale of unit normal vectors length is 2:1.

6. B-spline filtration and LSM comparison

In the case of LSM, the obtained characteristic radius r represents the only value which can be used for new input data and normal vectors calculation. The characteristic dimensions (i.e. radius r_{max} of the circumscribed sphere, radius r_{min} of the inscribed sphere and characteristic radius r) obtained by means of B-spline filtration method and by means of the least squares method are compared in Tab. 2, uncertainty zone $\Delta r = |r_{\text{max}} - r_{\text{min}}|$ including.

Characteristic dimension	B-spline filtration	Least squares method
$r_{\rm max} \ ({\rm mm})$	15.000365	15.000576
$r_{\min} (mm)$	14.999791	14.999706
$r (\rm{mm})$	15.000028	15.000036
$\Delta r \ (\mathrm{mm})$	0.000574	0.000870



Tab.2: Obtained results comparison

Fig.16: Characteristic dimensions and uncertainty zone for the B-spline filtration method and least square method (LSM)

It is obvious, that the B-spline uncertainty zone is thinner than the LSM uncertainty zone. The reason is visible from Fig. 16, where the orthogonal view of the B-spline surface and its control mesh from Fig. 9 is depicted.

The characteristic dimensions obtained by the LSM are calculated directly from the measured data. In the (φ, θ, ρ) space, the radius of the circumscribed sphere corresponds with the highest measured point – i.e. with the highest point of control mesh (designated as r_{max} – LSM in Fig. 16). Similarly, the radius of the inscribed sphere corresponds with the lowest measured point – i.e. with the lowest point of control mesh (designated as r_{min} – LSM in Fig. 16).

On the contrary, the characteristic dimensions obtained by B-spline filtration method are calculated from the filtered data. The filtered data is representing by analytic expression of B-spline surface lying in the convex hull of its control mesh (see 2.1 Bicubic B-spline surface) and approximating all control points. Therefore, the global extremes of B-spline surface are less than the global extremes of its control mesh. Specifically, the radius of circumscribed sphere corresponds with the global maximum of B-spline surface – i.e. with the highest point of B-spline surface (designated as $r_{\rm max}$ – B-spline in Fig. 16) and the radius of inscribed sphere corresponds with the global minimum of B-spline surface – i.e. with the

The differences between characteristic radii values (designated as r – B-spline and r – LSM in Fig. 16) obtained by B-spline filtration and the LSM are caused by different approaches to the calculation of these values from the different sets of input data.

lowest point of B-spline surface (designated as r_{\min} – B-spline in Fig. 16).

In the (φ, θ, ρ) space, the characteristic radius obtained by LSM represents the horizontal plane (fitted by LSM through all measured points) passing through the value 15.000036 on the ρ axis. Its orthogonal view is designated as r - LSM in Fig. 16. The characteristic radius obtained by B-spline method (designated as r - B-spline in Fig. 16) represents the upper base of the prism. The lower base of this prism is represented by top view of B-spline surface in the plane (φ, θ). The volume of this prism is equal to the volume of the prismatic solid with the same base in the plane (φ, θ) and bounded by the B-spline surface from above.

7. Conclusion

The method of B-spline filtration of the points measured on the sphere is realised as a bicubic B-spline surface approximating the input data. The bicubic B-spline surface is closed in u direction and open in v direction.

The procedure of B-spline filtration includes the following steps:

- Firstly, it is necessary to recalculate the input data (coordinates of measuring probe centres) to obtain points of contact between the sphere and the measuring probe.
- After that, the Cartesian coordinates [x, y, z] of points of contact are transformed into the spherical coordinates $[\varphi, \theta, \rho]$ to avoid the problems with B-spline approximation of convex data.
- Then, the points of contact are placed into the control mesh with respect to the conditions for closing and opening the resulting surface.
- Next, the calculation of B-spline surface is accomplished and its mathematical expression $\mathbf{S}(u, v) = (\varphi(u, v), \theta(u, v), \rho(u, v)), (u, v) \in [0, 1]^2$ is obtained.
- Finally, the minimal, maximal and characteristic radii of the sphere are determined. The minimal and maximal radii are calculated as the global extremes of coordinate function $\rho(u, v)$. The mathematical description of B-spline surface is realised in the space (φ, θ, ρ) , where the ideal sphere is mapped into the plane perpendicular to the ρ axis and passing through point $\rho = r$. Therefore, the characteristic radius can be determined as the altitude of the prism with the base in the plane (φ, θ) (top view of B-spline surface) and with the volume equal to the volume of the prismatic solid with the same base in the plane (φ, θ) and bounded by B-spline surface from above.
- Additionally, for the following metrological processing, the B-spline surface can be considered as an improvement of the initial estimation of measured surface given

by the original measured data. In this case, the Cartesian coordinates of points on B-spline surface and the normal vectors at these points are necessary to determine, because the directions of the measuring probe is identical with the directions of normal vectors.

The suggested method can be used for the filtration of the data measured on an arbitrary surface with known analytic expression of its ideal shape (e.g. cylindrical surface, conical surface, ...).

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