# COUPLED LONGITUDINAL 1–D THERMAL AND VISCOELASTIC WAVES IN MEDIA WITH TEMPERATURE DEPENDENT MATERIAL PROPERTIES

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A general analysis is formulated for the closed loop coupled thermal and displacement viscoelastic 1–D wave problem. The proper inclusion of the highly temperature sensitive viscoelastic material properties renders the problem nonlinear, even though the displacements and material properties continue to obey linear relations. Various categories of open and closed loop temperature-displacement couplings as well as boundary conditions are formulated and discussed. Comparisons between temperature independent and dependent material properties indicate that the nature of the latter nonlinear problem radically differs from the former steady-state temperature linear one.

Keywords: material temperature sensitivity, longitudinal waves, thermal waves, nonlinear variable coefficient integral PDEs

### 1. Introduction

Propelled by intense needs for the development of structural applications of high polymers, solid propellants, elevated temperature metals, composites, etc., fundamental research efforts during the last sixty plus years starting with [3] have moved linear viscoelasticity into the realm of mature sciences [10] and [3–17]. However, there remain a number of problems areas that need further research refinements, such as improved linear characterization, analysis and computational protocols and practically the entire nonlinear viscoelasticity area.

Analyses in the presence of temperature fields, whether static T(x) or fully dynamic T(x,t) where  $x = \{x_i\}$ , with i = 1, 2, 3 are Cartesian coordinates, generate their own set of intrinsic difficulties and fundamentally distinct responses. The temperature dependence of viscoelastic coefficients of viscosity  $\eta$  generally follows the Arrhenius relation [2]

$$\eta(\Theta) = \eta_0 \, \exp\left(-\frac{\mathcal{Q}}{R\Theta}\right) \,. \tag{1}$$

See Table 1 for symbol legend. A plot of this equation in Fig. 1 shows that  $\eta$  varies about one order of magnitude for every 20 °C. This extremely sensitive temperature dependence of viscoelastic material properties produces significant orders of magnitude changes (shifts) in relaxation moduli times as seen in Fig. 2. It is to be noted that three distinct, albeit related, temperature definitions  $(T, \vartheta, \Theta)$  are used in this paper as listed in Table 1.

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 Elastic constitutive relations are path independent and algebraic, while the viscoelastic ones are integral partial differential equations. Table 2 summarizes the coupling between displacements, temperature and material properties.

- It has been shown in [34] that steady state temperatures cannot arise from the coupled closed loop elastic relations nor are T(t) function possible solutions to the coupled governing relations. In other words, while these forms of temperature functions are certainly physically and mathematically realizable they do not lead to any elastic displacement-temperature coupling.
- However, in viscoelastic media even the most degenerate constant temperature case leads to coupling because of the relaxation moduli, such as  $E(x, t - t', T_0)$ , are inherently temperature dependent.



Fig.1: Viscoelastic viscosity temperature dependence



Fig.2: Relaxation modulus variation with temperature

Temperature	Definition	Where used	Units	Eq. no.	Scale
T	material	$E(T), E^{\mathrm{T}}(T),$	$^{\circ}\mathrm{C}$	(5)	$T=\Theta-273.16$
	temperature	$C(T), C^{\mathrm{T}}(T)$			
ϑ	temperature	$\epsilon_{ij}^{\rm T} = \alpha_{ij}  \vartheta$	$^{\circ}\mathrm{C}$	(3)	$\vartheta = T - T_0$
	relative to $T_0$				
$T_0$	temperature for	$\epsilon_{ij}^{\mathrm{T}} = 0$	$^{\circ}\mathrm{C}$	(3)	
Θ	absolute	$\eta(\Theta)$	Κ	(1)	$\Theta = T + 273.16$
	temperature				
$T_{\rm R}$ and $T_{\rm g}$	WLF reference	$a_{\mathrm{T}}(T,T_R)$	$^{\circ}\mathrm{C}$	(25)	$T_{\rm R} \ge T_{\rm g}$
	and glassy				
	temperatures				

Tab.1: Temperature definitions

Material	Displacement	Moduli	Density
elastic	yes	no	no
viscoelastic	yes	yes	yes

Tab.2: Temperature coupling

Material	Temperature/	Modulus	Coupling	Governing	Convolution	EVCP
	displacement			relations	in time space	
elastic	$\partial u_1^{\rm E}(x,t)/\partial t$	$E_0(x)$	weak	linear	n/a	n/a
elastic	T(x,t)	$E_0(x)$	strong	linear	n/a	n/a
viscoelastic	$\partial u_1(x,t)/\partial t$	$E[x,t,t^{\prime},T(x,t^{\prime})]$	strong	nonlinear	no	no
viscoelastic	T(x,t)	$E[x,t,t^{\prime},T(x,t^{\prime})]$	strong	nonlinear	no	no

Tab.3: Elastic and viscoelastic coupling

- For the ubiquitous worst case scenario when T(x,t), material properties effectively become non-homogeneous resulting in governing integral partial differential relations (IPDE) without convolution properties and with spatially and temporal variable coefficients, thus making even the linear problem not amenable to the elastic/viscoelastic correspondence principle (EVCP) analysis.
- In all cases where material properties are considered temperature dependent, the analysis of coupled temperature-displacement problems entails solutions of nonlinear governing relations (see Table 3) and, consequently, the EVCP is equally inapplicable.

There exists a substantial body of literature on separate thermal waves and elastic stress waves [18] [24] and for coupled thermal/elastic waves [27–32]. Uncoupled viscoelastic stress wave analyses may be found in [5] and [41–50]. The latter are typified by stress/displacement waves traveling at elastic wave speeds but with decaying amplitudes and a dissipating after-flow behind the wave front. The stability of the governing relations for pure viscoelastic deformation waves at constant temperatures is analyzed in [48]. Uncoupled (pure) thermal waves are independent of deformations and are, in effect, heated rigid body phenomena and may be considered as zero approximations to the coupled formulation.

Early formulations of effects of temperature dependent viscoelastic properties on the state of stress and strain may be found in [53–61].

One of the important factors in the study of 1–D wave propagations is that such analyses present analytical and experimental tools for the relatively simple and accurate determination of the instantaneous viscoelastic modulus  $E_0$  at either constant or variable temperatures.

# 2. Analysis

## 2.1. Constitutive relations

One of the most waxing problems yet remaining in linear thermo-viscoelasticity is the formulation of an efficient protocol for handling integral relations for temperature dependent material properties as the following analysis indicates. A MATLAB<sup>TM</sup> subroutine has been developed in [64] but further refinements are needed for extensions to large scale finite element procedure.

In a Cartesian coordinate system<sup>1</sup>,  $x = \{x_j\}$  (j = 1, 2, 3), the linear anisotropic nonhomogeneous non-isothermal linear stress-strain relations can be stated as

$$\text{elastic} \Longrightarrow \begin{cases} \sigma_{ij}^{\text{E}}(x,t) = E_{\underline{ij}kl}^{\text{E}}(x) \left[ \epsilon_{kl}^{\text{E}}(x,t) - \alpha_{\underline{ij}} \vartheta(x,t) \right] \\ \epsilon_{ij}^{\text{E}}(x,t) = C_{\underline{ij}kl}^{E}(x) \sigma_{kl}^{E}(x,t) + \alpha_{\underline{ij}} \vartheta(x,t) \end{cases},$$

$$(2)$$

viscoelastic  $\Longrightarrow$ 

$$\Longrightarrow \begin{cases} \sigma_{ij}(x,t) = \int_{-\infty}^{t} E^*_{ijkl}(x,t,t') \epsilon_{kl}(x,t') dt' - \int_{-\infty}^{t} E^{*T}_{\underline{ij}}(x,t,t') \underbrace{\alpha_{\underline{ij}} \vartheta(x,t')}_{= \epsilon^{T}_{\underline{ij}}(x,t')} dt' \\ \epsilon_{ij}(x,t) = \int_{-\infty}^{t} C^*_{ijkl}(x,t,t') \sigma_{kl}(x,t') dt' + \int_{-\infty}^{t} C^{*T}_{\underline{ij}}(x,t,t') \alpha_{\underline{ij}} \vartheta(x,t') dt' \end{cases}, \quad (3)$$

where  $E^*$ s and  $C^*$ s are respectively moduli and compliances with

$$E_{ijkl}^*(x,t) = \frac{\partial E_{ijkl}(x,t)}{\partial t} \quad \text{and} \quad C_{ijkl}^*(x,t) = \frac{\partial C_{ijkl}(x,t)}{\partial t} . \tag{4}$$

The temperature  $\vartheta$  is measured relative to a reference temperature  $T_0$  where the thermal strains are at rest, i.e.  $\epsilon_{ij}^T(T_0) = 0$ . Table 1 lists the various temperature definitions and their notational symbols.

Viscoelastic moduli and compliances are highly temperature sensitive, such that (see Fig. 1)

$$E_{ijkl}(x,t,t') \equiv E_{ijkl}[x,t,t',T(x,t')] \quad \text{and} \quad E_{ij}^T(x,t,t') \equiv E_{ij}^T[x,t,t',T(x,t')] \quad (5)$$

From Eqs. 3 it can be seen that zero stress conditions,  $\sigma_{ij}(x,t) = 0$ , and T = T(x) only will produce strains that creep in time. No such strain-time variations occur in elastic materials. For time independent temperatures the constitutive relations simplify to t-space convolution integrals with

$$E_{ijkl}(x,t,t') \equiv E_{ijkl}[x,t-t',T(x)]$$
 and  $E_{ij}^{T}(x,t,t') \equiv E_{ij}^{T}[x,t-t',T(x)]$ . (6)

<sup>&</sup>lt;sup>1</sup> The Einstein tensor notation applies throughout the paper. Pairs of repeated indices indicate summation while underlined indices signify no summations.

The explicit x dependence of the moduli and compliances indicates non-homogeneous materials in their own right exclusive of any temperature contributions. Additionally, strains are dependent on small displacements through the linear relations

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Alternately, the constitutive relations can be expressed in a differential form as

$$\boldsymbol{P}_{\underline{ij}}(x,t)\left\{\sigma_{\underline{ij}}(x,t)\right\} = \boldsymbol{Q}_{ijkl}(x,t)\left\{\epsilon_{kl}(x,t)\right\} - \boldsymbol{Q}_{ij}^{\mathrm{T}}(x,t)\left\{\alpha\,\vartheta(x,t)\right\}$$
(8)

with

$$\boldsymbol{P}_{ij}(x,t) = \sum_{n=1}^{r} a_{ij} \left[ x, T(x,t) \right] \frac{\partial^n}{\partial t^n} , \qquad \boldsymbol{Q}_{ijkl}(x,t) = \sum_{n=1}^{r} b_{ijkl} \left[ x, T(x,t) \right] \frac{\partial^n}{\partial t^n} , \qquad \boldsymbol{Q}_{ij}^{\mathrm{T}}(x,t) = \sum_{n=1}^{r} b_{ij}^{\mathrm{T}} \left[ x, T(x,t) \right] \frac{\partial^n}{\partial t^n}$$

$$(9)$$

and where the underscore indicates no summation over the so designated indices. Elastic material characterization is included with  $P_{ij} = 1$ ,  $Q_{ijkl} = E_{ijkl}^{E}$  and  $Q_{ij}^{T} = E_{ij}^{ET}$ . Eqs. (3) essentially are the Green's functions associated with the PDEs (8). With the higher and more numerous derivatives now present in the differential form, extreme care must be exercised to properly and consistently state the many initial conditions. Real material high polymer characterization may require derivatives up to orders of 25 to 30. Such higher order derivatives require additional initial conditions when compared to the integral relations, but the differential form may offer limited advantages when for instance Runge-Kutta techniques are employed, although in [64] a protocol is formulated for handling nonlinear integral differential equations in conjunction with the Runge-Kutta method. Integral-differential constitutive relations will exhibit at most second order time derivatives plus single temporal integrals.

The elastic relations are path independent algebraic equations while viscoelastic media exhibit Volterra hereditary behavior with energy dissipation and memory. In viscoelastic materials the thermal strains  $\alpha \vartheta$  can creep independently of the mechanical strains  $\epsilon_{kl}$ . This means that it is possible to creep due to thermal expansions alone with zero stresses present throughout the body. In elasticity both types of these strains follow the same moduli and stresses and strains are time invariant under static loading conditions.

Elastic materials are conservative and their loading and unloading paths are identical and history independent. Unless inertia effects are present, elastic stresses and strains, unlike viscoelastic ones, are always in phase. Viscoelastic materials dissipate energy starting at the initial condition time and continuing through the entire subsequent time period. This non-conservative action is initiated at any stress level and, therefore, neither material class possesses a yield point or yield stress.

Unlike elastic responses, another consequence of this viscoelastic behavior is the inability to produce stress-strain curves since all stress/strain responses are path and history specific. Any and all such data can be presented only in the form of stress vs. time and strain vs. time curves. Furthermore, any such responses are individually loading and temperature history dependent and are so reflected in these  $\sigma-t$  and  $\epsilon-t$  plots. The explicit dependence of relaxation moduli on positions  $x_i$  indicates that the material may be non-homogeneous in its own right in addition to the temperature T(x) or T(x,t) dependences.

Anisotropic isothermal non-homogeneous linear viscoelastic moduli can be expressed as Prony series [26] in the form<sup>2</sup>

$$E_{ijkl}(x,t) = E_{ijkl\infty}(x) + \sum_{n=1}^{N_{ijkl}} E_{\underline{ijkl}n}(x) \exp\left(-\frac{t}{\tau_{\underline{ijkl}n}(x)}\right)$$
(10)

with (Fig. 3)

$$\underbrace{E_{ijkl}^{E}(x) \equiv E_{ijkl0}(x)}_{\text{instantaneous elastic moduli}} = \underbrace{E_{ijkl\infty}(x)}_{\text{fully relaxed moduli}} + \underbrace{\sum_{n=1}^{N_{ijkl}} E_{ijkln}(x)}_{\text{viscoelastic contributions}} .$$
 (11)

The constitutive relations (3) can be rewritten as

$$\sigma_{ij}(x,t) = \underbrace{\underbrace{E_{ijkl}(x)}_{\text{instantaneous elastic response}}^{E_{ijkl}(x)} \epsilon_{kl}(x,t)}_{\text{instantaneous elastic response}} + \underbrace{\int_{0}^{t} E_{ijkl}(x,t-t') \frac{\partial \epsilon_{kl}(x,t')}{\partial t'} \, \mathrm{d}t'}_{\text{viscoelastic response}}$$
(12)

The dimensions<sup>3</sup> of the various parameters are given in Table 4. This includes symbols with and without subscripts n. Therefore,

$$E_n = E_n^* \tau_n$$
 and  $C_n = C_n^* \tau_n$  (13)

etc. Typical elastic and viscoelastic moduli curves are shown in Fig. 3.



Fig.3: Normalized isotropic isothermal relaxation moduli

 $<sup>^2</sup>$  Underlined indices indicate no summations. The Einstein tensor notation is used throughout this paper.  $^3$  [F] = force, [L] = length, [T] = time

Parameters	Dimensions
$E_{ijkln}^*, \ E_{ijn}^{*\mathrm{T}}$	$[F/(L^2 - T)]$
$E_{ijkln}, E_{ijn}^{\mathrm{T}}$	$[\mathrm{F}/\mathrm{L}^2]$
$C^*_{ijkln}, \ C^{*\mathrm{T}}_{ijn}$	$[L^2/(F-T)]$
$C_{ijkln}, \ C_{ijn}^{\mathrm{T}}$	$[L^2/F]$
$ au_{ m n}$	[T]

Tab.4: Viscoelastic parameters ([F] = force, [L] = length, [T] = time)

At  $t = -\infty$  and in the interval  $-\infty \le t < 0$  all state variables are at rest, i.e.

$$\sigma_{ij}(x,t) = \epsilon_{ij}(x,t) = u_i(x,t) = 0 \quad \text{for} \quad t \in [-\infty,0) \tag{14}$$

and initial conditions (ICs) at t = 0 are prescribed for individual problems for state variables  $u_i(x,t)$  and  $\sigma_{ij}(x,t)$  and their time derivatives.

Elastic materials posses algebraic constitutive relations and represent conservative systems. Viscoelastic ones, on the other hand, respond to time hereditary integrals, which imply energy dissipation, creep, relaxation phenomena, memory and time histories. Additionally, viscoelastic thermal expansions, unlike their elastic counter parts, can take place in time at sustained zero stress levels throughout the entire medium.



Fig.4: The prismatic bar

# 2.2. Elastic and viscoelastic coupled thermal governing relations

As an illustrative example and for the sake of simplicity, consider the 1–D problem of a long isotropic prismatic bar with a heat source and impact load at the end  $x_1 = 0$  (Fig. 4). Standing 1–D displacement and thermal waves will travel down the bar toward the end at  $x_1 = L$  for t > 0 and the solution will be of the form

$$u_1(x,t) = u_1(x_1,t) , (15a)$$

$$T(x,t) = T(x_1,t)$$
, (15b)

$$\rho[x, t, T(x, t)] = \rho(x_1, t) \quad \text{viscoelastic only} , \qquad (15c)$$

$$E[x, t, T(x, t)] = E(x_1, t) \quad \text{viscoelastic only} , \qquad (15d)$$
$$x_1 \in [0, L] \quad \text{and} \quad t \in [0, \infty] .$$

While it may be justifiable in some instances to assume the density  $\rho$  to be temperature independent, the same cannot equally defensibly be said about the relaxation moduli E, G, K and their counterpart compliances (see Fig.2). The state variables are identified in Eqs. (15a)–(15c) while the modulus E[x, t, T(x, t)] is a temperature dependent material property. Other state variables, such as  $\sigma_{ij}$ ,  $\epsilon_{ij}$ ,  $\sigma$  and  $\epsilon$  will be eliminated through substitution from the governing relations (17)–(24).

In the elastic case, the waves will not dissipate any energy and eventually reflect from the end  $x_1 = L$  and continue indefinitely back and forth between the two ends. For the viscoelastic material, dissipation will take place with possibly no reflection from  $x_1 = L$  or with at most a finite number of reflections at both ends.

The driver energy source comes from either or both an input load and/or heat flow at  $x_1 = 0$ . The initial conditions (ICs) for the study at hand are due to impulsive inputs at  $x_1 = 0^4$ 

$$F(0,0) = F_0 \,\delta(t) \,\,, \tag{16a}$$

$$q_1(0,0) = q_{10}\,\delta(t) \,\,,\tag{16b}$$

$$u_1(x_1, 0) = 0 , (16c)$$

$$T(x_1, 0) = T_0$$
, (16d)

$$\rho(x_1, 0) = \rho_0 . (16e)$$

The bar and the state variables are at rest for  $t \in [-\infty \ 0]$ .

For small deformations (7) and linear elastic or viscoelastic materials (2) and (3), the governing relations describing the coupled 1–D thermal and displacement wave motions are given by Eqs. (17)-(24)

conservation of mass 
$$\implies \mathcal{L}_{CM}(u_1, T) = \frac{\mathcal{D}[\rho[x_1, t, T(x_1, t)] \quad \underbrace{\operatorname{div} \mathbf{u}(x, t)}_{\mathcal{D}t}]}{\mathcal{D}t} = 0 \quad (17)$$

where  $\Theta = T + T_0^*$  is the absolute temperature. For a 1–D ideal material, the law reads in various forms as

equation of state 
$$\implies \mathcal{L}_{\mathrm{ES}}(u_1, \rho, T) =$$

$$\begin{cases}
f(p, \rho, V, \Theta) = f(\sigma_{11}, \rho, \epsilon, \Theta) = 0 \\ & \text{or} \\ \sigma(x_1, t) = \underbrace{\frac{\sigma_{11}(x_1, t)}{3}}_{= \text{pressure } p} - \rho(x_1, t) R \Theta(x_1, t) = 0 \\ & \text{or} \\ \frac{\sigma_{11}(x_1, t)}{9} \underbrace{\left[\frac{\partial u_1(x_1, t)}{\partial x_1} + 2 \frac{\partial u_2(x_1, x_2, t)}{\partial x_2}\right]}_{= 3 \epsilon(x_1, t) = \epsilon_{ii}(x_1, t)} - R \Theta(x_1, t) = 0
\end{cases}$$
(18)

where the displacement  $u_2$  has for consistency at most the permissible form

$$u_2(x,t) = u_3(x,t) = [a_0(t) + a_1(t) x_2] f_{u_2}(x_1,t)$$
(19)

 $<sup>^4</sup>$  The force per unit area F has the units of  $[F/L^2].$  See Table 4.

with

constitutive relations 
$$\implies \mathcal{L}_{CR1}(u_1, T) = \sigma_{11}(x_1, t) =$$
  
$$= \int_{-\infty}^{t} E[x_1, t, t', T(x_1, t')] \frac{\partial^2 u_1(x, t')}{\partial x_1 \partial t'} dt' - \int_{-\infty}^{t} E^T[x_1, t, t', T(x_1, t')] \frac{\partial [\alpha \, \vartheta(x_1, t')]}{\partial t'} dt' , \quad (20)$$

constitutive relations  $\implies \mathcal{L}_{CR2}(u_1, T) = \epsilon_{22}(x_1, t) = \frac{\partial u_2(x_1, x_2, t)}{\partial x_2} =$  $= \int_{-\infty}^{t} C_{1122}[x_1, t, t', T(x_1, t')] \frac{\partial \sigma_{11}(x_1, t')}{\partial t'} dt' + \int_{-\infty}^{t} C^{T}[x_1, t, t', T(x_1, t')] \frac{\partial [\alpha \, \vartheta(x_1, t')]}{\partial t'} dt' ,$ <sup>(21)</sup>

conservation of thermal energy  $\implies \mathcal{L}_{TH}(u_1, T) =$ 

$$=\underbrace{\frac{\partial}{\partial x_1}\left\{k[T(x_1,t)]\frac{\partial T(x_1,t)}{\partial x_1}\right\}}_{-\frac{\partial}{\partial t}}\left\{\rho[x_1,t,T(x_1,t)]c_v[T(x_1,t)]T(x_1,t)\right\}}_{-\frac{\partial}{\partial t}}\left\{\rho[x_1,t,T(x_1,t)]c_v[T(x_1,t)]T(x_1,t)\right\}}$$

uncoupled rigid body heat conduction law [28]

$$- \left\{ \underbrace{\underbrace{E^{\mathrm{E}} \alpha T_{0} \frac{\partial^{2} u_{1}^{\mathrm{E}}(x_{1}, t)}{\partial x_{1} \partial t}}_{\text{Coupling Term } T_{4\mathrm{EL}}: \text{ elastic volume changes}} \left\{ \underbrace{\alpha T_{0} \frac{\partial}{\partial t} \left( \int_{-\infty}^{t} E[x_{1}, t, t', T(x_{1}, t')] \frac{\partial^{2} u_{1}(x_{1}, t')}{\partial x_{1} \partial t'} \mathrm{d}t' \right)}_{\text{Coupling Term } T_{4\mathrm{VE}}: \text{ viscoelastic volume changes}} \right\} = 0 .$$

$$(22)$$

The heat flux Q in has been eliminated from Eq. (22) through

Fourier's heat conduction law  $\implies$ 

$$\mathcal{L}_{\mathrm{HC}}(T) = \underbrace{Q(x_1, t)}_{\text{heat flux }(T_3)} + \underbrace{\kappa[T(x_1, t)]}_{\text{Conductivity Term }T_5} \frac{\partial T(x_1, t)}{\partial x_1} = 0$$
(23)

although convection, radiation, etc., laws are equally applicable.

conservation of linear momentum  $\implies$ 

$$\mathcal{L}_{LM}(u_1,T) = -\frac{\partial}{\partial t} \left( \rho[x_1,t,T(x_1,t)] \frac{\partial u_1(x_1,t)}{\partial t} \right) + f_1(x_1,t) + \frac{\partial \sigma_{11}(x_1,t)}{\partial x_1} = 0 \implies \left\{ \underbrace{-\frac{\partial}{\partial t} \left( \rho[x_1,t,T(x_1,t)] \frac{\partial u_1^{\rm E}(x_1,t)}{\partial t} \right)}_{\text{inertia force } (T_{1\rm E2})} + \underbrace{\frac{\partial}{\partial x_1} \left( E^{\rm E}(x_1) \frac{\partial u_1^{\rm E}(x_1,t)}{\partial x_1} \right)}_{\text{internal elastic stresses } (T_{2\rm E2})} \right\} + \underbrace{\frac{\partial}{\partial t} \left( \rho[x_1,t,T(x_1,t)] \frac{\partial u_1(x_1,t)}{\partial t} \right)}_{\text{inertia force } (T_{1\rm VE2})} + \underbrace{\frac{\partial}{\partial x_1} \left( \int_{-\infty}^{t} E[x_1,t,t',T(x_1,t')] \frac{\partial^2 u_1(x_1,t')}{\partial x_1 \partial t'} \, dt' \right)}_{\text{internal viscoelastic stresses } (T_{2\rm VE2})} \right\} +$$

external heat flow or

$$+\underbrace{f_{1}(x_{1},t)}_{\substack{\text{body force}\\ \text{other than}\\ (T_{3B2})}} - \left\{ \underbrace{\underbrace{\frac{\partial}{\partial x_{1}} \left[ E^{\text{E}}(x_{1}) \alpha \vartheta(x_{1},t) \right]}_{\text{Coupling Term } T_{4\text{EL2}}: \text{ stresses due to elastic thermal expansions}}_{\substack{\frac{\partial}{\partial x_{1}} \left( \int_{-\infty}^{t} E^{\text{T}}[x_{1},t,t',T(x_{1},t')] \frac{\partial[\alpha \vartheta(x_{1},t')]}{\partial t'} dt' \right)}_{\text{Coupling Term } T_{4\text{VE2}}: \text{ stresses due to viscoelastic thermal expansions}} \right\} = 0 . \quad (24)$$

It must be noted that regardless of whether or not the heat flow term  $T_3$  in (22) is time dependent, the coupling of the governing elastic or viscoelastic relations can only lead to a temperature function T(x,t) and never produce a steady state one. The temperature consequences are displayed in Table 3.

Aside from the readily visible important fundamental differences in elastic and viscoelastic governing Eqs. (22) and (24), there remains the most significant matter of the temperature dependence of Young's and relaxation moduli. When one eliminates all relaxation/creep influences at elevated temperatures from Young's modulus experimental measurements, the remainder shows little variations of elastic moduli with temperature [38–40]. Viscoelastic relaxation moduli, on the other hand, show extreme sensitivity to temperature due to real material variations in viscosity coefficients of approximately one order of magnitude per 20 °C – see Fig. 1. The additional most significant effect of this temperature dependence is to change the kernel functions in the hereditary integrals from E(x, t - t')to E(x, t, t') = E[x, t, t', T(x, t')] thus destroying the convenient properties of the convolution integrals and making the governing relations nonlinear in the case of the coupled temperature-displacement formulation.



Fig.5: A typical temperature shift function

A large class of viscoelastic materials, known as thermo-rheologically simple materials (TSMs), has behavioral responses that admit the presence of the WLF<sup>5</sup> material property

<sup>&</sup>lt;sup>5</sup> Also known as the Williams-Landel-Ferry shift factor/function.

shift function  $a_T$  [51] empirically defined as (Fig. 5)

$$\log_{e} \left[ a_{\mathrm{T}}(T) \right] = \frac{-\widetilde{C}_{1}(T - T_{\mathrm{R}})}{\widetilde{C}_{2} + T - T_{\mathrm{R}}} \quad \in \quad (-\infty, \infty] , \qquad T \ge T_{\mathrm{g}}$$
(25)

with  $T_{\rm R}$  a conveniently chosen constant reference temperature in °C and  $T_{\rm g}$  is the glassy temperature. Typical values for high polymers are  $\tilde{C}_1 \approx 17.4$  and  $\tilde{C}_2 \approx 51.6$  °C. The reference temperature  $T_R$  may, but need not, be equated to

(a) the rest temperature  $T_0$  at which the thermal expansions  $\alpha \vartheta$  vanish,

- (b) the glassy temperature  $T_{\rm g}$ ,
- (c) any other value provided  $T_{\rm R} \ge T_{\rm g}$ .

In this paper the temperature  $T_{\rm R}$  is arbitrarily set to  $T_{\rm g}$ . By the above definition (25) it then follows that at  $T = T_{\rm R}$ ,  $a_{\rm T} = 1$  and the reduced time of Eq. (27) is equal to the real time,  $\xi = t$ , i.e. no temperature shift of the moduli and compliances and moduli and compliances are the reference functions, such that

$$E(t) = \widehat{E}(\xi) \Big|_{\substack{a_T = 1\\ \xi = t}} .$$
(26)

This empirical well working model for TSMs defines an associated reduced time  $\xi(x, t)$  as [51], [52]

$$\xi [x, t, T(x, t)] \equiv \xi(x, t) = \int_{0}^{t} a_{\rm T}[T(x, s)] \,\mathrm{d}s = \int_{0}^{t} \exp\left(\frac{-\widetilde{C}_{1}[T(x, s) - T_{\rm R}]}{\widetilde{C}_{2} + T(x, s) - T_{\rm R}}\right) \,\mathrm{d}s \tag{27}$$

and reduces all relaxation moduli curves at many divers temperatures to a single master relaxation curve for each TSM with  $\hat{E}(x,\xi) = E[x,t,T(x,t)]$  vs.  $\xi$ . Further, examinations reveal that

$$\int_{-\infty}^{t} E_{ijkl}[x,t,t',T(x,t')] \frac{\partial \epsilon_{kl}(x,t')}{\partial t'} dt' \equiv \int_{-\infty}^{\xi(x,t)} \widehat{E}_{ijkl}[x,\xi(x,t)-\xi'] \frac{\partial \widehat{\epsilon}_{kl}(x,\xi')}{\partial \xi'} d\xi'$$
(28)

and thus the convolution integrals are restored in the  $\xi$ -space. However, any success at recapturing an elastic-viscoelastic correspondence principle in the  $\xi$ -space is thwarted by the fact that the  $x_i$  derivatives acquire variable coefficients due to the  $\xi$  transformations, to whit

$$\frac{\partial}{\partial x_i} = \underbrace{\frac{\partial \xi(x,t)}{\partial x_i}}_{= Z_i(x,t) = \widehat{Z}_i(x,\xi)} \frac{\partial}{\partial \xi} .$$
(29)

The transformation into the  $\xi$ -space mandates that

$$E_{ijkl}(x,t) \equiv \widehat{E}_{ijkl}(x,\xi) = E_{ijkl}^{\infty}(x) + \sum_{n=1}^{N_{ijkl}} E_{\underline{ijkl}n}(x) \exp\left(-\frac{\xi(x,t)}{\tau_{\underline{ijkl}n}^0(x)}\right)$$
(30)

where  $\tau_{ijkln}^0(x) = \tau_{ijkln} [T_R(x)]$ . Eqs. (28) should be preferentially used in the governing relations as they simplify the 'bookkeeping' and numerical solutions when used. When substituting (30) into (24) the more convenient evaluative form results [64]

$$T_{2\text{VE2}} = \underbrace{\frac{\partial}{\partial x_1} \left( \int\limits_{-\infty}^{t} E[x_1, t, t', T(x_1, t')] \frac{\partial^2 u_1(x_1, t')}{\partial x_1 \partial t'} \, \mathrm{d}t' \right)}_{\text{internal viscoelastic stresses } (T_{2\text{VE2}})}$$
(31)

$$= Z_1(x_1,t) \frac{\partial}{\partial \xi} \left( \sum_{n=1}^N \left\{ E_n(x_1) \exp\left(-\frac{\xi}{\tau_n^0(x_1)}\right) \int_{-\infty}^{\xi} \exp\left(\frac{\xi'}{\tau_n^0(x_1)}\right) \widehat{Z}_1(x_1,\xi') \frac{\partial^2 \widehat{u}_1(x_1,\xi')}{\partial \xi'^2} \,\mathrm{d}\xi' \right\} \right) \,.$$

An examination of Eqs. (22) and (24) reveals that the thermo-viscoelastic coupling is three fold, namely through

- the material dependence on temperature,  $E(T) \implies$  non-linear differential-integral relations
- the displacement terms in the energy relation,  $u_1(x,t) \Longrightarrow$  fully coupled governing relations
- the thermal expansions,  $\alpha_{ij} \vartheta(x,t) \Longrightarrow$  fully coupled governing relations

Of course, the last two conditions are also ever present in thermo-elasticity, with a secondary effect stemming from the weak dependence of Young's modulus,  $E_0$  on T(x,t). Additionally, the density is coupled with the temperature and displacement through the conservation of mass and state equations.

#### 2.3. Solution protocols

1. If the density and material and thermodynamic properties are approximated as temperature independent, then the resulting linear constant coefficient integro-differential equations (17) to (24) are linear and can be solved by standard procedures, such as for instance Laplace (LT) or Fourier (FT) transforms or by series of separable terms

$$E(x_1, t) = \widehat{E}[\xi(x, t), t] = E_{\infty} + \sum_{n=1}^{N} E_n \exp\left(-\frac{\xi(x_1, t)}{\tau_n}\right) , \qquad (32a)$$

$$\rho(x_1, t) = \text{const.} = \rho_0 , \qquad (32b)$$

$$u_1(x_1,t) = \sum_{n=1}^{\infty} \mathcal{U}(t) \,\Xi^u(x_1) = \sum_{n=1}^{\infty} A_n^u(t) \,\exp\left(\mathrm{i}\,k_n^u \,x_1\right) + \sum_{n=1}^{\infty} B_n^u(t) \,\exp\left(-\mathrm{i}\,k_n^u \,x_1\right) \,, \quad (32\mathrm{c})$$

$$T(x_1, t) = \sum_{n=1}^{\infty} \mathcal{T}(t) \Xi^T(x_1) = \sum_{n=1}^{\infty} C_n^T(t) \exp\left(i k_n^T x_1\right) + \sum_{n=1}^{\infty} D_n^T(t) \exp\left(-i k_n^T x_1\right).$$
 (32d)

Such a formulation, particularly vis-à-vis temperature independent material properties, may be considered as a first approximation to the more realistic development in 3. below.

2. Since in the identical thermo-elastic problem the temperature influence on material properties is marginally weak compared to viscoelastic media, the same integral transform protocols as above can be employed. 3. In the more realistic viscoelastic formulation where material properties are indeed temperature dependent and the density is not a constant, Eq. (32a) is replaced by (30) and Eq. (32b) by (18). The  $\rho$  acquires a solution in the form

$$\rho(x_1, t) = \sum_{n=1}^{\infty} \mathcal{R}(t) \,\Xi^{\rho}(x_1) = \sum_{n=1}^{\infty} E_n^{\rho}(t) \exp\left(\mathrm{i}\,k_n^{\rho}\,x_1\right) + \sum_{n=1}^{\infty} F_n^{\rho}(t) \exp\left(-\mathrm{i}\,k_n^{\rho}\,x_1\right) \,. \tag{33}$$

- 4. If material and/or thermodynamic properties are inclusively prescribed as temperature dependent, then the governing relations are nonlinear and currently there is no hope for an analytical solution. Numerical procedure involving finite element and finite difference approaches are then useful [64].
- 5. If boundary conditions corresponding to say absorbing or reflecting boundaries at  $x_1 = 0$  and/or  $x_1 = L$  are introduced, then Galerkin's method may be applied to the governing relation's spatial dependences and thus resulting in ordinary, rather than partial, temporal integro-differential nonlinear equations.
- 6. The collocation method many be used at one or more points, However, here the solution and its accuracy are very sensitive to the ensemble of selected points.
- 7. A Runge-Kutta approach may also be employed [62–64] (see [73]).
- 8. Alternately, for the nonlinear relations (17) to (24), the Poincaré-Kuo method of successive approximations may be used to linearize the simultaneous PIDEs [65], [66].
- 9. An approximate analytical EVCP approach realized through averaging temperature dependent moduli/compliances over time interval [60], [61] may also be employed if approximate analytical solutions are desired.

In order to bring answers/solutions to the coupled 1–D thermo-viscoelastic problem, a minimum of three simultaneous governing relations – (17), (22) and (24) – must be solved for the unknowns  $\rho$ , T and  $u_1$  or alternately a minimum of two relations if the density  $\rho$ is assumed to be approximately constant. In the latter case by essentially considering the density-temperature variations as higher order, one may eliminate Eqs. (17) and (18) from the ensemble to be solved.

The material property dependence on the temperature T is given by (27). The order of some of the derivatives contained in the governing relations may be reduced by using the  $E^*$  definitions instead of the E ones and also by introducing  $\epsilon_{11}$  as the fourth unknown and substituting it for  $u_1$  where appropriate. This approach then brings into play the fourth governing simultaneous relation (7) and trades  $\epsilon_{11}$  for  $u_1$  as one of the unknowns of choice.

#### 2.4. An illustrative example problem

General Problem Statement and BCs at  $x_1 = 0$ . – Assume that the elastic instantaneous moduli  $E_0$  are independent of temperature, which in effect is a reality since the viscoelastic moduli are the ones that are predominantly affected by temperatures [40]. In a homogeneous medium the  $E_0$  is, therefore, a constant. The viscoelastic wave front velocity will in a linear medium be elastic and given by

$$c_0 = c^{\rm E} = \sqrt{\frac{E_0}{\rho}} \tag{34}$$

with the Young's (instantaneous) modulus  $E_0$  a constant and associated with the specific viscoelastic medium. The boundary conditions at  $x_1 = 0$  are those of a load and a thermal

impact<sup>6</sup> such that

$$\sigma_{11}(0,t) = -F_0 \,\delta(t), \qquad \epsilon_{11}(0,t) = \frac{\sigma_{11}(0,t)}{E_0} = \frac{\partial u_1(0,t)}{\partial x_1} \tag{35}$$

and

 $q_1(0,t) = q_{00}\,\delta(t)$  and  $T(x,0) = \widetilde{T}_0$ . (36)

While these BCs are mathematically consistent and valid statements, they are physically unrealizable as it takes a finite time and a finite acceleration to build up a load and/or temperature level. In Ref. [38] the severe influences of various BCs on a much simpler 1–D uncoupled physical and mathematical wave problem are examined. However, as a purely theoretical formulation, the mathematical model is valid and self consistent.

The initial conditions are given by Eqs. (16) as

$$u_1(x_1,0) = \epsilon_{11}(x_1,0) = 0$$
,  $T(x_1,0) = T_0$ ,  $x_1 \in [0,L]$ . (37)

BCs at Far End  $x_1 = L$  or  $x_1 = L_L$ . – Three, by no means all inclusive, physically realistic conditions are presented for conditions at the far end that lead to no reflections, namely

- (a)All mechanical and thermal energies are dissipated at  $x_1 = L_L \leq L$ .
- (b)Absorbing boundary conditions (ABCs) are prescribed at  $x_1 = L$ .
- (c)Thermal and mechanical properties are tailored such that their ensemble values lead to no reflections at  $x_1 = L$  [72].

It is to be noted that the last two BCs are at the prescribed bar length L, whereas the first condition requires the solution of the unknown length  $L_{\rm L}$  in addition to the other state variables.

Let the length of the bar  $L \ge L_{\rm L}$  be such that when the wave fronts reach  $x_1 = L_{\rm L}$  at a time  $t_{\rm L}$  all energy has been dissipated and no reflective wave or waves take place. Therefore,

$$\epsilon_{11}(L_{\rm L}, t_{\rm L}) = \left. \frac{\partial u_1(x_1, t_L)}{\partial x_1} \right|_{x_1 = L_{\rm L}} = 0 \quad \text{and} \\ T(L_{\rm L}, t_{\rm L}) = \widetilde{T}_0 \quad \text{with} \quad t_{\rm L} = \frac{L_{\rm L}}{c_0} \le \frac{L}{c_0} \;.$$

$$(38)$$

The value of  $L_L$  and, therefore,  $t_L$  as well are part of the solution and may have to be determined iteratively with a first approximation of  $L_L = L$ . These quantities can be determined from the far end condition that the dissipative energy per unit volume  $DE(L_L, t_L)$  vanishes, or

$$DE(L_{\rm L}, t_{\rm L}) = 0 = \sigma_{11}(L_{\rm L}, t_{\rm L}) \epsilon_{11}(L_{\rm L}, t_{\rm L}) - \frac{\sigma_{11}^2(L_{\rm L}, t_{\rm L})}{2G_0}$$
(39)

where

$$\epsilon_{11}(L_L, t_L) = \int_{-\infty}^{t_L} C(t_L, t') \frac{\partial \sigma_{11}(L_L, t')}{\partial t'} dt' = \frac{\partial u_1(x_1, t)}{\partial x_1} \bigg|_{\substack{x_1 = L_L \\ t = t_L}} .$$
(40)

<sup>&</sup>lt;sup>6</sup> See [41], [44], [5–20] for uncoupled viscoelastic stress wave studies and recall footnote 4.

Subsequent values can, of course, be  $L_{\rm L} \leq L$ . A solution form meeting these BCs term by term is

$$u_1(x_1,t) = \sum_{n=1}^{N_u} \mathcal{A}_n^u(t) \cos\left(\frac{n\pi x_1}{L_{\rm L}}\right) + \sum_{n=1}^{N_u} \mathcal{B}_n^u(t) \sin\left(\frac{n\pi x_1}{L_{\rm L}}\right) \tag{41}$$

$$T(x_1, t) = \sum_{n=1}^{N_T} \mathcal{A}_n^T(t) \cos\left(\frac{n \pi x_1}{L_L}\right) + \sum_{n=1}^{N_T} \mathcal{B}_n^T(t) \sin\left(\frac{n \pi x_1}{L_L}\right)$$
(42)

$$\rho(x_1, t) = \sum_{n=1}^{N_{\rho}} \mathcal{A}_n^{\rho}(t) \cos\left(\frac{n \pi x_1}{L_{\rm L}}\right) + \sum_{n=1}^{N_{\rho}} \mathcal{B}_n^{\rho}(t) \sin\left(\frac{n \pi x_1}{L_{\rm L}}\right) \quad \text{or} \quad \rho(x_1, t) \approx \rho_0 \tag{43}$$

for  $x_1 \in [0, L_L]$  and  $t \in [0, t_L]$  with  $t_L < \infty$  and  $L_L \le L < \infty$ .

Alternately, ABCs can be specified where the length of the bar is specified and independent of the solution. For a 100% ABC that does not generate reflected waves the requirement is

$$\left. \frac{\partial u_1(x_1,t)}{\partial x_1} \right|_{x_1=L} = M \, u_1(x_1,t) \bigg|_{x_1=L} \tag{44}$$

where M is a constant.

For convenience one can prescribe the summation limits to be

$$N_u = N_{\rm T} = N_\rho = N^* \tag{45}$$

without loss of generality.

#### 3. Discussion and conclusions

Elastic formulations, although far different from the viscoelastic ones, are the degenerate much simplified viscoelastic cases. Unfortunately, because of the latter's extremely strong material property dependence on temperature the elastic-viscoelastic correspondence principles (analogies) are inapplicable in these thermal instances in either the real time space t or the reduced time space  $\xi[x, t, T(x, t)]$ . See Eq. (27).

The thermo-viscoelastic wave problems can be divided into four categories (see Table 6). The first formulation applies to the present problem only as a poor approximation. In

Conditions for uncoupling $E_{1}$ (22) $1$ (24)	Exact formulation	
Eqs. $(22)$ and $(24)$	if and only if	
totally uncoupled	time independent temperature	
	spatially independent temperature	
	temperature independent material properties	
	time independent displacements	
temperature uncoupled	constant temperature and	
	temperature independent material properties	
displacements uncoupled	time independent displacements	

Tab.5: Temperature-disc placement coupling (T-u)

Wave type	Displacement and temperature	Comments
uncoupled	two independent equations	no material temperature dependence
coupled with $T(x)$	two coupled relations	nonlinear, nonhomogeneous, no EVCP
fully coupled with $T(x,t)$	two coupled relations with variable coefficients	nonlinear, temperature dependent material properties, no EVCP
fully coupled and $\varrho[x, t, T(x, t)]$	three coupled relations with variable coefficients	nonlinear, temperature dependent material properties, no EVCP

Tab.6: Types of 1-D thermo-viscoelastic waves

the second case, even though the temperature only has spatial dependencies, EVCP is inapplicable due to nonlinear IPDEs. The third example is more inclusive with a spatial as well as temporal temperature dependency, but with constant density. The fourth example represents the full fledged thermo-viscoelastic wave problem. Example 2 to 4 all encompass nonlinear governing relations.

Results of an example with constant density but coupled thermal and viscoelastic displacement waves, including temperature dependent material properties, are depicted in Figs. 6 to 8. These graphs respectively show carpet plots of stress and temperature vs. distance and time for the specific 1–D nonlinear example described above. Of interest is the fact that the two waves in a bar with temperature dependent viscoelastic properties exhibit decreases in amplitudes with time while not in phase and that the two waves are of considerably different shapes. Similar responses for 1–D displacement wave with temperature independent viscoelastic properties have been observed previously [5], [10], [41], [47], [49] to mention a few.



Fig.6: Coupled viscoelastic stress waves for delta function force and heat inputs

Fig. 7 shows the details of a few viscoelastic stress wave fronts with decreasing amplitudes and the corresponding after flow, which is not present in elastic waves. The after flow and amplitude degeneration are caused by the energy dissipation. The typical highly significant material temperature dependence displayed in Figs. 1 and 5 indicate that the additional temperature coupling due to the functionality of viscoelasticity moduli E[x, t, T(x, t)] is of primary importance and must be considered in the analysis.



Fig.7: Coupled viscoelastic stress waves with after-flow for different times



Fig.8: Coupled thermal waves for delta function force and heat inputs

The applied force F(t) at  $x_1 = 0$  depicted in Fig. 4 can be readily related to any loadings other than impact, such as acoustic noise inputs, quasi-static and dynamic loads, etc. Similarly, distinct thermal inputs may be specified.

The complexity, magnitude and details of specific problems determine the type of computer to be used, i.e. from the present solution on a MacBook Pro<sup>TM</sup> laptop to the new NCSA/NSF Blue Waters<sup>TM</sup> sustained peta scale 10<sup>15</sup> flops/sec super computer [70], [71]. Additionally, the advent of large scale supercomputers has facilitated both analytical and numerical protocols. Software, such as MATLAB<sup>TM</sup>, MAPLE<sup>TM</sup>, MATHEMATICA<sup>TM</sup>, etc., is capable of carrying out the numerous analytical Galerkin steps and subsequently solve the large number of coupled Kutta-Runge simultaneous relations for the solution series coefficients. (See Eqs. (41) to (43) and [73].) Alternately, in larger more complicated formulations, spatial finite element and temporal finite difference protocols may be used.

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