ANALYSIS OF NONUNIFORM BEAMS ON ELASTIC FOUNDATIONS USING RECURSIVE DIFFERENTATION METHOD

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Analytical solutions for static and dynamic stability parameters ($P_{\rm cr}$ and $\omega_{\rm n}$) of an axially loaded nonuniform beam resting on a two parameter foundation are obtained using the recursive differentiation method (RDM) along with automatic differentiation. The analysis includes all cases of beam end conditions and indicates that the foundation stiffness influence is noticeable on both the critical load $P_{\rm cr}$ and natural frequency $\omega_{\rm n}$ in the case of slender beams. Also, it is found that the effect of the end conditions decreases as the slenderness parameter of the beam increases. In addition, the analysis concludes that neither the critical load nor the natural frequency corresponding to the first mode is always the smallest one in the case of beams on elastic foundations. The obtained solutions are verified and used to investigate the significance of different parameters on the critical loads and natural frequencies.

Keywords: *critical load, natural frequency, recursive differentiation method, beams on elastic foundation*

1. Introduction

Several numerical while few analytical methods were being developed to obtain approximate solutions for the static and dynamic behaviour of nonuniform beams. Using Ritz method, Sato [1] studied the transverse vibration of linearly tapered beams. Lee and Ke [2] proposed a fundamental solution for nonuniform beams with general end conditions. Vibration of nonuniform beams was analysed by Abrate [3] by transforming the equation of motion of some nonuniform beams to that for a uniform beam to obtain the natural frequencies. Differential transform (DTM) was used by Boreyri, et al [4] to study the nonuniform beams with exponentially variable thickness.

On the other hand, numerical methods such as: finite element method (FEM) used by Naidu, et al [5], and differential quadrature method DQM used by Chen [6], Hsu [7] and Taha and Nassar [8] offer tractable alternatives for beams that involve complicated configurations and/or complex boundary condition.

Recently, recursive differentiation method (RDM) proposed by Taha [9] have been used to solve many types of boundary value problems. However, the analytical solution for nonuniform beams needs the solution of differential equations with variable coefficients. In such cases, the RDM algorithm of the derived analytical expressions will include recursive differentiations which need lengthy calculations. An efficient tool to overcome such difficulty is the automatic differentiation (AD) which uses exact formulas along with floating-point

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values, instead of expression strings as in symbolic differentiation, and it involves no approximation error as in numerical differentiation using difference quotients (Neidinger [10]). Also computational differentiation or algorithmic differentiation may be used.

As a matter of fact, in the static stability analysis of axially loaded beams (columns), the determination of the axial load at which the beam lose its static stability is the main design issue, while in the dynamic analysis of beams, the natural frequency of the beam is the important factor to avoid resonance phenomenon which leads to unbounded response.

In the present work, the formulation of the RDM will be extended to include the solution of differential equations with variable coefficients, then employed to investigate the static and dynamic stability parameters of nonuniform beams resting on two parameter foundations. Analytic expression for the amplitude of the lateral displacement will be derived, and then the applications of the boundary conditions at beam ends yield the corresponding eigen problem in two parameters ($P_{\rm cr}$, $\omega_{\rm n}$). The solution of the eigen problem yields either the critical loads (for $\omega_{\rm n} = 0$) or the natural frequencies for the case ($P < P_{\rm cr}$). The influences of the different beam-foundation parameters on both the critical load and natural frequency will be investigated.

2. Problem statement

2.1. Equilibrium equation

The equations of dynamic equilibrium of an infinitesimal element of the axially loaded nonuniform beam resting on two parameter foundation shown in Fig. 1:

$$\frac{\partial V}{\partial x} + q(x,t) - k_1\, y(x,t) + k_2 \frac{\partial^2 y}{\partial x^2} = \varrho\, A(x) \frac{\partial^2 y}{\partial t^2}\ , \tag{1}$$

$$V(x,t) + p \frac{\partial y}{\partial x} = \frac{\partial M}{\partial x}\ . \tag{2}$$

The beam flexure constitutive equation is:

$$M(x,t) = -E\,I(x) \frac{\partial^2 y}{\partial x^2}\ . \tag{3}$$

where $E\,I$ is the flexural stiffness of the beam, ϱ is the density, A is the area of the cross section, p is the axial applied load, k_1 and k_2 are the linear and shear foundation stiffness per unit length of the beam, $q(x,t)$ is the lateral excitation, E is the modulus of elasticity, I is the moment of inertia, $V(x,t)$ is the shear force, $M(x,t)$ is the bending moment, $y(x,t)$ is the lateral response of the beam, x is the coordinate along the beam and t is time.

Substitution of Eq. (2) and Eq. (3) into Eq. (1), then the equation of lateral response of a nonuniform beam is obtained as:

$$\frac{\partial^2}{\partial x^2}\left(E\,I(x) \frac{\partial^2 y}{\partial x^2}\right) + (p - k_2) \frac{\partial^2 y}{\partial x^2} + k_1\, y(x) + \varrho\, A(x) \frac{\partial^2 y}{\partial t^2} = q(x,t)\ . \tag{4}$$

For free vibration:

$$q(x,t) = 0\ . \tag{5}$$

Using the dimensionless variables:

$$\xi = \frac{x}{L} \qquad \text{and} \qquad \phi(\xi,t) = \frac{y(x,t)}{L}\ , \tag{6}$$

where $\phi(\xi,t)$ is the dimensionless lateral response and L is the beam length.

Then, a dimensionless form of Eq. (4) may be rewritten as:

$$\frac{\partial^2}{\partial\xi^2}\left(E\,I(x)\,\frac{\partial^2\phi}{\partial\xi^2}\right) + (p-k_2)\,L^2\,\frac{\partial^2\phi}{\partial\xi^2} + k_1\,L^4\,\phi(\xi) + \varrho\,A(\xi)\,L^4\,\frac{\partial^2\phi}{\partial t^2} = 0 \ . \qquad (7)$$

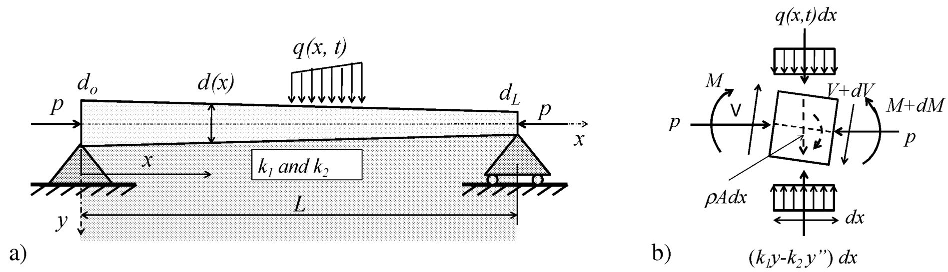

Fig.1: a) Nonuniform beam on elastic foundation, b) element forces

Linear variation of the beam depth or width is assumed in many researches, however in the present work, any continuous variation may be assumed. Assuming the width of the beam b is constant while the depth $d(\xi)$ varies in an exponential form as:

$$d(\xi) = d_0\,\mathrm{e}^{\alpha\,\xi} \ , \quad \text{then} \quad \alpha = \ln\left(\frac{d_\mathrm{L}}{d_0}\right) \ , \qquad (8)$$

where d_0 and d_L are the beam depths at $\xi = 0$ and $\xi = 1$ respectively. Practically, the exponential variation may be easily used to simulate various real cases of beam variations, such as the case of a beam with linearly varying depth where the exponential variation fits accurately the depth variations up to ($d_\mathrm{L}/d_0 = 1.5$). Using Eq. (8), then the moment of inertia $I(\xi)$ and the area $A(\xi)$ of the beam cross sectional are:

$$I(\xi) = I_0\,\mathrm{e}^{3\,\alpha\,\xi} \qquad \text{and} \qquad A(\xi) = A_0\,\mathrm{e}^{\alpha\,\xi} \ . \qquad (9)$$

Substitute Eq. (9) into Eq. (7), and assume the solution in the form $\phi(\xi,t) = w(\xi)\,\mathrm{e}^{\mathrm{i}\,\omega\,t}$, then the equation of mode functions of the beam free vibration may be expressed as:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}\xi^4} + 6\,\alpha\,\frac{\mathrm{d}^3 w}{\mathrm{d}\xi^3} + \left[(P-K_2)\,\mathrm{e}^{-3\,\alpha\,\xi} + 9\,\alpha^2\right]\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} + \left(K_1\,\mathrm{e}^{-3\,\alpha\,\xi} - \lambda^4\,\mathrm{e}^{-2\,\alpha\,\xi}\right) w(\xi) = 0 \ . \qquad (10)$$

where the following dimensionless parameters are defined:

$$K_1 = \frac{k_1\,L^4}{E\,I_0} \ , \quad K_2 = \frac{k_2\,L^2}{E\,I_0} \ , \quad P = \frac{p\,L^2}{E\,I_0} \ , \quad \lambda^4 = \frac{\varrho\,A_0\,\omega^2\,L^4}{E\,I_0} \ ,$$
$$\eta = \frac{L}{r} \quad \text{and} \quad r = \sqrt{\frac{I_0}{A_0}} \ . \qquad (11)$$

The parameters K_1 and K_2 are the foundation linear and shear stiffness parameters respectively, P is the axial load parameter, λ is the natural frequency parameter, η is the slenderness parameter and r is the radius of gyration of the beam cross section.

2.2. Boundary conditions

The boundary conditions in dimensionless forms may be expressed as:
For the case of the pinned-pinned (P-P) beams:

$$w(0) = w''(0) = 0 \quad \text{at} \quad \xi = 0 \qquad \text{and} \qquad w(1) = w''(1) = 0 \quad \text{at} \quad \xi = 1\,. \tag{12}$$

For the case of the clamped-pinned (C-P) beams:

$$w(0) = w'(0) = 0 \quad \text{at} \quad \xi = 0 \qquad \text{and} \qquad w(1) = w''(1) = 0 \quad \text{at} \quad \xi = 1\,. \tag{13}$$

For the case of the clamped-clamped (C-C) beams:

$$w(0) = w'(0) = 0 \quad \text{at} \quad \xi = 0 \qquad \text{and} \qquad w(1) = w'(1) = 0 \quad \text{at} \quad \xi = 1\,. \tag{14}$$

For the case of the clamped-free (C-F) beams:

$$\begin{aligned} &w(0) = w'(0) = 0 \quad \text{at} \quad \xi = 0 \qquad \text{and} \\ &w''(1) = 0\,, \quad w'''(1) = -P\,\mathrm{e}^{-3\,\alpha}\,w'(1) \quad \text{at} \quad \xi = 1\,. \end{aligned} \tag{15}$$

3. RDM solution

3.1. Recursive differentiation method (RDM)

Consider an nth-order non-homogeneous linear differential equation with variable coefficients is given in the form:

$$y^{(n)}(x) = \sum_{i=1}^{n} A_{1,i}(x)\,y^{(i-1)}(x) + F_1(x) \qquad a \le x \le b \tag{16}$$

with the boundary conditions:

$$B_i(y, y^{(1)}, \dots, y^{(n-1)}) = b_i\,, \quad i = n \tag{17}$$

where $y^{(n)}$ is the nth-derivative, $A_{1,i}(x)$ are the variable coefficients for the first recursion, $F_1(x)$ is the source function for the first recursion and b_i are constants. The solution of Eq. (16) using RDM, which based on Taylor's expansion, is given as [9]:

$$y(x) = \sum_{m=1}^{n} T_m\,R_m(x) + R_{\mathrm{F}}(x)\,, \tag{18}$$

where the recursive functions $R_m(x)$ and $R_{\mathrm{F}}(x)$ are given by:

$$R_m(x) = \frac{x^{m-1}}{(m-1)!} + \sum_{i=1}^{N-n} A_{i,m}(a)\,\frac{x^{n+i-1}}{(n+i-1)!}\,, \quad m = 1:n\,, \tag{19}$$

$$R_{\mathrm{F}}(x) = \sum_{i=1}^{N-n} F_i(a)\,\frac{x^{n+i-1}}{(n+i-1)!}\,. \tag{20}$$

and N is the truncation index that achieves the acceptable accuracy.

The recurrence formulae for the weighting coefficients $A_{k+1,i}(x)$ and $F_{k+1}(x)$ may be obtained as:

$$\begin{aligned} A_{k+1,1}(x) &= A^{(1)}_{k,1} + A_{k,n}\, A_{1,1} \ , \\ A_{k+1,r}(x) &= A_{k,r-1} + A^{(1)}_{k,r} + A_{k,n}\, A_{1,r} \ , \quad r = 2:n \\ F_{k+1}(x) &= F^{(1)}_{k} + A_{k,n}\, F_1(x) \ . \end{aligned} \tag{21}$$

The substitution of the boundary conditions at the beam ends into Eq. (18) yields a system of n algebraic equation in n unknowns; namely; the coefficients T_m, $m = 1:n$. However, the solution of such system yields T_m which may be substituted into Eq. (18) to obtain an analytical solution for the given differential equation.

In addition, in case of homogeneous differential equation ($F_1(x) = 0$), the substitution of the boundary conditions into Eq. (18) yields a homogeneous system of n algebraic equations in n unknowns. The nontrivial solution of such system requires that the determinant of the coefficients matrix should be vanished. The expansion of the determinant leads to the characteristic equation of the differential equation. The solution of such expansion is known as the eigen value problem of the system and yields the eigen values and the eigen vectors. Indeed, in the case of static analysis of beams, the eigen values represent the critical (buckling) loads; $P_{\text{cr-r}}$; while the eigen vectors represent the buckling modes $\varphi_r(x)$. In addition, for the case of dynamic analysis of beams, the eigen values represent the natural frequencies while the eigen vectors represent the mode shapes of free vibration $\varphi_r(x)$.

3.2. Application of RDM to the equation of the nonuniform beam vibration

To use the RDM, Eq. (10) is rewritten in the recursive form:

$$w^{(4)}(\xi) = A_{(1,1)}\, w^{(0)} + A_{(1,2)}\, w^{(1)} + A_{(1,3)}\, w^{(2)} + A_{(1,4)}\, w^{(3)} \ , \tag{22}$$

where:

$$\begin{aligned} A_{(1,1)} &= \lambda^4\, e^{-2\,\alpha\,\xi} - K_1\, e^{-3\,\alpha\,\xi} \ , \qquad & A_{(1,2)} &= 0 \ , \\ A_{(1,3)} &= (K_2 - P)\, e^{-3\,\alpha\,\xi} - 9\,\alpha^2 \ , \qquad & A_{(1,4)} &= -6\,\alpha \ . \end{aligned} \tag{23}$$

Using Eq. (15), then the solution of Eq. (22) may be expressed as:

$$w(\xi) = \sum_{i=1}^{4} T_m\, R_m(\xi) \ , \tag{24}$$

where the recursive functions $R_m(\xi)$, $m = 1:4$ are obtained from Eq. (19) as:

$$R_m(\xi) = \frac{\xi^{m-1}}{(m-1)!} + \sum_{i=1}^{N-4} A_{(i,m)}(0)\, \frac{\xi^{i+3}}{(i+3)!} \ , \quad m = 1:4 \ . \tag{25}$$

Substitution of Eq. (24) into the boundary conditions (Eq. (12):Eq. (15)) yields the characteristic equation for the corresponding case of the end conditions as:

For P-P case:

$$R_{20}\, R_{42} - R_{22}\, R_{40} = 0 \ , \tag{26}$$

for C-C case:

$$R_{30}\, R_{41} - R_{31}\, R_{40} = 0 \ , \tag{27}$$

for C-P case:

$$R_{30}\,R_{42} - R_{32}\,R_{40} = 0\ , \tag{28}$$

for C-F case:

$$R_{32}\,(R_{43} + P\,\mathrm{e}^{-3\,\alpha}\,R_{41}) - R_{42}\,(R_{33} + P\,\mathrm{e}^{-3\,\alpha}\,R_{31}) = 0\ , \tag{29}$$

where $R_{mk} = R_m^{(k)}$, $m = 1 : 4$ at $\xi = 1$.

Assembling a simple MATLAB code, using the automatic differentiation (Neidinger [10]), the coefficients $A_{k+1,i}(x)$ can be obtained up to the required accuracy. Hence, applying a proper iterative technique, the solution of the characteristic equations yields the fundamental (smallest) critical load P_{cr} in the static case ($\omega = 0$) for the corresponding end conditions. However, assuming any value for $P < P_{\mathrm{cr}}$, then the fundamental natural frequency ω_{n} for a nonuniform beam subjected to axial load may be obtained.

3.3. Verification

To verify the solutions obtained from the RDM, the natural frequency parameter λ_r of a nonuniform beam calculated from the RDM and those obtained from the 4$^{\mathrm{th}}$ order Runge-Kutta analytical solutions and DTM [4] are presented in Table 1 for different values of non-uniformity exponent α. It is clear that the RDM results are very close to those calculated from DTM as the two techniques are based on Taylor expansion.

α	Method	C-C			P-P		
		λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
−0.2000	DTM	4.5048	7.4713	10.4580	2.9948	5.9765	8.9628
	RK4	4.5069	7.4929	10.4889	2.9952	5.9881	9.0223
	RDM	4.5018	7.4706	10.458	2.9844	5.9752	8.9624
0.0000	DTM	4.7324	7.8537	10.9958	3.1496	6.2842	9.4251
	RK4	4.7335	7.8642	11.0473	3.1502	6.3001	9.4593
	RDM	4.73	7.8532	10.996	3.1416	6.2832	9.4248
0.3000	DTM	5.1062	8.4662	11.8480	3.3818	6.7692	10.1521
	RK4	5.1119	8.5277	12.0001	3.3824	6.7912	10.2850
	RDM	5.1046	8.4658	11.848	3.3762	6.7685	10.152
0.5000	DTM	5.3793	8.9020	12.4477	3.5350	7.1085	10.6597
	RK4	5.3855	8.9773	12.6389	3.5354	7.1341	10.8272
	RDM	5.378	8.9017	12.448	3.5305	7.108	10.66

Tab.1: Parameter of natural frequencies for nonuniform beams ($k_1 = k_2 = 0$)

4. Numerical Results and discussion

4.1. Values of the foundation stiffness parameters k_1 and k_2

The values of foundation stiffness (k_1, k_2) are calculated using the expressions proposed by Zhaohua et al. [11] for a rectangular beam resting on a two parameter foundation as:

$$k_1 = \frac{E_0\,b}{2\,(1-\nu_0^2)}\,\frac{\delta}{\chi} \qquad \text{and} \qquad k_2 = \frac{E_0\,b}{4\,(1-\nu)}\,\frac{\chi}{\delta}\ . \tag{30}$$

where:

$$\chi = \sqrt[3]{\frac{2\,E\,I\,(1-\nu_0^2)}{b\,E_0\,(1-\nu^2)}}\ , \qquad E_0 = \frac{E_{\mathrm{s}}}{1-\nu_{\mathrm{s}}^2} \qquad \text{and} \qquad \nu_0 = \frac{\nu_{\mathrm{s}}}{1-\nu_{\mathrm{s}}}\ ,$$

E and ν are the elastic modulus and Poisson ration of the beam respectively, E_s and ν_s for the foundation and δ is a parameter accounts for the beam-foundation loading configuration (it is a common practice to assume $\delta = 1$). Typical values of the elastic modulus and Poisson ratio for different types of foundations are given in Table 2.

Type of foundation	No foundation (NF)	Weak foundation (WF)	Medium foundation (MF)	Stiff foundation (SF)
E_s [N/m^2]	0	1×10^7	5×10^7	1×10^8
Poison ratio ν_s	0	0.40	0.35	0.30

Tab.2: Typical values for modulus of elasticity and Poisson ration for soils

The obtained solution expressions are derived in dimensionless forms for the sake of generality, however, in the present analysis, the properties of the beam are; concrete beam, $b = 0.2\,\mathrm{m}$, $h = 0.5\,\mathrm{m}$, $E = 2.1\times10^{10}\,\mathrm{Pa}$, Poisson ratio $\nu = 0.15$.

Actually, the smallest value of (P_{cr} or λ_f) is the most important value (or the fundamental value) which controls the stability behavior of the beam. In the present work, the used iteration technique picks P_{cr} (or λ_f) in ascending order according to the value not the mode number, then the first picked one is the most critical value. Furthermore, in the case of a beam on elastic foundation, the critical load corresponding to the first buckling mode is not always the smallest one as it is commonly known for beam without foundation. In addition, in the case of a beam on elastic foundation loaded by axial compression load, the natural frequency corresponding to the first vibration mode is not always the smallest natural frequency [9]. The distribution of the displacement along the beam may be traced and used to identify the mode number corresponding to the smallest picked value.

The above analysis is used to investigate the dependence of the stability parameters of the beam-foundation system (P_{cr} and λ_f) on the non-uniformity exponent α, the slenderness parameter η, the foundation stiffness parameters (K_1 and K_2) and the loading parameter γ for different beam end conditions. The non-uniformity exponent α and the loading parameter γ are defined as:

$$\gamma = \frac{p}{p_{cr}} \qquad \text{and} \qquad \alpha = \ln\left(\frac{d_L}{d_0}\right) . \tag{31}$$

where p and p_{cr} are the axial applied load and the critical load respectively and d_0 and d_L are the beam depths at $\xi = 0$ and $\xi = 1$ respectively.

4.2. Variation of critical load

The variations of P_{cr} with α for the case of a beam with $\eta = 50$ without foundation (NF) or resting on foundation of medium stiffness (MF) for different end conditions are shown in Fig. 2. It is obvious that as α increases P_{cr} increases while the effect of α on P_{cr} is negligible for C-F beams without foundations. For P-P beams on MF the interaction between the beam stiffness and the foundation stiffness depends on I_0 which leads to variation in the trend at $\alpha = 0$.

The variations of P_{cr} with η, for the case of NF and MF (no foundation and foundation of medium stiffness) and $\alpha = 0$, 0.5 for different end conditions are shown in Fig. 3. In Fig. 4, the variations of P_{cr} with η, for the case of WF and SF (weak stiffness foundation and stiff foundation) and $\alpha = 0.5$ for different end conditions is illustrated. It is found that

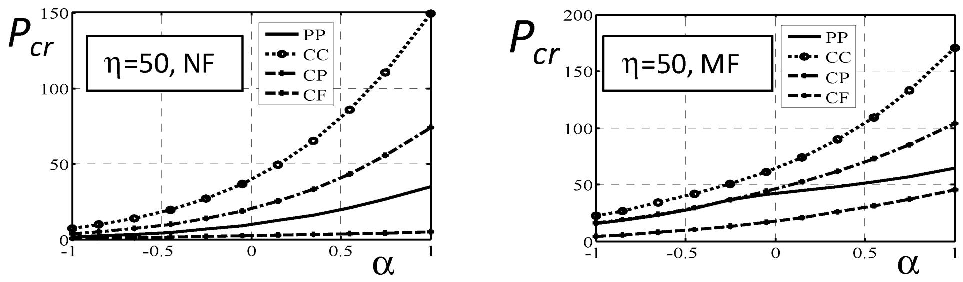

Fig.2: Variation of P_{cr} with α

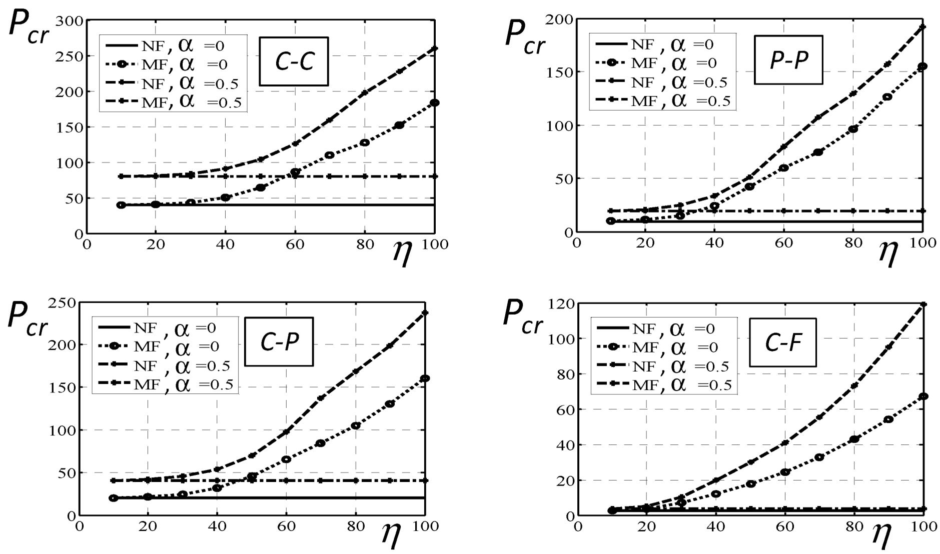

Fig.3: Variation of P_{cr} with η

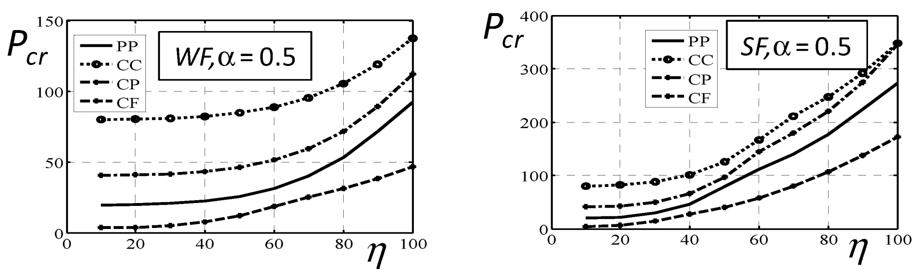

Fig.4: Variation of P_{cr} with η and the beam end conditions

the critical load increases as α increase. However, it should be noted that as η increases, the stiffness of the foundation relative to the beam increases, and the influence of the foundation stiffness on P_{cr} becomes more noticeable. The variation of α is more effective in the case of C-C beams. Also, the influence of the foundation stiffness is more effective in the case of C-F slender beams. The transition between different buckling modes is obvious in the case of SF. The irregularities in the $P_{\text{cr}} - \eta$ indicate the transitions of the beam from a buckling mode to the consequent higher one for the case of slender beams.

4.3. Variation of natural frequency

The variations of λ_f with α for the case of a beam with $\eta = 50$ and $\gamma = 0$, 0.5 resting on no foundation (NF) or foundation of medium stiffness (MF) for different end conditions are shown in Fig. 5. It is obvious that as α increases λ_f increases while the effect of α on λ_f is negligible for C-F beams when $\gamma = 0$. Also, it is found that as γ increases, λ_f decreases.

The variations of λ_f with η, $\alpha = 0.5$, for different foundation stiffness and different beam end conditions are shown in Fig. 6. It should be noted that as η increases, the foundation influence on λ_f increases, hence as α increases, λ_f increases.

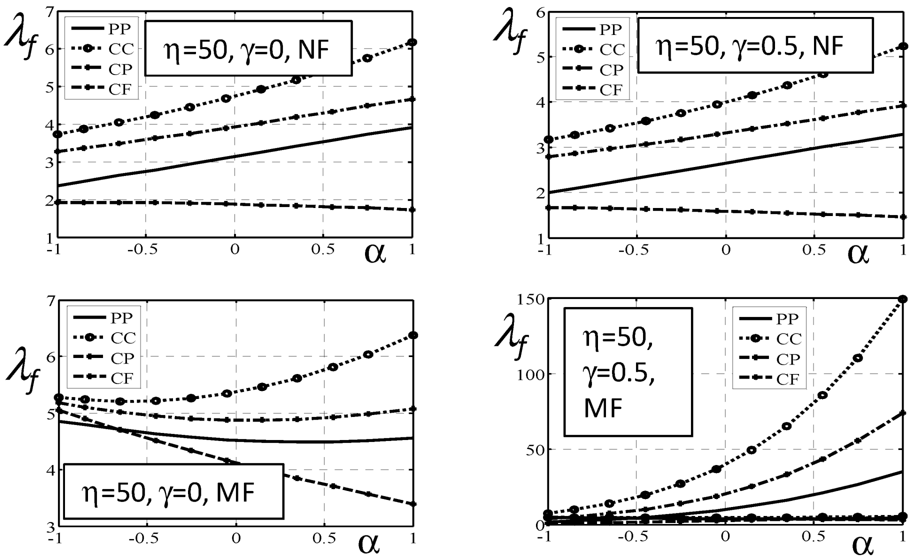

Fig.5: Variation of λ_f with α

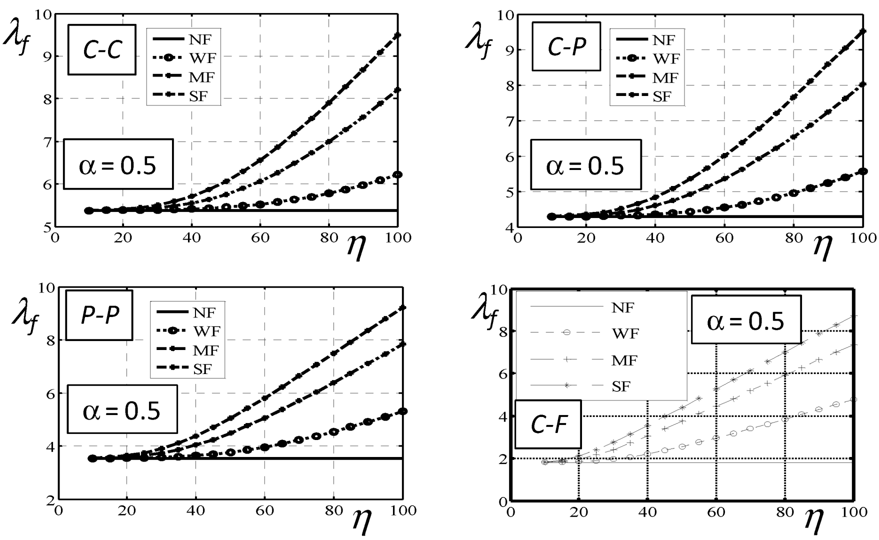

Fig.6: Variation of λ_f with the beam end condition and η

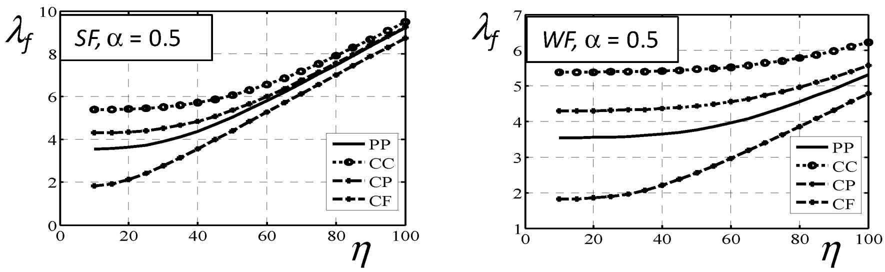

Fig.7: Variation of λ_f with η and the beam end conditions

In Fig. 7, the effect of η on λ_f is shown for the case of WF and SF for different end conditions. It is obvious that the effect of the end conditions decreases as η increases.

The effect of loading parameter γ on λ_f is shown on Fig. 8 to Fig. 11 for different beam-foundation system parameters. It is found that λ_f decreases as γ increases and the free vibration of the system die out as γ approaches one. The effect the foundation stiffness on the λ_f is insignificant for short beams on elastic foundations.

5. Conclusions

The static and dynamic stability behavior of nonuniform beams resting on two parameter foundations and subjected to axial load was analyzed using the recursive differentiation method (RDM). A simple MATLAB code for automatic differentiation is assembled and used to drive the weighting coefficients up to the required accuracy, then the recursive functions of the problem are derived. However, the application of the end conditions yields the corresponding eigen value problem in two parameter (P_{cr} and ω_n). The solution of the equivalent eigen problem yields either the critical loads or the natural frequencies of the

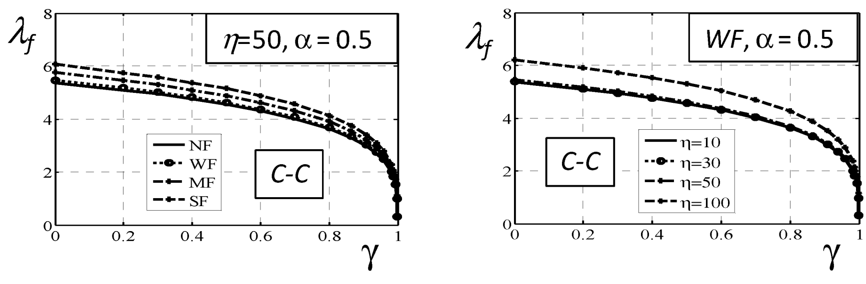

Fig.8: Variation of λ_f with γ and foundation stiffness for C-C beams

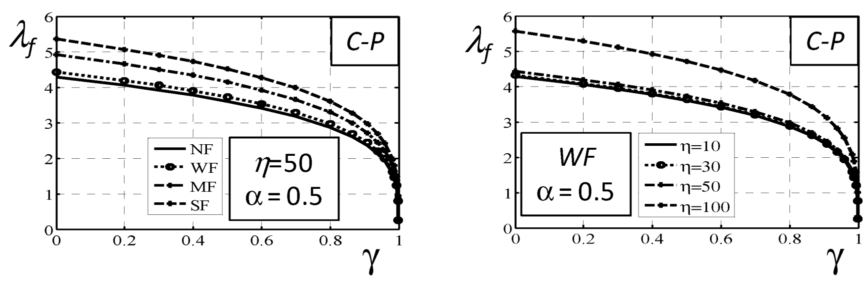

Fig.9: Variation of λ_f with γ and foundation stiffness for C-P beams

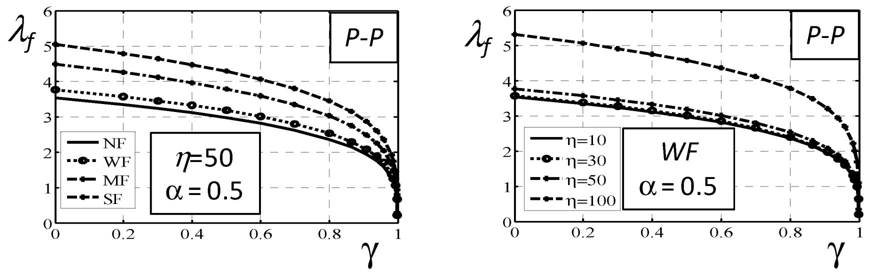

Fig.10: Variation of λ_f with γ and foundation stiffness for P-P beams

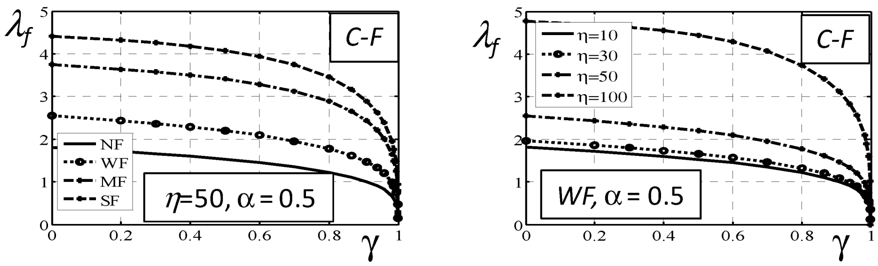

Fig.11: Variation of λ_f with γ and foundation stiffness for C-F beams

beam-foundation system for different end conditions. Both the smallest critical load and the smallest natural frequency are investigated to indicate the significance of the slenderness parameter of the beam (η), the foundation stiffness relative to the beam (K_1 and K_2), the non-uniformity exponent (α), the applied axial load parameter (γ) and the beam end conditions. The influence of foundation controls the behavior of the system in the case of slender beams ($\eta > 40$) while the end condition controls the behavior of short beams ($\eta < 40$). It is observed that, both the natural frequencies and the critical loads to be increase with the increase of α and as γ increases, the natural frequency decreases till the free vibration die out at $\gamma = 1$. Also, it is found that, as α increases, the stiffness of the foundation relative to the beam decreases; hence, the total stiffness of the system decreases in the case slender beams where the foundation stiffness is the dominant. The effect of the non-uniformity exponent α on the critical load is more significant in the case of C-C beams while for C-F beams, the effect of α is noticeable only for slender beams ($\eta > 40$). It should be noted that the relative stiffness of the C-F beam decreases as α increases and vice versa for the F-C beams.

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